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THE *MARĪCI* COMMENTARY ON THE *JYOTPATTI**

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The *Jyotpatti* composed by Bhāskara II (*circa* 1150 A.D.) in Sanskrit may be considered as a short tract on ancient Indian trigonometry. Besides a sort of summary of Hindu trigonometry upto the time, it contains several results which make their first appearance in India through this work, for example, the exact values of the sines of 18 and 36 degrees and the addition and subtraction theorems for the Sine function. However, the work, or the author's own commentary on it, does not contain any demonstration of these new results, a fact which led M. M. Sudhakara Dvivedi to suspect the workto be based on foreign material.

Munīśvara's *Marīci* (1638 A.D.) is a comprehensive commentary on the *Jyotpatti*. It not only contains a variety of rationales of trigonometrical rules found in *Jyotpatti*, but also important historical material reflecting interesting facts, for instance, a criticism of some earlier wrong derivations of the rules, and references to works of foreign authors. However, no mention or reference to the significant developments of Indian trigonometry by the late Āryabhaṭa School is found in the *Marīci* commentary.

This may indicate that the trigonometrical novelties of South India were not known in the Northern parts of the country which seems to be experiencing a decadence of indigenous science and culture under the Muslim rules since about 1200 A.D. But the North India has the advantage of getting opportunity to become familiar with Arabic material, or Arabic version of Greek material, on trigonometry culminating in Jagannātha's Sanskrit translation (eighteenth century) of an Arabic version of Ptolemy's *Almagest*.

1. INTRODUCTION

The famous Bhāskarācārya (born 1036 *śaka*=1114 A.D.), son of Maheśvara, was a great Indian astronomer and mathematician. He is designated as Bhāskara II to distinguish him from his namesake, Bhāskara I, who lived in the seventh century A.D. Bhāskara II is the author of *Līlāvatī* (the most populat book of Hindu mathematics), *Bījagaṇita* (or Algebra), *Siddhānta-Śiromaṇi* (= *SS*), etc. The *SS* (composed at the age of 36 years) is devoted to astronomy and is supplied with author's own commentary called *Vāsanā-Bhāṣya* (=*VB*).

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The *SS* consists of two parts, the *Graha-gaṇita* and the *Golādhyāya* (=*SSG*), which may be considered as two distinct works. In fact the *Līlāvatī* (devoted to Arithmetic, mensuration, etc.), *Algebra*, *Graha-gaṇita*, and *SSG* are often taken to form a single but voluminous work which is designated as *SS*, but the four parts are really four separate works each starting with its own benedictory verse and frequently found in independent manuscript as well as in printed form.

Any way, the *SSG* treats of a topic called *Jyotpatti* at two places. The first occasion is the fifth chapter, called *Sphuṭa-gati-vāsanā*, where the first six stanzas are devoted to it. These contain some basic definitions and rules of Hindu trigonometry. The *VB* on these ends with the words[1]

ato'vaśiṣṭām jyotpattimagre vakṣyāmaḥ

'The remaining *Jyotpatti* we shall deal ahead'.

True to this promise, the author dealt with *Jyotpatti* again at the end of *SSG*. This *Jyotpatti* work consists of 25 Sanskrit stanzas and may be considered to be the last (i.e. 14th) chapter of *SSG*, but it is better to regard it as an appendix to *SSG* which, strictly speaking and according to author's own statement[2], starts with the word *siddhi* and ends with the word *vṛddhim* after which only the *Jyotpatti* starts. Moreover, this *Jyotpatti* may very well be considered as a separate small tract on ancient Indian trigonometry. In fact the *Marīci* commentary (=*MC*), written by Munīśvara (also called Viśvarūpa) about 1638, on *SSG* regards it so and deals with it just after the earlier or smaller *Jyotpatti* portion (of 6 stanzas) in the fifth chapter itself[3].

The *Jyotpatti* tract (of 25 stanzas) has many trigonometrical rules which make their first appearance in India through this work. These include:

(*i*) The exact values of Sines of 18° and 36°.

(*ii*) Addition and Subtraction Theorems for the Sine function.

Since the author, contrary to his usual practice, has not given any explicit derivation or demonstration of these rules (in the text or in the *VB*), M. M. Sudhakara Dvivedi[4] suspected foreign origin of the material and pointed out its unjustified inclusion in the *SSG* which has been formally ended by the word *vṛddhim* by the author (see above).

The *MC* does supply the demonstrations of the above rules and much additional information which has several interesting implications. Details follow.

