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# THEOREM OF SQUARE ON THE DIAGONAL IN VEDIC GEOMETRY AND ITS APPLICATION

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Mahāvedi is considered as a popular altar and earliest geometrical structure in vedic rituals. We study the geometry of this (atleast 5000 years old) Mahāvedi, and show that it contains 36 Pythagorean triplets of which 18 are the mirror images of the corresponding 18.

**Key Words:** Mahāvedi, Pythagorean triplets, Śulbasūtra

## INTRODUCTION

According to Jaina canon, geometry is the lotus of mathematics. Ancient Indian savants are now credited for fundamental pioneering work in geometry. A very deep analysis by Seidenberg (cf. 11) concludes "– Greek geometry (especially the theorem of Pythagoras) did not somehow make its way into Vedic geometry as Greek geometry is only supposed to have started about 600 BC"[14]. Discussions on right triangles via rectangles/squares may be found in *Brāhmaṇas, Saṃhitās* and in more elaborated form in *Śulbasūtras*[7,10]

*Theorem of square on the diagonal*

*BSS* (1.12) (see the note at the end of this paper) enunciates the above theorem in the following stich:

*dīrghacaturaśrasyākṣṇayārajjuḥ pārśvamānī tiryaṅmānī ca yatpṛthagbhūte kurutastadubhayaṃkaroti.*

This means (cf. 12):

The areas (of the squares) produced separately by the length and breadth of a rectangle together equal the area (of the square) produced by the diagonal.

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That is, in the rectangle ABCD, $AC^2=AB^2+BC^2$. This is also available in subsequent compositions, viz, *ASS* (1.4), *KSS* (2.7) and *MSS* (10.10) However, *MSS* (10.10) utters the same in slightly different wordings (cf. 12): Multiply the length (of a right-angled triangle) by the (same) length and breadth by the breadth; the square-root of the sum of these two (results) gives the hypotenuse; this is already known to the scholars.

A direct proof of the theorem seems to appear first in Bhāskara II's *Bījagaṇita* (*see* Sudhakara Dvivedi's ed., 1888, p. 70).

Here it is remarkable indeed that, before giving a general statement *BSS* (1.12) hints that a systematic attempt was made to understand the theorem as is evident from its verses 1.9 and 1.10-1.11. BSS (1.9) states that the diagonal of a square produces double (its) area (and it is $\sqrt{2}$) while *BSS* (1.10-1.11) vindicates the same statement from the diagonal of a rectangle of breadth $\sqrt{2}$ and height 1 (and it is $\sqrt{3}$), and the *sūtra*(1.13) says that 1.12 can easily be understood for rectangle having sides (3, 4, 5), (12, 5, 13), (15, 8, 17), (7, 24, 25), (12, 35, 37) and (15, 36, 39). The geometrical and arithmetical treatment, by Bag [3], with the help of diagrams drawn on the basis of hints available in the *Śulbasūtras* justify possibly the truth of the statements 1.9, 1.10-1.11 (and hence of 1.12).

If ABC is a right-angled triangle at B, then the hypotenuse will be given by $AC^2=AB^2+BC^2$. In other words, area of the square on the hypotenuse will be equal to the combined area of the squares on the other sides This statement, generally known as *Pythagoras theorem*, is attributed to Greek philosopher and mathematician Pythagoras (c. 540 BC), although it was known in Babylonia, China, Egypt as well as India much earlier than his time. Its proof was later on given by Euclid (300 BC) (cf. *Elements, Book* 1, *Proposition* 47; see also [1, pp. 350-351], [4, pp. 156-162], [5, p. 123] & [9, pp. 99-100].

The exact practical form of Pythagoras theorem, is stated to have been known to *Brāhmaṇas* (c. 2500 BC), e.g., *Śatapatha, Taittirīya*-and *Garga-Saṃhitās* (see 7). In a more recent attempt, Seidenberg (cf. 8) has shown that *Taittirīya-Saṃhitā* knows not only the algebraic or computational aspect of the (Pythagoras) theorem but also geometric or constructive aspect which was not known to Babylonians of about 1700 BC. It is also evident from certain Babylonian cuneiform tablets that the practical use of the theorem was current in old Babylonian times. Neugebaur (cf. 2, p. 126) proved conclusively, on this basis, that Pythagoras derived the theorem from cuneiform tablets. Moreover, Burk (cf. 2, p. 124) concluded that the theorem was known and proved in all its generality by Indians long before Pythagoras time; and, went to the extent, that he probably obtained his theory from India.

