Indian Journal of History of Science, 47.4 (2012) 561–587

SIGNIFICANCE OF *ASAKṚT-KARMA* IN FINDING *MANDA-KARṆA*

Deepak P. Kaundinya, K. Ramasubramanian*
and M. S. Sriram†

(Received 20 August 2012; revised 20 September 2012)

Asakṛt-karma (also known as *aviśeṣa-karma*) refers to the mathematical technique employed by Indian astronomers in order to deal with the problem of *anyonyāśraya* (interdependence) among the variables involved in planetary computations. This technique is essentially an iterative process that may also be conceived as the method of successive approximation that converges to a limiting value. The aim of the present paper is three-fold: One, to provide a clear exposition of this procedure as described by Bhāskara I (c. 629 AD) in his commentary to *Āryabhaṭīya* in the context of computing the *manda-sphuṭa-graha*—the longitude of the planet corrected for the eccentricity of its orbit; two, to study the convergence of the iterative process prescribed by Bhāskara; and three, to understand the physical significance of doing *asakṛt-karma* in terms of the geometry of the planetary orbit.

Key words: *Asakṛt-karma*, *Aviśeṣa-karma*, Eccentricity, Equation of centre, Iterative process, *Bhujā, Koṭi, Manda-karṇa, Manda-saṃskāra, Sakṛt-karma.*

*Department of Humanities and Social Sciences, IIT Bombay; Email: kramas@iitb.ac.in

†Department of Theoretical Physics, University of Madras, Guindy Campus, Chennai – 600 025; Email: *sriram.physics@gmail.com*

Introduction

Ancient astronomers devised the eccentric and epicyclic models to explain the non-uniform motion of the Sun, Moon and the planets in the background of the stars. The details of epicycle models in India differed from those in Greko-European and Islamic traditions, and also from each other within the Indian tradition. An interesting feature of the epicycle model used for finding the 'equation of centre' (*mandaphala*)[1] in the school of Āryabhaṭa—as explained in the works of Bhāskara I, and later in the works of many other Indian astronomers—was that the radius of the epicycle was not a constant, but was proportional to the 'true' distance of the planet from the centre of mean motion, which is known as '*mandakarṇa*'. But the expression for the *mandakarṇa* involves the radius of the epicycle. So the *mandakarṇa* and the 'epicycle radius' are dependent on each other (*itaretarāśraya*). How to find them, then?

In his *Mahābhāskarīya*, Bhāskara I has explained the *asakṛt-karma* or the 'iterative process', through which the problem is solved.[2] The same procedure is used in *Tantrasaṅgraha* of Nīlakaṇṭha Somayājī (c. 1500 AD),[3] and explained in detail in his *Āryabhaṭīya-bhāṣya* and *Siddhānta-darpaṇa-vyākhyā* as well as in the *Gaṇita-yukti-bhāṣā* of Jyeṣṭhadeva (c. 1530 AD).[4] These works also describe a method to find the *mandakarṇa* directly, without the iterative process. This direct or one-step method for obtaining *mandakarṇa* is ascribed by Nīlakaṇṭha to Mādhava of Saṅgamagrāma.[5] In this paper, besides discussing the iterative process and its convergence in detail, we will also compare the planetary orbit obtained by the iterative process with that obtained using Kepler's model (ellipse).

Calculation of the manda-sphuṭa from the mean longitude

The scheme adopted by Indian astronomers for the computation of the geocentric longitude *sphuṭa-graha* of a planet essentially consists of two steps: the computation of the *madhyama-graha* (mean planet) and the computation of the *sphuṭa-graha* (true planet) from the former.[6]

The term mean planet is used to refer to the mean longitude of the planet. It is calculated from the *Ahargaṇa* (number of civil days elapsed since th epoch) by multiplying it with the mean daily motion of the planet. Having obtained this, a correction known as *manda-saṃskāra* is applied to it. This correction takes care of the eccentricity of the planetary orbit. In modern astronomy, the equivalent of this correction is termed the *equation of centre*. The longitude obtained by applying this *saṃskāra* is known as the *manda-sphuṭa-graha* or simply *manda-sphuṭa*.

For the Sun, the *manda-sphuṭa* is the true geocentric longitude. For the Moon, it is essentially so, if one ignores the 'evection' term and other small corrections. In the case of the other five planets (Mercury, Venus, Mars, Jupiter and Saturn), the *manda-sphuṭa* can be identified with the true heliocentric longitude. In this connection, it may be mentioned that it was Nīlakaṇṭha who gave the correct procedure for the application of the *manda-saṃskāra* in the case of the interior planets.[7] For *tāra-grahas* (the other five planets) one more correction, termed *śīghra-saṃskāra*, needs to be applied in order to get the true geocentric longitude (*śīghra-sphuṭa*) from the true heliocentric longitude (*manda-sphuṭa*).

The formula presented by the Indian astronomical texts for the calculation of the *manda-sphuṭa* using the epicycle approach can be understood with the help of Figure 1. Here, O is the centre of the *kakṣyāmaṇḍala* on which the mean planet P_0 moves with a uniform velocity. $O\Gamma$ is the reference line usually chosen to be the direction of *Meṣādi*, from which the longitudes are measured. The *kakṣyāmaṇḍala* is taken to be of radius R, known as *trijyā*. The longitude of the mean planet P_0 moving on this circle is given by

$$\Gamma\hat{O}P_0 = \textit{madhyama-graha} = \theta_0. \tag{1}$$

OU represents the direction of the *mandocca* whose longitude is given by

$$\Gamma\hat{O}U = \textit{mandocca} = \theta_m. \tag{2}$$

The modern equivalent of the *mandocca* is apoapsis-apogee in the case of the Sun and the Moon, and the aphelion in the case of the five planets.

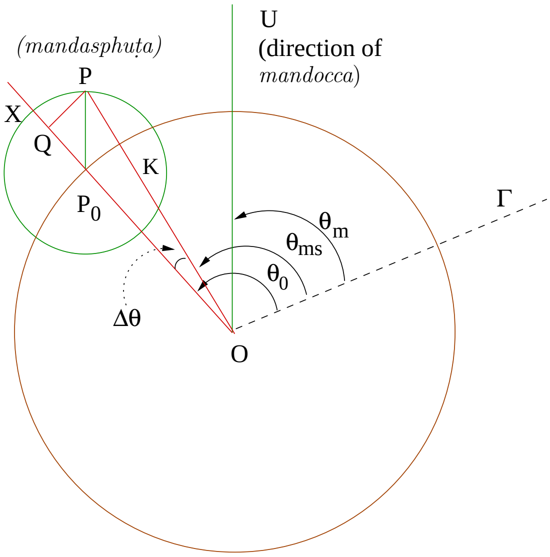

Fig. 1: Geometrical construction underlying the rule for obtaining the *manda-sphuṭa* from the *madhyama* using the epicycle approach.