2. Informations from MC

The opening line of the *Jyotpatti* says that by knowing the *Jyotpatti* one acquires the rank of *ācārya*. The *MC* (p. 141) explains that this is because the subject matter (of trigonometry) was considered to be *atidurgam* (very difficult to acquire)!

Verses 2 and 3 describe the graphical method of obtaining various Sine values by actually drawing the circle by taking the desired *Sinus totus* (or radius) in *aṅgulas* (digits or finger-breaths). This simple method of direct measurement was already described by Brahmagupta (628 A.D.) and his commentator Pṛthūdaka (*c.* 860) asks us to draw the circle of radius 3270 (which is Brahmagupta's *Sinus totus*) *aṅgulas* with a pair of compasses (*karkaṭa*)[5]. If the *aṅgula* mentioned be taken to be equivalent to three-fourth of an inch, where from to get the big compass of more than 100 feet arm![6] The *MC* does not provide an answer.

Verse 7 contains the exact value

$$\text{Sin } 36^\circ \ (=R \sin 36^\circ) = \sqrt{\frac{5R^2-\sqrt{5R^4}}{8}} \quad \ldots (1)$$

This result occurs in India for the first time in Bhāskara II's work but without any demonstration. The *MC* (p. 142) quotes a wrong derivation as given by Lakṣmīdāsa Miśra (*c.* 1500) :

$$\text{Sin } 30^\circ = \frac{R}{2} = \sqrt{\frac{4R^2-\sqrt{4R^4}}{8}}$$

$$\text{Sin } 60^\circ = \frac{\sqrt{3}R}{2} = \sqrt{\frac{9R^2-\sqrt{9R^4}}{8}}$$

from which it is seen that when the angle is 30°, the coefficient in its Sine value (in the numerator under the square root signs) is 4 (both for R^2 and R^4). And when the angle is 60°, the coefficient is 9. Hence it was argued (by Miśra) that when the angle is 36°, the coefficient, by the Rule of Three (or linear proportion) will be

$$4+(9-4)\times(36-30)/60-30) = 5$$

which gives the required result (1).

The *MC* rightly criticizes this absurd proof raising the objection (*āpatti*) that if such argument is allowed we ought to have (by extrapolation)

$$\text{Sin } 90^\circ = \sqrt{\frac{14R^2-\sqrt{14R^4}}{8}} > R$$

which is not so. However, Miśra's manipulative imagination should be appreciated.

A similar invalid derivation by Lakṣmīdāsa Miśra for Sine of 18° is quoted in the *MC* (p. 145) under verse 9 which contains its exact value (for the first time in India) but without any demonstration[7].

The *MC* (p. 144) gives a correct derivation of (1) from Sine of 18° by using

$$(\text{Sin } 36°)^2 = (R. \text{ Vers } 72°)/2$$
$$= (R/2).(R - \text{Sin } 18°).$$

The *MC* also gives a geometrical proof of the exact value of Sine of 18° based on certain lemma whose detailed demonstration is said to be found in a foreign work

Yavanagranthe savistaraṃ pratipāditā

as the *MC* puts it.[8]

Verse 10 which contains the standard and old Hindu subduplication formulas, is an exact repetition of the earlier lines at *SSG*, v, 4b-5a, and, surprisingly, is reproduced verbatim by Kamalākara (about 1658) in his *Siddhānta-tattva-viveka*, III, 78 (p. 136)[9].

The method of computing 24 tabular Sines, as given under verse 12, is described by the author in his *VB* as "new", but the same is already found in the *Mahā-siddhānta*, III, 2-3, composed 200 years earlier by Āryabhaṭa II (950 A.D.).[10]

However, by combining the rules of verses 12 and 14, Bhāskara II was able to point out the method of getting the Sine of 3 degrees and further explained in his *VB* under verse 15 the method of preparing a table of 30 Sines (interval 3°) which seems to be a new thing.

Verses 21-23a contain the correct statement of the famous Addition and Subtraction Theorems for the Indian Sine

$$\text{Sin } (A \pm B) = (R \sin A \,.\, R \cos B)/R \pm (R \cos A.R \sin B)/R. \qquad (2)$$

These are their first appearance in India but no explicit proof is given here. However, good applications of these rules are made by the author in deriving handy formulas (verses 16-20) for computing tables of 24 and 90 Sines[11].

The *MC* on verses 21-25 is a long excursus. It contains various proofs (including Bhāskara's approach through *vargaprakṛti*) of the Theorems (2) and related material[12]. It also contains a method for computing Sine of a trisected angle[13].