The Chinese also knew the geometrical aspect of the Pythagoras theorem in more eleborated form than Babylonians. Besides other triplets those found in Babylonian texts are also discovered in Chou Pei (fourth century BC) and was in use from the time of its first commentator Chao Chun Chhing (third century AD) (see 2, p. 125\6). The Chinese might have received and developed the Babylonian materials in their own style (3).

The Egyptians (c. 1900 BC) had also some knowledge of the theorem which they implied mainly in the foundation ceremony of temples. Berlin Papyrus no. 6619 involves the following problem based on this theorem (cf. 3): A square and a second square whose side is one-half and one-fourth of the first square, have together an area of 100, show how to calculate this.

## Some Geometrical Constructions

The following are a few vedic constructions using the theorem of square.

(i) *Construction of a line perpendicular to a given line:*

Following *KSS* (1.4-1.5), this is done by *nirañchana* method which seems to be based on the converse of Pythagoras theorem.

(ii) *Construction of a square equal to the sum or difference of two different squares:*

For details, refer to *ASS* (2.4), *BSS* (2.1.-2.2) and *KSS* (2.13 & 3.1). See also (6, pp. 76-79) and (8).

(iii) For different constructions of right triangles from Pythagoras theorem, as expressed by Brahmagupta (b. 598 AD), Mahāvīra (c. 850 AD) and Bhāskara II (b. 598 AD), refer to (2, p. 165). Note that Fibonacci (1202 AD) and Vieta (1580 AD) (cf. 2, p. 165) trace the same results as those given by Brahmagupta and Mahāvīra but without quoting them. *KSS* 6.7 applies the theorem to transform a square equal to $n$ times the given square (see also 6, pp. 72-74; 133-136; 178-180).

(iv) *Construction of an isosceles triangle:*

Such triangles are based ont he juxtaposition of two equal right triangles with a common leg. Refer (13, p. 262) for details.

(v) *Construction of a scalene triangle:*

A scalene triangle is formed by joining two equal sides of any two right triangles (see 2, p. 166 and 13, p. 263). The same problem was also discussed in Europe by Buchet (1621 AD) and Cunliffe (cf. 2, p. 166).

(vi) *Construction of a quadrilateral:*

A quadrilateral ca be constructed by (a) joining two scalene triangles of equal bases along their base; or by (b) joining four such right triangles in a way that each pair forms a sclene triangle with equal bases. Such pair can be constructed by suitable selection of right triangles. (See 2, pp. 166-167 and 13, pp. 267-268).

(vii) *Construction of an isosceles trapezium:*

Brahmagupta deals with such constructions (see 13, pp. 264-265).

(viii) *Construction of a trapezium with three equal sides:*

Brahmagupta discusses such constructions (refer 13, p. 266 for details).

Besides the above applications, the theorem has been widely employed by the *Śulbakāras* and other ancient and medieval mathematicians for construction and transformation of various geometrical figures. For details, refer to (6) and (12).

## Mahāvedi and right triangles

Mahāvedi ABCD is an isosceles trapezium of width 36 and parallel sides 24 and 30 units (fig-1). This is constructed by first making the central-axis (EW= east-west line), points W and E and then drawing the four segments WD, WC, EA and EB along the other cardinal directions (south and north).

Descriptions of *Mahāvedi* appear in *Brāhmaṇas* as well as in *saṃhitās* (see 7, 8, 11 & 14). Note that the same *Mahāvedi* is stated in *BSS* (4.3) and also in ASS (5.1) (Mahāvedi is called *Saumikīvedi* is ASS). For its construction, refer *ASS* 5.2-5.5. ASS (5.3-5.5) in connection with the construction of this altar mentions the following triplets: (3, 4, 5), (5, 12, 13), (8, 15, 17), (12, 16, 20), (12, 35, 37), (15, 20, 25) and (15, 36, 39).

Gupta (7) testifies that the under mentioned triplets may also be extended to lie within the *mahāvedi:* (6, 8, 10), (7, 24, 25), (9, 12, 15), (18, 24, 30), (21, 28, 35), (24, 10, 26), (24, 32, 40), (27, 36, 45) and (30, 16, 34).

