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FOLDING METHOD OF NĀRĀYAṆA PAṆḌITA FOR THE CONSTRUCTION OF *SAMAGARBHA* AND *VIṢAMA* MAGIC SQUARES

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A general mathematical treatment of the subject of magic squares is found in the celebrated work *Gaṇitakaumudī* (c.1356 AD) of Nārāyaṇa Paṇḍita. The last or XIV chapter of this work, entitled *bhadra-gaṇita* (auspicious mathematics), presents a detailed discussion of this subject. Nārāyaṇa discusses general methods of construction of magic squares depending upon whether the square is *samagarbha* (doubly-even), *viṣamagarbha* (singly-even) or *viṣama* (odd). In the case of *samagarbha* and *viṣama* squares, Nārāyaṇa develops a new method for their construction by means of folding two magic squares which are constructed by a simple prescription. In this paper we shall discuss the mathematical basis of this *sampuṭavidhi* or folding method of Nārāyaṇa. We shall show that in the *samagargbha* or the doubly-even case, the method always leads to a pan-diagonal magic square. In the case of *viṣama* or odd square, Nārāyaṇa's folding method leads to a magic square which is not pan-diagonal. However, whenever the order of the square is not divisible by 3, Nārāyaṇa's method can be slightly modified so that the resulting square is always pan-diagonal.

Key words: *Bhadragaṇita*, Magic squares, Pan-diagonal magic squares, *Samagarbha*, *Sampuṭavidhi* or folding method, *Viṣama*

Introduction

Let us begin by recalling that an n x n square array consisting of n^2 numbers is called a semi-magic square if all the rows and columns add up to the same number. If the principal diagonals of the square also add up to

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that number, then it is called a magic square. Further, if the sums of all the "broken diagonals" also add up to the same number, then it is called a pan-diagonal magic square.

The study of magic squares in India has a long history, going back to the very ancient times. For example, the work of the ancient seer Garga is supposed to contain several 3x3 magic squares. Later, a general class of 4x4 magic squares has been attributed to the Buddhist philosopher Nàgārjuna (c. 2nd Century AD). In the great compendium *Bṛhatsaṃhitā* of Varāhamihira (c. 550 AD), we find a description of a 4x4 magic square (actually a pan-diagonal magic square), referred to as *sarvatobhadra* (auspicious all around). There have been several instances of magic squares inscribed in temples. One famous example is the 4x4 pan-diagonal magic square which is inscribed at the entrance of a 12th century Jaina Temple at Khajuraho; the same magic square is also found in an inscription at Dudhai in Jhansi District.

The Prākṛta work, *Gaṇitasārakaumudī* of Ṭhakkura Pheru (c.1300) seems to be the first available text of Indian mathematics which deals with the subject of magic squares. In a very brief treatment of the subject, involving only 10 verses of *Jantrādhikāra* of Chapter IV of *Gaṇitasārakaumudī*, Pheru presents the classification of nxn magic squares into following three types: (1) *Samagarbha*, where n is doubly-even or of the form 4m, where m is a positive integer, (2) *Viṣamagarbha*, where n is singly-even or of the form $4m + 2$, where m is a positive integer, and (3) *Viṣama* where n is odd. Pheru also briefly indicates methods of constructing *samagarbha* and *viṣama* magic squares.

A general mathematical treatment of the subject of magic squares is found in the celebrated work *Gaṇitakaumudī* (c.1356 AD) of Nārāyaṇa Paṇḍita. The last or XIV chapter of this work, entitled *bhadra-gaṇita* (auspicious mathematics), presents a detailed discussion of this subject by means of about 60 *sūtra* verses (rules or algorithms) and 17 *udāharaṇa* verses (examples). Nārāyaṇa seems to have been the first mathematician to emphasize that a general theory of magic squares can be developed if we assume that the squares are filled by an arithmetic sequence (or a collection of arithmetic sequences). He uses the method of *kuṭṭaka* (known to Indian mathematicians at least since the time of Āryabhaṭa (c.499 AD) to find out the initial term a and the constant difference d of the arithmetic sequence to be used to fill a n x n square to get sum S, by solving the linear indeterminate equation