Around the mean planet P_0, a circle of radius r is to be drawn. The value of r is as yet unspecified. This circle is known as the *manda-nīcocca-vṛtta* or simply the *manda-vṛtta* (epicyle). At any given instant of time, the *mandasphuṭa-graha* P is to be located on the *manda-vṛtta* by drawing a line from P_0 along the direction of the *mandocca* (parallel to OU). The point of intersection of this line with the *manda-vṛtta* gives the location of the *manda-sphuṭa* P. In the figure, $\theta_{ms} = \Gamma\hat{O}P$ represents the *manda-sphuṭa* which is to be determined from the position of the mean planet (*madhyama-graha*) P_0. Clearly,

$$\begin{aligned} \theta_{ms} &= \Gamma\hat{O}P \\ &= \Gamma\hat{O}P_0 - P\hat{O}P_0 \\ &= \theta_0 - \Delta\theta. \end{aligned} \tag{3}$$

Since the mean longitude of the planet, θ_0 is known (obtained from the *ahargaṇa* and the mean motion of the planet), the *manda-sphuṭa*, θ_{ms} is obtained by simply subtracting $\Delta\theta$ from the *madhyama*. The expression for $\Delta\theta$ can be obtained by making the following geometrical construction. We extend the line OP_0, which is the line joining the centre of the *kakṣyāmaṇḍala* and the mean planet, to meet the epicycle at X. From P drop the perpendicular PQ onto OX. Then

$$\begin{aligned} U\hat{O}P_0 &= \Gamma\hat{O}P_0 - \Gamma\hat{O}U \\ &= \theta_0 - \theta_m, \end{aligned} \tag{4}$$

is the *manda-kendra* (*madhyama* − *mandocca*), whose magnitude determines the magnitude of $\Delta\theta$ (see (7)). Also, since P_0P is parallel to OU (by construction), $P\hat{P_0}Q = (\theta_0 - \theta_m)$. Hence, $PQ = r\sin(\theta_0 - \theta_m)$ and $P_0Q = r\cos(\theta_0 - \theta_m)$. Since the triangle OPQ is right-angled at Q, the hypotenuse $OP = K$ (known as the *manda-karṇa*) is given by

$$\begin{aligned} K = OP &= \sqrt{OQ^2 + OP^2} \\ &= \sqrt{(OP_0 + P_0Q)^2 + OP^2} \\ &= \sqrt{\{R + r\cos(\theta_0 - \theta_m)\}^2 + r^2\sin^2(\theta_0 - \theta_m)}. \end{aligned} \tag{5}$$

Again from the triangle POQ, we have

$$\begin{aligned} K\sin\Delta\theta &= PQ \\ &= r\sin(\theta_0 - \theta_m). \end{aligned} \tag{6}$$

Multiplying the above by R and dividing by K we have

$$R\sin\Delta\theta = \frac{r}{K}\, R\sin(\theta_0 - \theta_m). \tag{7}$$

It is a significant feature of the Āryabhaṭan school, that the radius of the *manda* epicycle is not constant. As mentioned earlier, this is explained for instance by Bhāskara in his *Mahābhāskarīya*. This is also stated in *Tantrasaṅgraha* of Nīlakaṇṭha, and explained in detail in his *Āryabhaṭīya-bhāṣya* as also in the *Gaṇita-yukti-bhāṣā* of Jyeṣṭhadeva.

According to these texts r varies continuously in consonance with the hypotenuse, the *manda-karṇa* (K), in such a way that their ratio is always maintained constant and is equal to the ratio of the mean epicycle radius (r_0)—whose value is specified in the texts—and the radius of the deferent circle (R).[8] In fact, it has been shown by K. S. Shukla that apart from the astronomers belonging to the Āryabhaṭan school like Bhāskara I, Govindasvāmin, Parameśvara, Nīlakaṇṭha Somayājī, etc., others like Brahmagupta, Śrīpati, Bhāskara-II and the (anonymous) author of Sūryasiddhānta have all adopted a variable epicycle, whose radius r is proportional to the *manda-karṇa* K such that their ratio is a constant—Caturveda Pṛthūdakasvāmin being the only exception.[9]

Thus, according to Bhāskara I and several other astronomers, as far as the *manda* process is concerned, the motion of the planet on the epicycle is such that the following equation is always satisfied:

$$\frac{r}{K} = \frac{r_0}{R}. \tag{8}$$

Then the relation (7) reduces to

$$R\sin\Delta\theta \;\; = \;\; \frac{r_0}{R}\, R\sin(\theta_0 - \theta_m), \tag{9}$$

where r_0 is the mean or tabulated value of the radius of the *manda* epicycle.

Further, according to *Āryabhaṭīya*, r_0 itself is not a constant but varies with anomaly. We will not discuss this aspect in this paper.

The problem of interdependency or anyonyāśraya

It may be noted in (8) that while RHS is known, both the quantities r and K in the LHS are unknown. Hence, to know the value of the *mandakarṇa* K, we need to know the value of the radius of the *manda-nīcocca-vṛtta* r, and vice versa. This problem of mutual or interdependency among the variables (here r and K) is referred to as *itaretarāśraya* or *anyonyāśraya* in Sanskrit.

In his *Āryabhaṭīya-bhāṣya* Nīlakaṇṭha describes the problem of mutual dependency as follows:[10]

तत्र मन्दकर्मणि कियांश्चिद्विशेषः स्यात्, यतः, 'कक्ष्यायां ग्रहवेग' इत्यादिनार्यार्धेन मन्दकर्ण-

> वृद्धिह्रासानुरूपमुच्चनीचवृत्तस्यापि महत्वमल्पत्वं च वक्ष्यते, ततो मन्दकर्णस्याविशेषः कार्यः। कर्णे ज्ञाते एव उच्चनीचवृत्तव्यासार्धं ज्ञेयं, तस्मिंश्च ज्ञाते एव कर्णो ज्ञेय इतीतरेतराश्रय-परिहारायाविशेषणं क्रियते।
>
> In the process of *manda* correction, there is some speciality, because, by the half of the *āryā* (verse) '*kakṣyāyāṃ grahavega*' etc., the decrease and increase in the [dimension of] *manda* circle, will be explained, on account of the increase and decrease of the *karṇa*. Hence, iteration has to be done for the *karṇa*. To circumvent the [problem of] mutual dependency [that is], if only the *karṇa* is known, the radius of the *uccanīcavṛtta* (manda epicycle) is known, and only if that (the radius of the *uccanīcavṛtta*) is known, then the *karṇa* is known, the process of iteration (*aviśeṣaṇaṃ*) is done.

What is of particular interest to us in the above passage is the quotation '*kakṣyāyāṃ grahavega*' given by Nīlakaṇṭha from *Āryabhaṭīya* itself in support of the variable epicycle model. By making the statement *ityādinā āryādhena ... vakṣyate*', Nīlakaṇṭha in no uncertain terms points to the fact that the variation in radius of the *manda* epicycle in accordance with *karṇa*, was not a concept that was newly introduced by the later astronomers into the planetary theory, but was something that was present in the Āryabhaṭa's conception of planetary model itself. This indeed is in corroboration with the prescription of iterative process by Bhāskara I—whose works came nearly 150 years after *Āryabhaṭīya*—which we will be describing at great length in the following sections.

Solving the problem of itaretarāśraya by asakṛt-karma

To solve this problem of *itaretarāśraya* (interdependency) Bhāskara has proposed an iterative procedure by which both r and K are simultaneously obtained. This iterative procedure has been referred to as *asakṛt-karma*[11] or *aviśeṣa-karma* (operation by which the difference is reduced to zero), and the hypotenuse obtained thereby is called *asakṛt-karṇa* or *aviśeṣa-karṇa*.[12]

The term *asakṛt-karma*–literally meaning 'doing more than once'–refers to a certain interesting mathematical technique that has been employed by Indian astronomers in their planetary computations in order to obtain the value of a certain physical quantity by means of an iterative process. It is also called *aviśeṣa-karma* in the sense that this iterative process is to be carried out till we get consecutive values that are very close to each other. That is, the successive values do not differ from each other (*aviśeṣa*), upto a specified degree of accuracy.

Among the astronomical works that are extant today, the earliest one that provides a systematic exposition of the method of *asakṛt-karma*, is *Mahā-bhāskarīya* of Bhāskara.[13] Here Bhāskara has explained the procedure in just four verses [chapter 4, 9–12]. In what follows, we provide these verses along with their English translation. A detailed mathematical explanation with necessary geometric constructions will be presented in the ensuing sections.