The last part of the *MC* (pp. 163-165) on the line 25b discusses the *Līlāvatī* rule for computing (approximate) lengths of chords subtending given angles at the centre of any circle. The rule is ultimately based on the famous ancient Indian formula for approximating Sine algebraically[14]

$$\sin \phi^\circ = \frac{4\,\phi\,(180-\phi)}{40500-\phi(180-\phi)} \tag{3}$$

However, it is disappointing to find that even the *MC* contains no proper derivation of this most typically Indian rule (which has not been found elsewhere in the world) already more than a thousand years old at that time.

3. Concluding Remarks

In the late Āryabhaṭa School, the theorems (2) were known as *Jīveparaspara Nyāya* and attributed to Mādhava of Saṅgamagrāma (*c*. 1340-1425) as if Bhāskara II's *Jyotpatti* was not known to that School. However, Mādhava also gave some other forms of (2), e.g.[15]

$$\text{Sin}\,(A\pm B) = \sqrt{[(\text{Sin}\,A)^2 - L^2]} \pm \sqrt{[(\text{Sin}\,B)^2 - L^2]} \tag{4}$$

where

$$L = (\text{Sin}\,A)\,.\,(\text{Sin}\,B)/R$$

It is noteworthy that the Arab Abúl Wafā (Baghdad, 940-998) is stated to have already known both the forms, (2) and (4) with proofs, two centuries before Bhāskara II and about five centuries before Mādhava[16].

Another point worth noting is that the *MC* does not mention any of the significant developments of Indian trigonometry achieved in the late Āryabhaṭa School, for instance the trigonometrical series and the accurate computation of tabular Sines for traditional radius and interval. This may indicate that the trigonometrical novelties of South India were not known in the Northern parts of the country which seems to be experiencing a decadence of indigenous science and culture under the Muslim rule since about 1200 A.D.

But the North India had the advantage of getting opportunity to become familiar with foreign material through contacts with Islamic neighbours. In fact, some Arabic material or Arabic version of Greek material was available to court *paṇḍits* in India. The *Yantrarāja* composed by Mahendra Suri (*c*. 1370) is full of non-Indian material. The *Samrāt Siddhānta* is the Sanskrit translation made by Jagannātha (eighteenth century) from an Arabic version of Ptolemy's *Almagest*, the famous Greek astronomical work (*c*. 150 A.D.)

References and Notes

[1] The *SS* edited by Bapu Deva Sastri and revised by Ganapati Deva Sastri, Benares, 1929, p. 199 (Kashi Sanskrit Series No. 72).

[2] Ibid., p. 175.

[3] *The SSG* with *MC* edited by D. V. Apte, Part I, Poona, 1943, pp. 129-165 (Ananda Asrama Sanskrit Series No. 122).

[4] Dvivedi, S. *Gaṇaka Taraṅgiṇī* (in Sanskrit) edited by P. Dvivedi, Benares, 1933, p. 37.

[5] The *Brāhmasphuṭa-siddhānta* edited by R. S. Sharma and his team, New Delhi, 1966, Vol. IV, p. 1627.

[6] Gupta, R. C. Indian Values of the Sinus Totus, paper presented at the XVth International Congress of the History of Science, Edinburgh, 1977.

[7] ———— Sine of Eighteen Degrees in India upto the Eighteenth Century, *Indian J. Hist. Sci*,. Vol. 11 (1976), pp. 1-10.

[8] Ibid., pp. 5-6.

[9] In the edition, with notes of S. Dvivedi, by M. Jha, fasc. I, Benares, 1924 (Benares Sanskrit Series No. 1).

[10] See p. 54 in fasc. I in the edition by S. Dvivedi, Benares, 1910 (Braj Bhushan Das and Co.).

[11] Gupta, R. C. Addition and Subtraction Theorems for the Sine and Their Use etc., *The Math. Educ.* Vol. VIII (1974), Sec. B, pp. 43-46.

[12] ———— Addition and Subtraction Theorems for the Sine and Cosine etc., *Indian J. Hist. Sci.*, Vol. 9 (1974), 164-177.

[13] The author of the present paper proposes to publish a separate paper on the topic. Some information on the topic is given in R. C. Gupta, Sines of Sub-multiple Arcs etc., *Ranchi University Math. Journal*, Vol. 5 (1974), pp. 21-27,

[14] Gupta, R. C. Bhāskara I's Approximation to Sine, *Indian J. Hist. Sci.*, Vol. 2 (1967), pp. 121-136.

[15] Gupta, *op. cit.* (under ref. 12 above), p. 166.

[16] Bond, J. D. The Development of trigonometric methods etc, *ISIS*, Vol. 4 (1921-22), p. 308.