$$nS = (n^2/2)\ [a + a+(n^2- 1)\ d\ ] \tag{1}$$

or, equivalently, the equation

$$S = na + (n/2)(n^2- 1)\ d \tag{2}$$

Nārāyaṇa then gives a general method of constructing 4x4 pan-diagonal magic squares. He in fact displays 24 pan-diagonal 4x4 magic squares, with different cells being filled by different numbers from the arithmetic sequence 1, 2, ..., 16, the top left entry being 1. Nārāyaṇa also remarks that (by permuting the rows and columns cyclically) we can construct 384 pan-diagonal 4x4 magic squares with entries 1, 2, ..., 16. In 1938, Rosser and Walker proved that this is in fact the exact number of 4x4 pan-diagonal magic squares with entries 1, 2, ..., 16. Vijayaraghavan (1941) gave a much simpler proof of this result, which have been outlined by Sridharan and Srinivas (2011, pp.389-391),

Nārāyaṇa goes on to discuss general methods of construction of magic squares depending upon whether the square is *samagarbha* (doubly-even), *viṣamagarbha* (singly-even) or *viṣama* (odd). In the case of *samagarbha* and *viṣama* squares, apart from discussing the traditionally well-known methods of construction (indicated for instance in Pheru's work earlier), Nārāyaṇa presents an entirely new method known as *sampuṭa-vidhi* (method of folding). This is a general method of construction of magic squares by composing or folding two magic squares constructed suitably[1].

In this paper we shall discuss the mathematical basis of the folding method of Nārāyaṇa for the construction of *samagarbha* and *viṣama* magic squares. We shall show that in the *samagargbha* or the doubly-even case, the method always leads to a pan-diagonal magic square. In the case of *viśama* or odd square, Nārāyaṇa's folding method leads to a magic square which is not pan-diagonal. However, whenever the order of the square is not divisible by 3, Nārāyaṇa's method can be slightly modified so that the resulting square is always pan-diagonal.

Nārāyaṇa's Folding Method for the Construction of *Samagarbha* Magic squares

Nārāyaṇa's *sampuṭavidhi* (folding method) involves the construction of two auxiliary magic squares, which are called the *chādya* (covered) and *chādaka* (coverer). As he states (see below), the process of folding involves covering of the *chādya* by the *chādaka* like in the folding of the palms.

In what follows we shall adopt the convention that the columns (rows) of an nxn square array are indexed from the left (top) by the integers 0,1,2...,*n*-1. We shall denote the element at the intersection of the *i*-th column and *j*-th row of the array M by M(*i*,*j*). Now, if M_1 and M_2 are two nxn square arrays, then the process of folding results in the square array M, whose (*i*,*j*)-th element is given by

$$M(i,j) = M_1(i,j) + M_2(n-1-i,j) \qquad (3)$$

for all $0 \leq i, j \leq n-1$.

Nārāyaṇa outlines the folding method for *samagarbha* magic squares as follows:

समगर्भे द्वे कार्ये छादकसंज्ञं तयोर्भवेदेकम्।
छाद्याभिधानमन्यत्करसंपुटवच्च संपुटो ज्ञेयः॥
इष्टादीष्टचयाङ्का भद्रमिता मूलपङ्क्तिसंज्ञाद्या।
तद्वदभीप्सितमुखचयपङ्क्तिश्चान्या पराख्या स्यात्॥
मूलाख्यपङ्क्तियोगोनितं फलं परसमाससंभक्तम्।
लब्धहता परपङ्क्तिर्गुणजाख्या सा भवेत् पङ्क्तिः॥
मूलगुणाख्ये पङ्क्ती ये ते भद्रार्धतस्तु परिवृत्ते।
ऊर्ध्वस्थितैस्तदङ्कैश्छादकसंछाद्ययोः पृथग्यानि॥
तिर्यक्कोष्ठान्याद्येऽन्यतरस्मिन्नूर्ध्वगानि कोष्ठानि।
भद्रस्यार्धं क्रमगैरुत्क्रमगैः पूरयेदर्धम्॥
भद्रानामिहसम्पुटविधिरुक्तो नृहरितनयेन।