आद्ये पदे चतुर्थे च व्यासार्धे कोटिसाधनम्।
क्षिप्यते शोध्यते चैव शेषयोः कोटिका भवेत् ॥९॥

In the first and the fourth quadrants, *koṭiphala* is added to the radius and in the remaining quadrants (second and fourth), it is subtracted from the radius. The resultant (sum or difference) will be the *koṭi*.

तद्बाहुवर्गयोगस्य मूलं कर्णः प्रकीर्तितः।
बाहुकोटिफलाभ्यस्ते कर्णे व्यासार्धभाजिते॥१०॥
भुजाकोटिफले स्यातां ताभ्यां कर्णश्च पूर्ववत्।
भूयः पूर्वफलाभ्यस्ते कर्णे त्रिज्याविभाजिते॥११॥
एवं पुनः पुनः कुर्यात् कर्णः पूर्वोक्तकर्मणा।
यावत्तुल्यो भवेत्कर्णः पूर्वोक्तविधिनाऽमुना॥१२॥

The square root of the sum of the squares of that *koṭi* and the *bāhu* is known as *karṇa* (hypotenuse) [of the triangle

> formed by the three sides namely, *bāhu*, *koṭi* and *karṇa*]. If the *karṇa* is multiplied by *bāhuphala* and *koṭiphala* and divided by *vyāsārdha* (radius), the results will still be [called] the *bāhuphala* and the *koṭiphala* [but new ones from the previous iterated value]. Again from them the new hypotenuse has to be obtained. Further, when this hypotenuse is multiplied bythe previous *phalas* (the initial *bāhuphala* and *koṭiphala*} and divided by the radius [we obtain the next iterated value of the *bāhuphala* and *koṭiphala*]. Following the above procedure, the [new] hypotenuse should be obtained again and again till they [two successive values] become equal (*yāvat tulyo bhavet karṇaḥ*).

Bhāskara first defines the hypotenuse (*karṇa*) of the triangle formed by the three sides, namely *koṭi*, *bāhu* and *karṇa*.[14] He then proposes an iterative procedure to obtain the hypotenuse which represents the true distances of planets from the centre of the *kakṣyāmaṇḍala*. The procedure prescribed here may be outlined as follows: First we need to obtain an approximate value of the hypotenuse (denoted by K_0) using the initial values of *koṭiphala* and *bhujāphala*, which are obtained in terms of the mean epicycle radius (r_0). Now the *bāhuphala* (b_0) and *koṭiphala* (k_0) are to be multiplied by this hypotenuse (K_0) and divided by the radius R to obtain subsequent values of the *bāhuphala* (b_1) and the *koṭiphala* (k_1). Using these values b_1 and k_1, the first iterated hypotenuse (K_1) is obtained. Multiplying b_0 and k_0 by K_1 and dividing by R, b_2 and k_2 are obtained. From b_2 and k_2, the second iterated hypotenuse K_2 is obtained. This process is to be continued till the difference between two successive hypotenuses, say K_i and K_{i-1} become negligible so that the two values may be treated to be almost equal. This is what is indicated by the phrase *yāvattulyo-bhavet-karṇaḥ* in the third quarter of verse 12.

To summarize, we start with an initial approximate value of the hypotenuse K_0, continue to obtain successive iterates, $K_1, K_2, \ldots, K_n, \ldots$ until $K_n - K_{n-1} \approx 0$. The same procedure is given by Nīlakaṇṭha in verses 41, 42 of his *Tantra-saṅgraha*:

दोः कोटिफलनिघ्नाद्वो कर्णात् त्रिज्याहृते फले ॥४१॥

ताभ्यां कर्णः पुनस्साध्यः भूयः पूर्वफलाहतात्।
तत्तत्कर्णात् त्रिभज्याप्तफलाभ्यामविषेशयेत्॥४२॥

The *doḥphala* and the *koṭiphala* [initially obtained] are multiplied by the *karṇa* and divided by *trijyā*. From these resulting *phalas*, the *karṇa* has to be obtained again. Further, the previous *phalas* must be multiplied by the corresponding *karṇas* and divided by the *trijyā*, and the process has to be repeated to get the *aviṣeśa-karṇa* (the hypotenuse which does not change on iteration).

We now move on to explain in great detail the *aviśeṣa-karma*, prescribed by *Bhāskara* in his *Mahābhāskarīya*, with the help of geometrical figures and mathematical equations. Let R and r be radii of the deferent circle and the epicycle respectively. It was shown earlier that

$$K = \left[(R + r\cos(\theta_0 - \theta_m))^2 + (r\sin(\theta_0 - \theta_m))^2\right]. \qquad (10)$$

Here the radius of the epicycle r itself is proportional to *karṇa* K and therefore needs to be determined along with K iteratively. In the first approximation, r is set equal to r_0. In Figure 2, U is in the direction of *mandocca* and the mean planet is at P_0. $U\hat{O}P_0$ is the *manda-kendra*, $\theta_0 - \theta_m$. Draw a circle of radius r_0 around P_0. This is the epicycle in the first approximation. P_1 is a point on this circle, such that P_0P_1 is parallel to OU, P_1 is the true planet in this approximation. OP_0 is extended to N. Draw P_1N_1 perpendicular to ON. Then,

$$\begin{aligned} b_0 &= P_1N_1 = r_0\sin(\theta_0 - \theta_m) \\ \text{and} \qquad k_0 &= P_0N_1 = r_0\cos(\theta_0 - \theta_m), \end{aligned} \qquad (11)$$

are the *bāhuphala* and *koṭiphala* in this approximation. In the same approximation, OP_1 is the *karṇa*, which represents the distance of the planet from the centre of *kakṣyāvṛtta* and is given by

$$\begin{aligned} OP_1 = K_0 &= \left[ON_1^2 + P_1N_1^2\right]\frac{1}{2} \\ &= \left[(R + k_0)^2 + b_0^2\right]^{\frac{1}{2}}. \end{aligned} \qquad (12)$$

In the next approximation, the radius of the epicycle is

$$P_0P_2 = r_1 = r_0\frac{K_0}{R}. \tag{13}$$

and the planet is at P_2. Then the *bāhuphala* (b_1) and *koṭiphala* (k_1) are obtained using (13) and (11) as follows:

$$\begin{aligned} b_1 &= P_2N_2 = P_0P_2\sin(\theta_0-\theta_m) = b_0\frac{K_0}{R} \\ \text{and} \qquad k_1 &= P_0N_2 = P_0P_2\cos(\theta_0-\theta_m) = k_0\frac{K_0}{R}. \end{aligned} \tag{14}$$

In this approximation the *manda-karṇa* is given by

$$K_1 = OP_2 = \left[(R+k_1)^2 + b_1^2\right]^{\frac{1}{2}}. \tag{15}$$

In the next approximation, the planet is at P_3 such that the radius of the epicycle is

$$P_0P_3 = r_2 = r_0\frac{K_1}{R}. \tag{16}$$

Now, the *bāhuphala* $b_2 = P_3N_3$, the *koṭiphala* $k_2 = P_0N_3$ and the *manda-karṇa*, $K_2 = OP_3$ are given by

$$\begin{aligned} b_2 &= b_0\frac{K_1}{R} \\ k_2 &= k_0\frac{K_1}{R} \\ \text{and} \qquad K_2 &= \left[(R+k_2)^2 + b_2^2\right]^{\frac{1}{2}}. \end{aligned} \tag{17}$$