## Mirror Image Concept

Let x, y, z (z, the largest) be the sides of a right-angled triangle. Clearly (x, y, z) is a solution of

(1A) $$x^2+y^2+z^2$$

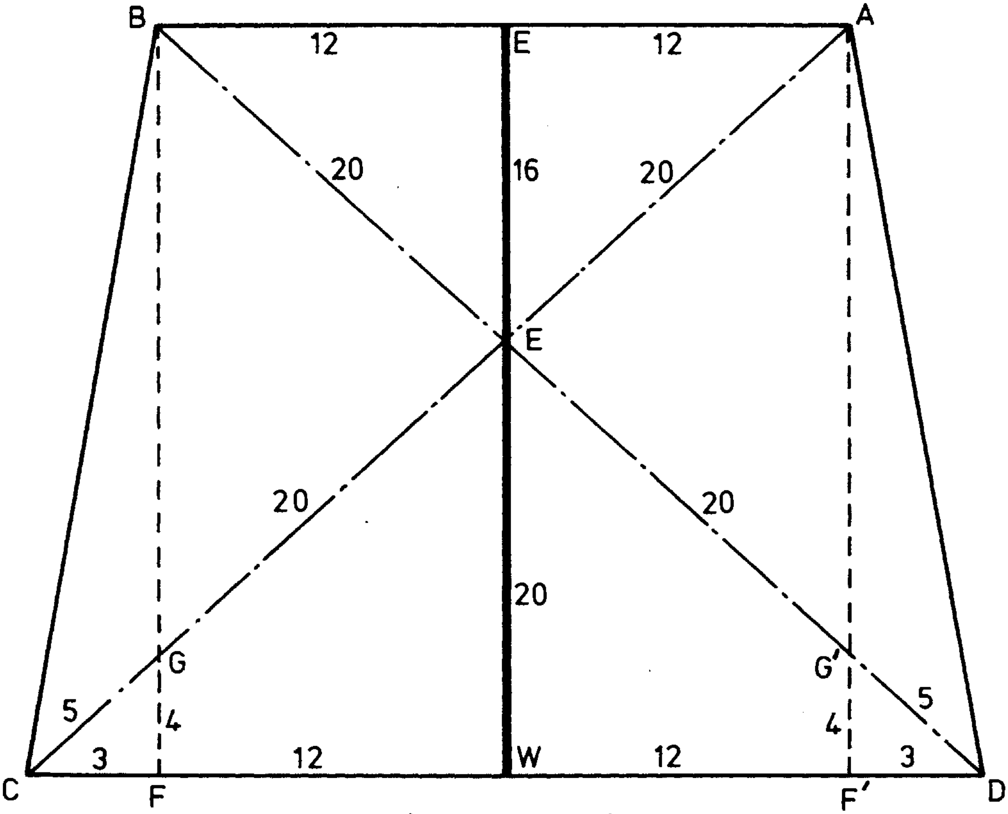


Fig. 1. Mahāvedi, central East-West line, EW acts as a mirror

and that (y, x, z) is a solution of

(1B) $$y^2+x^2=z^2.$$

(x, y, z) and (y, x, z) can be represented by the arrangements I and Í respectively. Arrangement Í is the mirror image of arrangement I and vice-versa (Fig. 2)

## MATHEMATICAL ANALYSIS

Let (a, a+p, a+q) be the solution of (1A). Therefore, $a^2+(a^2+ap+p^2)= a^2+2aq+q^2$

