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Book Review

Lennart Berggren, Jonathan Borwein and Peter Borwein, ***Pi: A Source Book,*** Third Edition, Springer-Verlag, New York, 2004, Pages xix+797. (ISBN 0-387-20571-3)

The prominent number which is denoted by the Greek letter π (1st letter of the word perimeter or periphery) is really the most wonderful number in the mathematical world. Its importance lies in the fact that it frequently appears in various branches of physical and mathematical sciences including even seemingly unrelated fields such as vibration theory, statistics, theory of numbers, probability theory and acturial theory. Even in the geometrical situations, its use is not confined to the circle (to which it was originally related) but extends to various curves, surfaces, and solids.

Surprisingly the nature of the number π which is so universal and popular, is not simple in that π is neither rational (i.e. ratio of two integrers) nor algebraic (i.e. root of a polynomial equation with integral coefficients). That is, π is not only irrational but transcendental.

The ancient problem of mutual transformation of circle and square arose from the religious needs of peoples (especially in India). The old geometrical problem of quadrature of circle (i.e. finding a square equal in area to a given circle) exercised a greater and longer attraction than any other mathematical problem. And although the final settlement (arising from the 1882 proof of the transcendence of π) gave a death blow to the problem, the enthusiastic circle-squarers continue to suffer from *morbus cyclometricus*.

The problems related to various aspects of mensuration of circle has been engaging the attention of mathematicians in all cultures since the earliest times of human civilization. It is often remarked that the quality and accuracy of the various approximations of π used in any culture can be taken to be a measure or index of the development of mathematics in that culture.

The history of the genesis, growth and development, and of the discoveries of the various properties of π form a good part of the origin and development of mathematical concepts. The development includes interplay of some of the most varietyful aspects of mathematics from trifling empirical guesses to highly refined theoretical considerations. The "story of pi reflects the most seminal, the most serious, and sometimes the most whimsical aspects of mathematics". A history of π has to contain an exposition of mathematics from simple concepts and conjectures to advanced theorems and proofs. It has to deal with finding values using methods of ordinary classical calculations to most sophisticated computations.

The present book *Pi: A Source Book* is the most standard compilation available so far on the subject. It is quite comprehensive and authentic and will serve a very useful reference work for researchers as well as for others. In fact the *Source Book* can be said to be a mini-encyclopedia of significant documents which expose history, nature, and computation of π. Although it is, as the authors admit (p. vii) not a history of π, it does present the most relevant and very extensive literature related to historical and other aspects of π.

The main part of the book comprises of a collection of 70 research items (or articles and papers) by various writers from "The Rhind Mathematical Papyrus Problem 50 (c. 1650 BC) (pp.1-2) to "On the Rapid computation of various Polylogarithmic Constants" (1997) (pp. 663-676). About one-third of the items occupy four or less pages each. Then there are four useful appendices on Early History of Pi, A Computational Chronology of Pi, Selected Formulae for Pi, and Translations of Viete and Huygens. These are followed by a Bibliography, Credits (i.e. acknowledgements), and "A Pamphlet on Pi" (pp.721-780).

The source material presented in this voluminous work can be classified in several ways. Chronologically the authors consider the past 4000 years to be divided into three periods namely the Pre-Newtonian (items 1 to 15), from Newton to Hilbert (next 10 items), and the 20th Century (items 26 to 70). From computational point of view, the three significant methods are the precalculus (Archimedean exhaustion), the calculus method (arctangent formulas and classical series), and the modern methods using elliptic and modular functions. For greater details, a more appropriate consideration will be to classify the material according to its nature and the topics covered by items.

Translations of source material from ancient and medieval culture-areas include the Greek (Archimedean) measurement of a circle (3rd cent. BC) (pp. 7-14); Circle measurements in Ancient China (from Liu Hui, 3rd cent. AD and Zu Chongzhi, AD 429-500) (pp.20-35); and the Arabic Banu Musa: The measurement of Plane and Solid Figures (AD 850) (pp.36-44).

Item no. 7 (pp.45-50) entitled "Mādhava: The Power Series for Arctan and Pi (c. 1400)" (see p.xi) is the first one on Indian material. It is K. Balagangadharan's Appendix to the paper "On the Hindu Quadrature of the Circle" by K. Mukunda Marar and C.T. Rajagopal. Details of this paper are not given but the reviewer could find that it was published in the *Journal of the Bombay Branch of the Royal Asiatic Society* (N.S.), Vol. 20 (1944), pp.65-82.

However it must be pointed out that the works *Tantra-Saṅgraha* and *Karaṇa-paddhati* (from which the Sanskrit verses are quoted) are not from the pen of Mādhava. Actually when the paper was written (60 years ago!), our knowledge of the Late Āryabhaṭa School starting with Mādhava was not adequate. Now the scholars of the School are better known. I give some relevant information:

(i) Mādhava of Saṅgamagrāma (c. 1350-1425), founder of the School.