Proceeding in this manner, the i^{th} iterated value $(i = 1, 2, 3, \ldots)$ of the epicycle radius, r_i, *bāhuphala* b_i, *koṭiphala* k_i and the *manda-karṇa* are given by

$$\begin{aligned} r_i &= r_0\frac{K_{i-1}}{R} \\ b_i &= b_0\frac{K_{i-1}}{R} \\ k_i &= k_0\frac{K_{i-1}}{R} \\ \text{and} \qquad K_i &= \sqrt{(R+k_i)^2 + b_i^2}. \end{aligned} \tag{18}$$

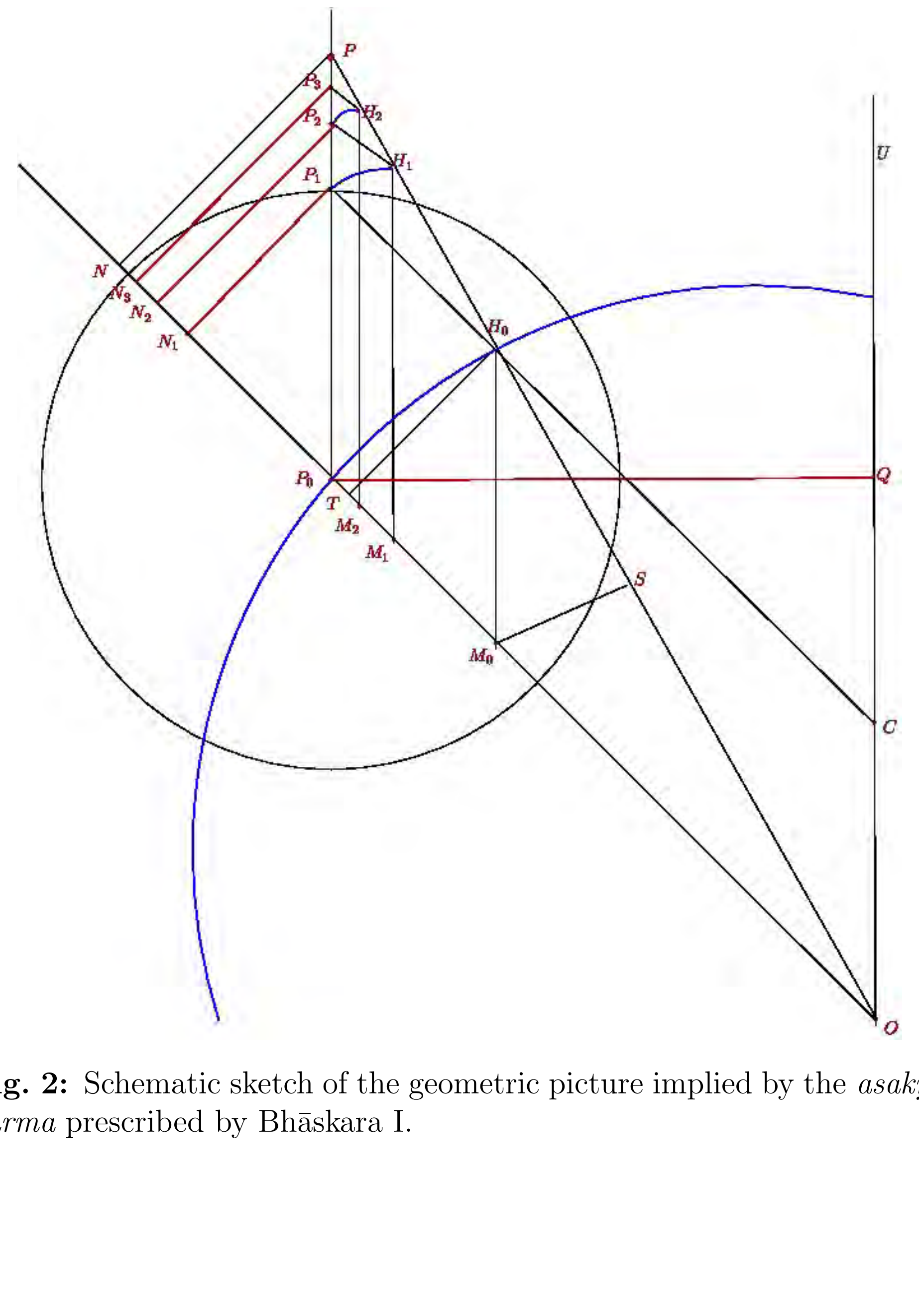

Fig. 2: Schematic sketch of the geometric picture implied by the *asakṛt-karma* prescribed by Bhāskara I.

The above process is carried out until $K_i \approx K_{i-1}$, to the desired accuracy. When this happens, K_i is referred to as the *aviśeṣa-karṇa*. When $K_i \approx K_{i-1}$, we see from (18) that

$$\frac{r_i}{K_i} = \frac{r_0}{R}\frac{K_{i-1}}{K_i} \approx \frac{r_0}{R}. \tag{19}$$

In the following section we will use geometrical arguments to show that K_i converges to a definite value K, P_i converges to a point P, and r_i to a value r such that

$$\frac{P_0P}{OP} = \frac{r}{K} = \frac{r_0}{R}. \tag{20}$$

Geometric picture implied by asakṛt-karma

The discussion on the geometrical picture implied by the iterative process and the convergence of r and K presented here, is based upon K. S. Shukla's explanatory notes on *Mahābhāskarīya*.[15]. In Figure 2, a part of the *kakṣyāmaṇḍala* (deferent circle) with radius R, centered at O is shown. U represents the *mandocca* and P_0 is the mean planet. Draw a circle of radius r_0, which is the mean value of the radius of epicycle as given in the text, around P_0, explained earlier. r_0 is the first approximation to the actual value of the epicycle radius, r. Choose P_1 on this circle such that P_0P_1 is parallel to OU. Then P_1 is the first approximation to the position of the true planet and $OP_1 = K_0$ is the *manda-karṇa*, in the first approximation. We then choose a point C along the line OU such that $OC = r_0$. Draw a line from C parallel to OP_0. This line would intersect the circumference of the epicycle at P_1 as shown in Figure 2. The line CP_1 intersects the deferent at point H_0. Join OH_0 and extend it such that it meets the extended P_0P_1 at P. We will show shortly that this point P is the true position of the planet and OP is the true value of *manda-karṇa* K.

For this, we first draw H_0M_0 such that it is parallel to OU. Then, drop a perpendicular P_1N_1 from P_1 on the extended OP_0 at N_1. Now,

P_1N_1 and P_0N_1 are the *bāhuphala* and the *koṭiphala* in the first approximation. With O as centre and OP_1 as the radius draw an arc such that it intersects OP at H_1. From H_1 draw H_1M_1 to meet OP_0 at M_1 such that H_1M_1 parallel to OU. Then, draw H_1P_2 parallel to OP_0 meeting P_0P at P_2. From P_2 draw P_2N_2 perpendicular to extended OP_0 intersecting it at N_2. Again with O as centre and OP_2 as the radius draw an arc such that it intersects OP at H_2. From H_2 draw a line parallel to OU such that it meets OP_0 at M_2. Then, draw H_2P_3 parallel to OP_0 meeting P_0P at P_3 as shown in the figure.