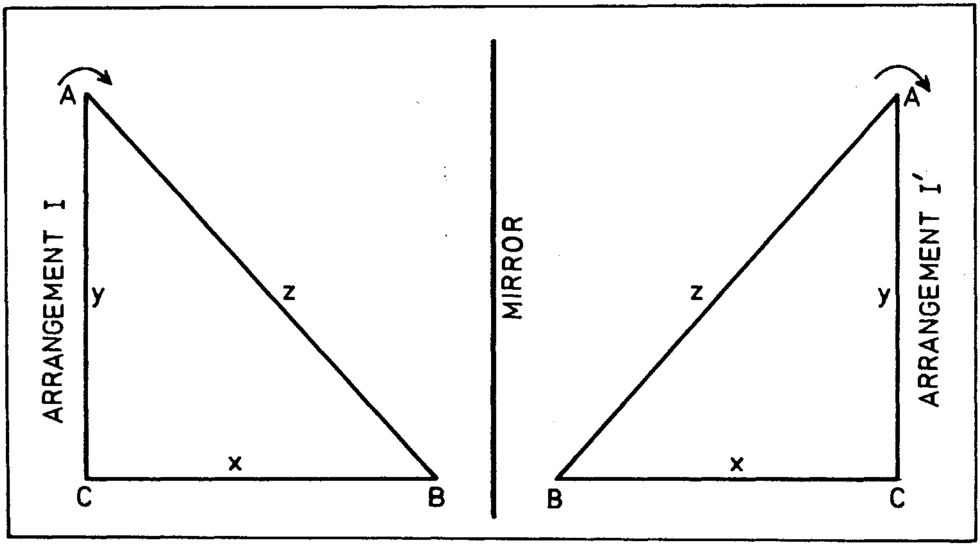


Fig. 2. Mirror Image of Triangle

i.e., $a^2-2a(q-p)= (q-p)\ (q+p)$

i.e., $(q-p)\ (q+p+2a)= a^2$.

If $q-p=L1$ and $q+p+2a=L_2$, then $L_1<L_2$ and $a^2= L_1L_2$.

Evidently $L_1$ and $L_2$ are either both even or both odd.

Observe that $(L_2+L_1)/2$ can not exceed the dimetrical length (viz., AC or BD in the case of *Mahāvedi).*

Solving the preceding two equations, we obtain

$$a+p=(L_2-L_1)/2 \text{ and } a+q=(L_2+L_1)/2.$$

In the case of *Mahāvedi,* $(L_2+L_1)/2=45$ [$=\sqrt{27^2+36^2}$)], and evidently we have the following sets of triplets within this restriction:

(1) (3, 4, 5),

(2) (4, 3, 5) [similar to (1)].

(3) (5, 12, 13).

(4) (6, 8, 10) [two times (1)].

(5) (7, 24, 25).

(6a) (8, 6, 10) [two times (2)];

(6b) (8, 15, 17).

(7a) (9, 12, 15) [three times (1)];

(7b) (9, 40, 41).

(8a) (10, 24, 26) [two time (3)].

(9a) (12, 9, 15) [three times (2)];

(9b) (12, 16, 20) [four times (1)];

(9c) (12, 35, 37);

(9d) (12, 5, 13) [similar to (3)].

(10a) (15, 8, 17) [similar to (6b)];

(10b) (15, 36, 39) [three times (3)];

(10c) (15, 20, 25) [five times (1)].

(11a) (16, 12, 20) [four times (2)];

(11b) (16, 30, 34) [two times (6b)].

(12) (18, 24, 30) [six times (1)];

(13a) (20, 15, 25) [five times (2)];

(13b) (20, 21, 29).

(14a) (21,28, 35) [seven times (1)];

(14b) (21, 20, 29) [similar to (13b)].

(15a) (24, 7, 25) [similar to (5)];

(15b) (24, 10, 26) [tow times (9d)];

(15c) (24, 18, 30) [six times (2)].
(24, 32, 40) [eight times (1)].

(16) (27, 36, 45) [nine times (10].

(17) (28, 21, 35) [seven times (2)].

(18) (30, 16, 34) [two time (10a)].

(19) (32, 24, 40) [eight times (2)].

(20) (35, 12, 37) [similar to (9c)].

(21a) (36, 15, 39) [there times (9d)];

(21b) (36, 27, 45) [nine times (2)].

(22) (40, 9, 41) [similar to (7b)].

## Conclusion

The number of Pythagorean triplets in *Mahāvedi* may be summarized as follows:

(3, 4, 5) and its mirror image (4, 3, 5) each recur in the form i (3, 4, 5) and i (4, 3, 5); i = 1, 2, ............. 9.

Similarly (5, 12, 13) and (8, 15, 17) along with their mirror images recur in the form j (5, 12, 13), j 912, 5, 13), k 98, 15, 17) and k (15, 8, 17); j = 1, 2, 3; k = 1, 2. All others viz, (7, 24, 25), (9, 40, 41), (12, 35, 37) and (20, 21, 29) along with their mirror images appear only once (Fig. 2).

Thus the *Mahāvedi,* simplest geometrical structure amongst the vedic ritualistic altars, is loaded with a total of 18 (= 9+3+2+4) Pythagorean triplets with the same number of corresponding mirror images.

## ABBREVIATIONS

*Āpastamba Śulbasūtra* (= *ASS*), *Baudhāyana Śulbasūtra* (= *BSS*), *Kātyāyana Śulbasūtra* (= *KSS*) and *Mānava Śulbasūtra* (= *MSS*) are according to Sen and Bag edition (12); wherein, for example, *BSS* (6.12) stands for the 12th verse of the sixth chapter of BSS.

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