(ii) Parameśvara (c. 1360-1460) (pupil of Mādhava): He wrote, among other works, commentaries on the *Āryabhaṭīya* and *Līlāvatī*, and a super-commentary on the *Mahābhāskarīya*.

(iii) Nīlakantha Somayāji (born 1444) (pupil of Dāmodara, son of Parameśvara): He wrote the famous *Tantra-Saṅgraha* (1500 AD), an extensive commentary on the *Āryabhaṭīya*, and many other works.

(iv) Jyeṣṭhadeva (16th century) (pupil of Dāmodara): He wrote the Malayam *Yuktibhāṣā* (c.1530), an exposition of *Tantra-Saṅgraha* etc.

(v) *Śaṅkara Vāriyar* (c. 1500-1560) (pupil of *Nīlakantha*): He wrote two commentaries on *Tantra-Saṅgraha*, and the famous commentary *Kriyā kramakari* (on *Līlāvatī*) which was completed by Nārāyaṇa (c. 1500-1575) after the demise of the former.

(vi) Putumana Somayāji (c. 1660-1740) was the author of *Karaṇa paddhati* (1732) which is rather a compilation of earlier material. This information may help to fix dates of the quoted Sanskrit rules also.

The Source Book under review has paid due attention to Lambert's 1761 proof of the irrationality of π (pp. 129-140), and to the proofs of transcendence of π by Hermite (1873), Lindemann (1882), Weierstrass (1885), and Hilbert (1893) (items 21 to 24). For a mathematics student it is easy to remember:

(a) It is shown that if x ($\neq 0$) is rational, then tan x cannot be so. Since tan (π/4) =1, it follows that π/4 (and hence π) must be irrational.

(b) Similarly it has been proved that if x is algebraic, the equation $e^{ix} + 1 = 0$ cannot be satisfied. But $e^{ix} + 1 = 0$, so π must be transcendental.

(c) Of course, if π is non-algebraic, it cannot be rational.

Some popular noteworthy contributions collected in the present book include those from Viete (first infinite expression, other than series, for π 1593); Wallis (infinite product for π, 1655); Brounker (continued fraction for 4/π, 1655); William Jones (first use of the symbol π, 1706), Euler (many manipulated series for π and its powers, 1748); Shanks (computation of π to 607 decimal places, 1853); and the notorius Indian House Bill (1897) to legislate the value of π (pp.231-235), etc.

In fact the *Source Book* has collected very profuse and remarkable material from accumulated research literature of past 4000 years which consists of historical studies of π and also which "comprises the treatments of π that are fanciful, satirical, or whimsical, or just wrongheaded" (p.viii).

The book has rightly devoted its major position to the computational aspect of π. Before the beginning of the last quarter of the last century, almost all computer evaluations of π were based on calssical series and formulas usually involving arctan functions. Around 1910, the Indian genius Srinivasa Ramanujan discovered some new infinite series formulas "but these were not well-known until quite recently when his writings were widely published" (p.725). Thus special and great attention has to be necessarily paid to his modern work and the book has rightly done so.

Ramanujan's 1914 seminal paper on "Modular Equations and Approximations to π" (*Quarterly Journal of Mathematics*, Vol. 45, pp. 350-372) (reprinted here as item no. 29, pp. 241-257, from his *Collected Papers*, New York, 1962, pp.23-39) includes several "striking series and algebraic approximations for π e.g. (p.249)

$$\pi = \left(12/\sqrt{190}\right)\log\left[\left(2\sqrt{2}+\sqrt{10}\right)\left(3+\sqrt{10}\right)\right]$$

which yields π correct to 18 decimal places, and (p.254)

$$\frac{4}{\pi} = 1 + \frac{7}{4}\left(\frac{1}{2}\right)^3 + \frac{13}{4^2}\left(\frac{1.3}{2.4}\right)^3 + \frac{19}{4^3}\left(\frac{1.3.5}{2.4.6}\right)^3 + \ldots\ldots$$

It is remarked (p.xix) that the paper "presents an extraordinary set of approximations to pi via 'singular values' of elliptic integrals". And that the second half of the paper "which presents marvelous series for pi" was decoded and applied only recently (i.e. more than 70 years after it was written). Three of these 'decoding' papers are:

(1) Borwein and Borwein, "Ramanujan and Pi" (1988).

(2) D.V. Chudnovsky and G.V. Chudnovsky, "Approximations and complex Multiplication according to Ramanujan" (1988).

(3) Borwein, Borwein, and D.H. Bailey,"Ramanujan, Modular Equations, and Approximations to Pi or How to compute One Billion Digits of Pi" (1989).

These papers give sufficient credit to the originality of the Indian genius and are included in the book (items 62 to 64). Ramanujan's amazing sum (p.623 etc.)

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801}\sum_{n=0}^{\infty}\frac{(4n)!}{(n!)^4}\frac{(1103+26390n)}{396^{4n}}$$

will add 8 correct digits per additional term (p.725). A variation of Ramanujan's above formula has been also found (see p.726). His method of calculating π with extraordinary efficiency finds creditable place in computer algorithms yielding millions of digits of π.