Now $P_0P_1 = r_0$ and $OP_1 = K_0$. By construction, $H_0M_0 = P_0P_1 = r_0$, $P_0P_2 = H_1M_1$ and $OH_1 = OP_1 = K_0$. As the triangles H_1M_1O and H_0M_0O are similar, we have

$$P_0P_2 = H_1M_1 = H_0M_0\frac{OH_1}{OH_0} = r_0\frac{K_0}{R}. \tag{21}$$

As the radius of the epicycle in the second approximation, $r_1 = r_0\frac{K_0}{R}$, it is clear that P_2 represents the true planet in the second approximation which is situated on an epicycle of radius $P_0P_2 = r_1$. In this approximation, the *manda-karṇa* is

$$OP_2 = K_1 = (ON_2^2 + P_2N_2^2)^{\frac{1}{2}} = [(OP_0 + P_0N_2)^2 + P_2N_2^2]^{\frac{1}{2}}, \tag{22}$$

where P_2N_2 and P_0N_2 are the *bāhuphala* and the *koṭiphala* in this approximation. In the same manner, it can be seen that P_3 represents the true planet in the third approximation, and $P_0P_3 = H_2M_2 = r_2 = r_0\frac{K_1}{R}$ and $OP_3 = K_2$ represent the epicycle radius and the *mandakarṇa* in this approximation. Continuing in this manner, it is clear that $P_1, P_2, P_3, \ldots$ represent the true planet in the first, second, third steps in the iterative process.

We first consider the case when the *manda-karṇa* is in the first or the fourth quadrant ($0 \leq \theta_0 - \theta_m \leq 90°$ or $270° \leq \theta_0 - \theta_m \leq 360°$). It may be noted that, by construction,

$$r_i < r_{i+1} \qquad \text{and} \qquad K_i < K_{i+1}.$$

Moreover, $r_0, r_1, r_2, \ldots$ are each less than P_0P which is the upper bound of the sequence $\{r_i\}$, and $K_0, K_1, K_2, \ldots$ are each less than OP which is the upper bound of the sequence $\{K_i\}$. That is,

$$r_0 < r_1 < r_2 < r_3 \ldots < r_n < \ldots < P_0P$$
$$\text{and} \quad K_0 < K_1 < K_2 < K_3 < \ldots < K_n < \ldots < OP.$$

Since the sequences $\{r_i\}$ and $\{K_i\}$ are each monotonic and bounded, they are convergent. If OP is the convergent value of the *karṇa* K and P_0P is the convergent value of the epicycle radius, r, they ought to satisfy $\frac{r}{K} = \frac{r_0}{R}$. This is indeed so, as the traingles OP_0P and OM_0H_0 are similar, and

$$\frac{r}{K} = \frac{P_0P}{OP} = \frac{H_0M_0}{OH_0} = \frac{P_0P_1}{OH_0} = \frac{r_0}{R}.$$

This confirms that the point P indeed represents the true planet.

Strictly speaking, the above argument is valid only in the first and the fourth quadrants. When the *kendra*, $\theta - \theta_m$ is in the second or the third quadrants ($90° \leq \theta_0 - \theta_m \leq 270°$), the sequences $\{r_i\}$ and $\{K_i\}$ are oscillatory. That is,

$$r_1 < r_0, \quad r_2 > r_1, \quad r_3 < r_2, \ldots$$
$$\text{and} \quad K_1 < K_0, \quad K_2 > K_1, \quad K_3 < K_2, \ldots.$$

However, even in this case it can be shown that $|K_i - K_{i-1}| \to 0$, and $|r_i - r_{i-1}| \to 0$, as $i \to \infty$, (as will be clear in the next section). Then P_i converges to P, where P is found in the above manner geometrically, and the sequences $\{r_i\}$ and $\{K_i\}$ converge to $P_0P = r$ and $EP = K$ respectively, such that

$$\frac{r}{K} = \frac{r_0}{R}. \tag{23}$$

Analytic expression for the mandakarṇa in the iterative process

We now obtain the explicit expression for the *manda-karṇa* at the various steps of iteration. For convenience, we use the symbols $\beta = \frac{r_0}{R}$, which is the ratio of the 'stated value' of the radius of the epicycle and *trijyā*, and $\phi = \theta_0 - \theta_m$ for the anomaly. Then, using the binomial expansion,

$$\begin{aligned}
K_0 &= \left[(R + r_0\cos(\theta_0 - \theta_m))^2 + r_0^2\sin^2(\theta_0 - \theta_m)\right]^{\frac{1}{2}} \\
&= R\left[1 + 2\beta\cos\phi + \beta^2\right]^{\frac{1}{2}} \\
&= R[1 + \frac{1}{2}(2\beta\cos\phi + \beta^2) + \frac{\frac{1}{2}(\frac{1}{2}-1)}{1.2}(2\beta\cos\phi + \beta^2)^2 \\
&\quad + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{1.2.3}(2\beta\cos\phi + \beta^2)^3 + \ldots] \\
&= R[1 + \beta\cos\phi + \frac{\beta^2}{2} - \frac{1}{8}\beta^2(2\cos\phi + \beta)^2 \\
&\quad + \frac{1}{16}\beta^3(2\cos\phi + \beta)^3 - \frac{5}{16.8}\beta^4(2\cos\phi + \beta)^4 + \ldots]
\end{aligned} \tag{24}$$

Collecting all the terms proportional to β, β^2, β^3 and β^4 together, we have,

$$\begin{aligned}
K_0 &= R[1 + \beta\cos\phi + \frac{\beta^2}{2}\sin^2\phi - \frac{\beta^3}{2}\cos\phi\sin^2\phi \\
&\quad - \frac{\beta^4}{8}\sin^2\phi(1 - 5\sin^2\phi) + O(\beta^5)].
\end{aligned} \tag{25}$$

Now, from (18) we have

$$\begin{aligned}
K_i &= \left[\left(R + r_0\frac{K_{i-1}}{R}\cos(\theta_0 - \theta_m)\right)^2 + r_0^2(\frac{K_{i-1}}{R})^2\sin^2(\theta_0 - \theta_m)\right]^{\frac{1}{2}} \\
&= R\left[\left(1 + \beta\frac{K_{i-1}}{R}\cos\phi\right)^2 + \left(\beta\frac{K_{i-1}}{R}\sin\phi\right)^2\right]^{\frac{1}{2}}
\end{aligned}$$

Therefore, K_i can be obtained from the expression for K_0 by replacing β by $\beta.\frac{K_{i-1}}{R}$. Hence,

$$\begin{aligned}
K_i &= R[1 + \beta\frac{K_{i-1}}{R}\cos\phi + \frac{1}{2}(\beta\frac{K_{i-1}}{R})^2\sin^2\phi \\
&\quad - \frac{1}{2}(\beta\frac{K_{i-1}}{R})^3\cos\phi\sin^2\phi - \ldots + O(\beta^5)].
\end{aligned} \tag{26}$$

Here, in the RHS, $\frac{K_{i-1}}{R}$ is also a series in powers of β. Noting this, and again collecting terms proporional to $\beta, \beta^2, \beta^3, \beta^4$ together, we have,

$$\begin{aligned}
K_1 &= R[1 + \beta\cos\phi + \beta^2(\cos^2\phi + \frac{\sin^2\phi}{2}) + \beta^3\cos\phi\sin^2\phi \qquad (27)\\
&\quad + \frac{\beta^4}{8}\sin^2\phi(3\sin^2\phi - 8\cos^2\phi) + O(\beta^5)]\\
K_2 &= R[1 + \beta\cos\phi + \beta^2(\cos^2\phi + \frac{\sin^2\phi}{2}) + \beta^3\cos\phi\\
&\quad + \frac{\beta^4}{8}\sin^2\phi(13\cos^2\phi - 1) + O(\beta^5)]\\
K_3 &= R[1 + \beta\cos\phi + \beta^2(\cos^2\phi + \frac{\sin^2\phi}{2}) + \beta^3\cos\phi\\
&\quad + \beta^4\left(\cos^2\phi + \frac{3}{8}\sin^2\phi + \frac{1}{8}\sin^2\phi\cos^2\phi\right) + O(\beta^5)],
\end{aligned}$$

and so on. From these expressions, it is clear that

$$K_{i+1} - K_i = O(\beta^{i+2}). \qquad (28)$$

In the expressions for K_i, the coefficients of the powers of β in the various terms are of $O(1)$.$\beta = \frac{r_0}{R} \ll 1$. Hence it is clear that the series $K_0, K_1, \ldots K_n$ converges to some definite value. Towards the end of the next section we will see that as a series expansion in powers of β, K_i is correct to $O(\beta^{i+1})$, and also find the limiting value to which it converges.

Mādhava's formula for the aviśeṣa-karṇa through sakṛt-karma

Having analysed the geometry of the problem, Mādhava of *Saṅgamagrāma*, the renowned astronomer-mathematician of the 14th century, came up with a brilliant formula that gives the value of the *asakṛt-karṇa* (the iterated value of the hypotenuse), through *sakṛt-karma* (a single operation), that circumvents the iterative process prescribed by Bhāskara.

Mādhava's procedure for determining the *aviśiṣṭa-mandakarṇa* involves finding a new quantity called the *viparyayakarṇa* or *viparītakarṇa*. The term *viparītakarṇa* literally means 'inverse hypotenuse', and is nothing but the radius of the *kakṣyāvṛtta* when the *mandakarṇa* is taken to be the *trijyā*, R. The following verses from *Tantrasaṅgraha* (II, 43–44) present the way of obtaining the *aviśiṣṭa-mandakarṇa* proposed by Mādhava that circumvents the iterative process.

विस्तृतिदलदोःफलकृतिवियुतिपदं कोटिफलविहीनयुतम् ।
केन्द्रे मृगकर्किगते स खलु विपर्ययकृतो भवेत् कर्णः ॥
तेन हृता त्रिज्याकृतिः अयत्नविहितोऽविशेषकर्णः स्यात् ।
इति वा कर्णः साध्यः मान्दे सकृदेव माधवप्रोक्तः ॥

The square of the *doḥphala* is subtracted from the square of the *trijyā* and its square root is taken. The *koṭiphala* is added to or subtracted from this depending upon whether the *kendra* (anomaly) is within 6 signs beginning from *Karki* (Cancer) or *Mṛga* (Capricorn). This gives the *viparyayakarṇa*. The square of the *trijyā* divided by this *viparyayakarṇa* is the *aviśeṣakarṇa* (iterated hypotenuse) obtained without any effort [of iteration]. This is another way by which the [*aviśeṣa*]-*karṇa* in the *manda* process can be obtained as enunciated by Mādhava.

Here, the *viparyaya-karṇa* is given as

$$R_v = \sqrt{R^2 - r_0^2 \sin^2(\theta_0 - \theta_m)^2} - r_0 \cos(\theta_0 - \theta_m) \tag{29}$$

and the *manda-karṇa* is stated to be:

$$K = \frac{R^2}{R_v} = \frac{R^2}{\left[\sqrt{R^2 - r_0^2 \sin^2(\theta_0 - \theta_m)^2} - r_0 \cos(\theta_0 - \theta_m)\right]}. \tag{30}$$

The rationale behind the above formula is outlined in *Gaṇita-yukti-bhāṣā* and can be understood from Figure 2. Here, the triangle

OPP_0 and OH_0M_0 are similar. Hence,

$$\frac{OP}{OP_0} = \frac{OH_0}{OM_0}. \tag{31}$$

Now $OP = K$ and $OP_0 = OH_0 = R$. As we shall explain below, $OM_0 = R_v$. Hence

$$\frac{K}{R} = \frac{R}{R_v} \qquad \text{or} \qquad K = \frac{R^2}{R_v}. \tag{32}$$

From the above relation, $OM_0 = R_v$ can be understood as the value of *trijyā*, when the *manda-karṇa*, K is taken to be *trijyā* R. That is why it is termed the *viparyaya-karṇa* or *viparīta-karṇa* or the 'inverse hypotenuse'. From H_0, drop the perpendicular H_0T on the OP_0. It is clear that

$$\begin{aligned} H_0T &= P_1N_1 = r_0\sin(\theta_0 - \theta_m) \qquad (33)\\ \text{and} \qquad H_0T &= P_0N_1 = r_0\cos(\theta_0 - \theta_m). \end{aligned}$$

Then, in the right angled triangle OH_0T, we have

$$OT = \sqrt{OH_0^2 - H_0T^2} = \sqrt{R^2 - r_0^2\sin^2(\theta_0 - \theta_m)} \tag{34}$$

Hence,

$$\begin{aligned} R_v = OM_0 &= OT - M_0T \\ &= \sqrt{R^2 - r_0^2\sin^2(\theta_0-\theta_m)} - r_0\cos(\theta_0-\theta_m) \\ \text{and} \qquad K &= \frac{R^2}{R_v} \\ &= \frac{R^2}{\sqrt{R^2 - r_0^2\sin^2(\theta_0-\theta_m)^2} - r_0\cos(\theta_0-\theta_m)}, \qquad (35) \end{aligned}$$

as required. Using our notation, $\beta = \frac{r_0}{R}$ and $\phi = (\theta_0 - \theta_m)$, we have

$$K = R\left[(1-\beta^2\sin^2\phi)^{\frac{1}{2}} - \beta\cos\phi\right]^{-1} \tag{36}$$

Using binomial expansion, and including terms only to $O(\beta^4)$, we can write

$$K = R\left[1 - \frac{1}{2}\beta^2\sin^2\phi + \frac{\frac{1}{2}(\frac{1}{2}-1)}{1.2}\beta^4\sin^4\phi - \beta\cos\phi\right]^{-1} + O(\beta^5)$$

$$
\begin{aligned}
&= R\left[1 - \beta\cos\phi - \frac{1}{2}\beta^2\sin^2\phi - \frac{1}{8}\beta^4\sin^4\phi\right]^{-1} + O(\beta^5) \\
&= R[1 + \beta\left(\cos\phi + \frac{1}{2}\beta\sin^2\phi + \frac{1}{8}\beta^3\sin^4\phi\right) \\
&\quad + \beta^2\left(\cos\phi + \frac{1}{2}\beta\sin^2\phi + \frac{1}{8}\beta^3\sin^4\phi\right)^2 \\
&\quad + \beta^3\left(\cos\phi + \frac{1}{2}\beta\sin^2\phi + \frac{1}{8}\beta^3\sin^4\phi\right)^3 \\
&\quad + \beta^4(\cos\phi + \frac{1}{2}\beta\sin^2\phi + \frac{1}{8}\beta^3\sin^4\phi)^3] + O(\beta^5). \qquad (37)
\end{aligned}
$$

After some simplifications, we obtain

$$
\begin{aligned}
K &= R[1 + \beta\cos\phi + \beta^2(\frac{1}{2}\sin^2\phi + \cos\phi) + \beta^3\cos\phi \\
&\quad + \beta^4(\cos^2\phi + \frac{3}{8}\sin^2\phi + \frac{1}{8}\sin^2\phi\cos^2\phi)] + O(\beta^5). \qquad (38)
\end{aligned}
$$

This is the expression for the *manda-karṇa* obtained without using an iterative process correct to the fourth power of β, where ϕ is the anomaly ($\phi = \theta_0 - \theta_m$).

Comparing this expression with those for $K_0, K_1, K_2, \ldots$ obtained through the iterative process of Bhāskara I, we find that $K_0, K_1, K_2, \ldots$ are correct to $O(\beta)$, $O(\beta^2)$, $O(\beta^3)$ and $O(\beta^4)$ respectively. In fact, the $(i+1)^{th}$ iterated value K_i would be correct to $O(\beta^{i+1})$, again emphasizing the validity of the iterative process of Bhāskara I.