Regarding early history of π, as pointed out by A. J. E. M. Smeur in his 1970 paper (*Archive for History of Exact Sciences*, Vol 6, pp. 249-270), it is technically desirable to distinguish between π_1 (ratio of area of circumfrerence of any circle to its diameter) and π_2 (ratio of area of a circle to the square of its radius). We know that $\pi_1 = \pi_2$, but this equality was not necessarily known in all cases and situations of ancient times, especially prior to the discovery of the rule; Area = (circumf.) (diam.)/4.

The authors of the *Source Book* are aware of the distinction being made between π_1 and π_2 (see p. 677). But the task of differentiating between various relevant cases dealt and described is left to the interested reader. Thus it may be pointed out, for illustration, that the simple constructions given by Jakob de Gelder in 1849 (see pp. 299 and 404) are for $\pi_1 = 355/113$, while that given by Ramanujan (see pp. 240 and 253) for the same approximation, is for $\pi_2 = 355/113$.

The most important addition in the 3rd edition of the book (over its 2nd ed.) is the Supplement (pp. 721-777) entitled "A Pamphlet on Pi" in four chapters. It starts with 'A Recent History of Pi' which presents the most exciting piece on modern and current computational techniques related to π. Using a variation of Ramanujan's formula, π was computed in 1994 to over 4 billion decimal digits (1 billion $=10^9$). In 1999 it was computed to over 206 billion places using some algorithms. Surprisingly, in 2002 Y. Kanada and his team computed π to over 1240 billion places by using two arctan formulas one of which was found by Carl Stormer as early as in 1896. This is a good example of the use of historical material for modern research. Apart from curiosity and intrinsic fascination, the monstrously huge set of digits of π may be used for studying their pattern and distribution, and also for checking some results.

There are two articles (items 52 and 53) on the Biblical value of π which is also mentioned at other places (see Index and p. 753). But the information provided is very small than found in R.C.Gupta, "The Molten Sea and the Value of π", *The Jewish Bible Quarterly*, Vol. XIX, No.2 (1990-1991), pp.127-135.

Minor tidbits about p mentioned in the book include mnemonics, puzzles, Pi-song, etc. The reviewer would like to add the beautiful figure of π-lotus with magical circles (Fig. 1). The numbers on eight intersection points of every circle add to the same magical sum 324, while the numbers on the outermost intersections represent digits of π.

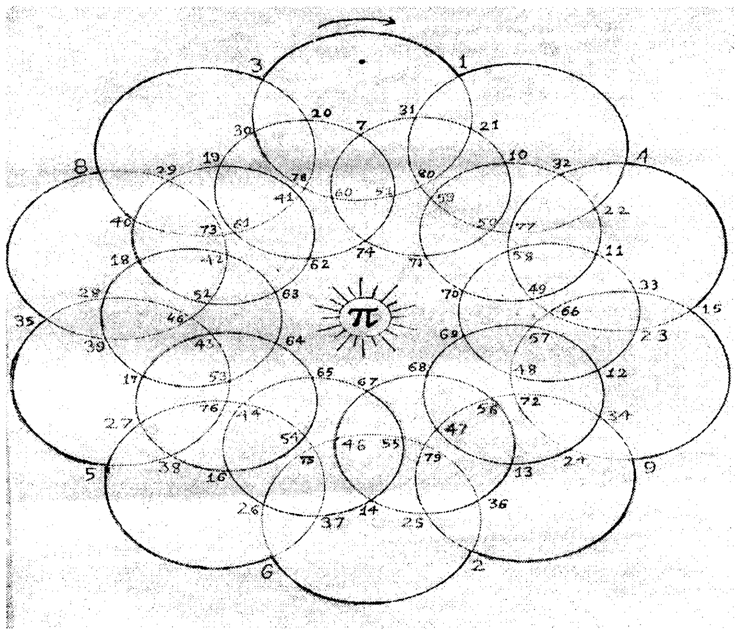

Fig.1: PI LOTUS (with magical sum 324 on each side)

The Bibliography towards the end of the *Source Book* should be richer and upgraded to match with the great and voluminous book. Following may serve as samples:

(i) G. Almkvist and B. C. Berndt, " Gauss, Landen, Ramanujan, the AGM, Ellipses, π and etc.", *Amer. Math. Monthly*, 94 (1987), 585-608.

(ii) R. C. Gupta, "Rectification of Ellipse from Mahāvīra to Ramanujan", *Gaṇita-Bhāratī*, 15 (1993), 14-40.

(iii) T. Hayashi et al., *Studies in Indian Mathematics: Series, Pi and Trigonometry* (in Japanese), Tokyo, 1997.

(iv) G. M. Phillips, *Two Millennia of Mathematics*, Springer, 2000.

The Index at the end needs corrections (e.g. check entries of Champernowne and Catalan's constant), additions (e.g. Song about Pi; Ubiquitous π, p.656), and fullness (e.g.p.728 for Stormer formula).

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