Āryabhaṭa's orbit and the Keplerian orbit

In this section we compare the planetary orbit in the Āryabhaṭan model with the standard Keplerian orbit.[16] The *manda-karṇa* K represents the distance of the planet from the centre of the mean planet's orbit (Earth in the case of the Sun and the Moon, and the mean sun in the case of the five planets, Mercury, Venus, Mars, Jupiter and Saturn). For discussing the orbit, we should express K in terms of the true longitude θ, rather than the mean longitude θ_0 as in equation (35). In fact, K is given as a function of θ also in *Tantra-saṅgraha* and *Gaṇita-yukti-bhāṣā*.

In Figure 2, $H_0\hat{O}U = \theta - \theta_m$ is the 'true anomaly'. In the triangle M_0H_0S where M_0S is drawn perpendicular to OH_0, $M_0\hat{H}_0S = H_0\hat{O}U = \theta - \theta_m$, and $M_0H_0 = r_0$. Hence,

$$\begin{aligned} M_0S &= r_0 \sin(\theta - \theta_m) \\ \text{and} \quad H_0S &= r_0 \cos(\theta - \theta_m) \end{aligned} \tag{39}$$

Then, the *viparīta-karṇa* R_v is given by

$$\begin{aligned} R_v = M_0O &= \sqrt{(OS)^2 + (M_0S)^2} \\ &= \sqrt{(OH_0 - H_0S)^2 + (M_0S)^2} \\ &= \sqrt{(R - r_0\cos(\theta-\theta_m))^2 + r_0^2 \sin^2(\theta - \theta_m)} \\ &= \sqrt{R^2 - 2r_0R\cos(\theta - \theta_m) + r_0^2}. \end{aligned} \tag{40}$$

Hence,

$$\begin{aligned} K &= \frac{R^2}{R_v} \\ &= \frac{R}{\left[1 - \frac{2r_0\cos(\theta-\theta_m)}{R} + \frac{r_0^2}{R^2}\right]^{\frac{1}{2}}} \end{aligned}$$

Introducing the symbol $\psi = \theta - \theta_m$, for the 'true' anomaly and remembering that $\beta = \frac{r_0}{R}$, and using the binomial expansion,

$$\begin{aligned} K &= R[1 - 2\beta\cos\psi + \beta^2]^{-\frac{1}{2}} \\ &= R[1 + \beta\cos\psi - \frac{1}{2}\beta^2(1 - 3\cos^2\psi) + O(\beta^3)]. \end{aligned} \tag{41}$$

In a model with a constant value for the radius of the epicycle r_0, the true planet[17] would be at P_1 and the true anomaly would be $\psi = P_1\hat{O}U = O\hat{P}_1P_0$, the *karṇa* would be $OP_1 = K_0$. Then in the triangle OP_1P_0,

$$\begin{aligned} R^2 = OP_0^2 &= OP_1^2 + P_0P_1^2 - 2OP_1.P_0P_1.\cos(O\hat{P}_1P_0) \\ &= K_0^2 + r_0^2 - 2K_0r_0\cos\psi. \end{aligned} \tag{42}$$

This is a quadratic equation in K_0, whose solution is given by

$$\begin{aligned} K_0 &= \frac{2r_0\cos\psi + \sqrt{4r_0^2\cos^2\psi - 4(r_0^2 - R^2)}}{2} \\ \text{or} \quad K_0 &= R[\beta\cos\psi + \sqrt{1 - \beta^2\sin^2\psi} \\ &= R[1 + \beta\cos\psi - \frac{1}{2}\beta^2\sin^2\psi + O(\beta^3)]. \end{aligned} \tag{43}$$

In this case, we know that the true planet moves in a circle of radius R with C as the centre, where C is situated at a distance of $CO = r_0$, from O. Comparing the expression for K with K_0, it is clear that the orbit in the Āryabhaṭan model with a variable epicycle radius would be close to an eccentric circle whose centre is at a distance r_0 from O with departure from circularity of $O(\beta^2) = O(r_0^2/R^2)$. Now $\frac{r_0}{R} = \frac{13.5}{360} = 0.0375$ for the Sun and $\frac{r_0}{R} = \frac{31.5}{360} = 0.0875$ for the Moon in Āryabhaṭīya. Then this departure from the circularity would be ≈0.14% for the Sun and ≈0.77% for the Moon. The orbit of motion in Kepler's model is an ellipse whose equation is given by

$$\frac{l}{r} = [1 - e\cos(\theta - \theta_m)], \tag{44}$$

where r is the distance from one of the foci; θ is the angle with respect to any reference line, $l = a(1 - e^2)$ with a as the semi-major axis, e is the eccentricity and θ_m is the longitude of the apogee/aphelion. Now normalizing the distance in the model to the value of R when $\theta - \theta_m = 90°$, by setting $l = R$, and denoting the distance by K_{kepler}, we have

$$\begin{aligned} K_{\text{(Kepler)}} &= \frac{R}{(1 - e\cos\psi)} \\ &= R[1 + e\cos\psi + e^2\cos^2\psi + O(e^3)]. \end{aligned} \tag{45}$$

For small values of e, this is an ellipse which is very close to a circle of radius R, whose centre is at a distance of eR from O.

Now, it can be shown that the longitude θ is given by the following expression in the Āryabhaṭan and Keplerian models:

$$\begin{aligned} \theta_{\text{(Āryabhaṭa)}} &= \theta_0 - \frac{r_0}{R}\sin(\theta_0 - \theta_m) + O((\frac{r_0}{R})^2) \qquad (46) \\ &= \theta_0 - \beta\sin(\theta_0 - \theta_m) + O(\beta^2) \\ \text{and} \qquad \theta_{\text{(Kepler)}} &= \theta_0 - 2e\sin(\theta_0 - \theta_m) + O(e^2). \end{aligned}$$

The expression in the higher orders are quite different in the two models, but it is clear that $\beta = \frac{r_0}{R}$ should be close to $2e$, if the true longitude in the Āryabhaṭan model is to be close to the one in the Kepler model. The modern values of $2e$ are 0.034 and 0.11 for the Sun and the Moon respectively, compared to the values of 0.0375 and 0.0875 for $\beta = \frac{r_0}{R}$ in the Āryabhaṭan model.

Hence, we see that the 'centre' of the orbit in the Āryabhaṭan model is at a distance from O (centre of the mean planet's orbit) which is nearly twice the corresponding distance in the Kepler model. Note that this has very little to do with the variation of the epicycle radius, as the functional dependence of the true planet's distance from O in this case is same as the one in a model with a constant epicycle radius to $O(\beta)$ or $O(e)$. Rather, it is the fact that the variation of the planet's distance in the epicycle model cannot be matched with the corresponding distance in the Kepler model, if the true longitudes are matched.

Discussion

Bhāskara I in his *Mahā-Bhāskarīya* while discussing the epicycle model for the equation of centre, apart from providing a description as to how the radius of the epicycle varies in accordance with the *manda-karṇa*, has also outlined the procedure for obtaining the *aviśiṣṭa-manda-karna* through the *aviśeṣa-karma*. Commenting on why the *aviśeṣa-karma* is done, Bhāskara in his *Āryabhaṭīya-bhāṣya* observes:[18]

तत्सूक्ष्मार्थिभिरविशिष्यते, प्रतिमण्डलकर्णस्य वृद्धिह्रासवशात् दृष्टिर्भिद्यत इति।

> Since the observation does not tally [with the computed value] due to the increase and decrease in the dimension of the *pratimaṇḍalakarṇa*,[19] by those who seek for more accurate results, the *aviśeṣa* process is adopted.

In short, according to *Bhāskara*, the *aviśeṣa-karma* is for improving the accuracy of the computed values, so that there is a better agreement between the computed and the observed values. This is a little difficult to understand because—as seen from our discussion in the paper (see section 6) the difference between the expressions for the equation of centre in the epicycle models with the fixed and the variable radii is of order $(\frac{r_0}{R})^2$, and it would have required far more accurate observations than what would have been possible at that time, to chose one model over the other.

However, it must be noted that this prescription, namely the radius of the epicycle being proportional to the *manda-karṇa*—as indicated in equation (9), and which seems to be a feature adopted in most Indian works on astronomy—leads to a simple formula for the equation of centre, which does not involve the *manda-karṇa*. This tempts us to think that this could probably the reason behind the choice of the curious relation between r and K. In any case, as the expression for the *manda-karṇa* involves the radius of the epicycle, and the latter is stated to be proportional to the former, they are mutually dependent. Both of them are found using an iterative procedure outlined by Bhāskara I himself. The same procedure is stated and explained in *Tantrasaṅgraha* of Nīlakaṇṭha Somayājī, and other Kerala works.

In this paper, we have elaborated on the iterative process, and showed that the *manda-karṇa* obtained through this process converges to the direct expression for the *manda-karṇa* given in *Tantra-saṅgraha* and *Yukti-bhāṣā*, which is ascribed to Mādhava of *Saṅgamagrāma*. In fact, there are very many occasions where iterative processes are prescribed in the Indian astronomical texts. For instance, calculations pertaining to eclipses are based on iterative processes. Another instance is to find the arc from the Rsine for small values of the arc, which is discussed in another article in this volume.

We have also explored the nature of the planet's orbit in the Āryabhaṭan model. This model with a variable radius of the epicycle leads to a non-circular orbit, though the departure from circularity is understandably small. In the Greeko-European tradition, before Kepler, this would have been unthinkable, as all motions are conceived as combinations of uniform circular motions.[20] In the Indian texts, there are no such preconceived notions[21] and a variable epicycle model would have been perfectly legitimate, as long as it served a useful purpose.

Acknowledgements: The authors are grateful to Prof. M. D. Srinivas, Centre for Policy Studies, Chennai for making several valuable suggestions to improve the manuscript.

Notes

[1]This refers to the correction employed to the mean longitude of a planet (*madhyama-graha*) to take the eccentricity of its orbit into account.

[2]The geometrical picture implied by the iterative process is discussed in Shukla K S, 1960.

[3]See for instance, Sarma K V, 1977; and Ramasubramanian K and Sriram M S, 2011.

[4]Sarma K V et.al., 2008.

[5]Having described the procedure, Nīlakaṇṭha states: मान्दे सकृदेव माधवप्रोक्तः।

[6]See for instance, Appendix F in Ramasubramanian K and Sriram M S, 2011.

[7]Ramasubramanian K et.al., 1994.

[8]For instance, in *Tantrasaṅgraha* the ratio of $\frac{r_0}{R}$ is given to be $\frac{3}{80}$ and $\frac{7}{80}$ in the case of the Sun and the Moon respectively.

[9]Shukla K S, 1973.

[10]See Sambasiva Sastri, 1931, p. 35.

[11]For a discussion on *asakṛt-karma*, see also Plofker, 2010.

[12]For example, in his *Mahā·abhāskarīya*, Bhāskara uses the word *aviśeṣa* for iteration – 'स्वाविशेषेण कर्णेन स्फुटभुक्तिरवाप्यते'। See Shukla K S 1960, p. 23.

[13]Shukla K S, 1960.

[14]In the case of the triangle OP_iN_i in Figure 2 that is being discussed here, since one of its sides, namely P_iN_i is commonly shared with the other triangle $P_0P_iN_i$ this side is interchangeably referred to as *bāhu* or *bāhuphala* in these set of verses by the author.

[15]Shukla K S, 1960

[16]See also Duke, Dennis.

[17]Note that the true planet is located at a different position here, so that the expression for ψ in terms of the mean planet θ_0 would be different from the one in the variable epicycle model.

[18]See Shukla K S, 1976, p. 223.

[19]In certain instances, as here, Bhāskara seems to be using the term *pratimaṇḍalakarṇa* to refer to what is generally referred as *mandakarṇa*.

[20]See for instance, Evans James, 1998.

[21]See Srinivas M D, 2002; and Srinivas M D, 2010

Bibliography

Duke, Dennis
Computer animations of planetary orbits according to Ptolemy's model, Āryabhaṭan model as well as models proposed by some of the Islamic Astronomers, at *https://people.scs.fsu.edu/ dduke/models.htm*

Evans, James 1998
The History and Practice of Ancient Astronomy, Oxford University Press, New York.

Plofker, Kim 2010
'Why did Sanskrit Mathematics Ignore Asakṛt Methods?' in *Proceedings of the 13th World Sanskrit Conference*, S. R. Sarma and Gyula Wojtilla (eds.), Motilal Banarsidass, New Delhi.

Ramasubramanian K., et.al, 1994
K. Ramasubramanian, M. D. Srinivas and M. S. Sriram, 'Modifications of the Ancient Planetary Model by the Astronomers of the Kerala School and the Implied Heliocentric Picture of the Planetary Motion', *Current Science* 66, 784–790.

Ramasubramanian K. and Sriram M. S., 2011
Tantrasaṅgraha of Nīlakaṇṭha Somayājī, with Eng. tr. and detailed Explanatory Notes, Hindustan Book Agency, New Delhi; (repr. Springer 2011).

Sambasiva Sastri K., 1931
(ed.) *Āryabhaṭīya* of Āryabhaṭācārya with the *Mahābhāṣya* of Nīlakaṇṭha Somasutvan, Part II *Kālakriyāpāda* ed. by Sāmbaśiva Śāstrī, Trivandrum Sanskrit Series 110, Trivandrum.

Sarma K. V., 1977
Tantrasaṅgraha of Nīlakaṇṭha Somayājī, ed. with *Yuktidīpikā* (for Chapters I–IV) and *Laghuvivṛti* (for Chapters V–VIII) of Saṅkara Vāriyar, Hoshiarpur.

Sarma K. V., et.al., 2008
Gaṇita-yukti-bhāṣā of Jyeṣṭhadeva, ed. and tr. by K. V. Sarma with Explanatory Notes by K. Ramasubramanian, M. D. Srinivas and M. S. Sriram, Hindustan Book Agency, New Delhi; (Repr. Springer 2009).

Shukla K. S., 1960
(ed.) *Mahābhāskarīya* of Bhāskara I, with tr. with Explanatory Notes, Lucknow.

Shukla K. S., 1973
'Use of Hypotenuse in the Computation of Equation of Centre under the Epicyclic Theory in the School of Aryabhata I' *Indian Journal of History of Science*, 8, 43–57.

Shukla K. S., 1976
(ed.) *Āryabhaṭīya* of Āryabhaṭa with the commentaries of Bhāskara I, and Someśvara with Introduction and Appendices, Indian National Science Academy, Delhi.

Srinivas M. D., 2002
'Geometrical Picture of Planetary Motion According to Nilakantha' in *500 Years of Tantrasangraha: A Landmark in the History of Astronomy*, M. S. Sriram, K. Ramasubramanian and M. D. Srinivas (eds.), Indian Institute of Advanced Study, Shimla.

Srinivas M. D., 2010
'Indian Planetary Models from Aryabhata to Nilakantha' in *Indian Heritage Science Philolophy Art and Architecture*, B. V. Subbarayappa and N. S. Rajaram (eds.), Bangalore, pp. 96–140.