Two *samagarbha* squares known as the coverer and the covered are to be made. Their combination is to be understood in the same manner as the folding of palms. The *mūlapaṅkti* (base sequence) has an arbitrary first term and constant difference and number of terms equal to the order of the magic square. Another similar sequence is called the *parapaṅkti* (other sequence). The quotient of *phala* (desired magic sum) decreased by the sum of the *mūlapaṅkti* when divided by the sum of the *parapaṅkti* [is the *guṇa*]. The elements of the *parapaṅkti* multiplied by that gives the *guṇapaṅkti*. The two sequences *mūlapaṅkti* and *guṇapaṅkti* are reversed after half of the square is filled. The cells of the coverer are filled horizontally and those of the covered vertically. Half of the square is filled [by the sequence] in order and the other half in reverse order. The way of combining magic square is here enunciated by the son of Nṛhari.

In order to elucidate the above method, we consider two examples presented by Nārāyaṇa.

Nārāyaṇa's Example 1: 4x4 Magic Square with Sum 40

Nārāyaṇa takes the sequence 0, 1, 2, 3 as the base sequence (*mūlapaṅkti*), which is also called the first sequence; and the sequence 0, 1, 2, 3 as the other sequence (*parapaṅkti*), which is also called the second sequence. The sum of the first sequence is 10. When this is subtracted from 40, or the desired magic sum (*phala*), we get 30. When this is divided by the sum of the second sequence, namely 6, we get 5 as the factor (*guṇa*). Multiplying each of the terms of the second sequence by this factor, we get the product sequence (*guṇapaṅkti*) 0, 5, 10, 15. From these sequences, Nārāyaṇa forms the covered (*chādya*) and the coverer (*chādaka*) squares as shown in Fig. 1.

2	3	2	3
1	4	1	4
3	2	3	2
4	1	4	1

5	0	10	15
10	15	5	0
5	0	10	15
10	15	5	0

Fig. 1. The *Chādya* and *Chādaka* Squares

Fig. 2 displays the process of folding (*sampuṭavidhi*) leading to the desired 4x4 magic square with magic sum 40.

2+15	3+10	2+0	3+5		17	13	2	8
1+0	4+5	1+15	4+10		1	9	16	14
3+15	2+10	3+0	2+5	=	18	12	3	7
4+0	1+5	4+15	1+10		4	6	19	11

Fig. 2. Folding Process for Construction of 4x4 Magic Square with Sum 40

Nārāyaṇa also displays another magic square which is obtained by interchanging the covered and the coverer squares as shown in Fig. 3.

5+3	0+2	10+3	15+2		8	2	13	17
10+4	15+1	5+4	0+1		14	16	9	1
5+2	0+3	10+2	15+3	=	7	3	12	18
10+1	15+4	5+1	0+4		11	19	6	4

Fig. 3. Folding Process for Construction of 4x4 Magic Square with Sum 40

We see that both the squares displayed in Figures 2, 3 are in fact pan-diagonal magic squares with sum 40.

Nārāyaṇa's Example 2: 8x8 Magic Square with Sum 260

In this case, Nārāyaṇa takes 1, 2, 3, 4, 5, 6, 7, 8 as the first sequences and 0, 1, 2, 3, 4, 5, 6, 7 as the second sequence. The sum of the first sequence is 36. Subtracting this from 260 gives 224. This, when divided by 28, which is the sum of the second sequence, gives 8 as the factor. Thus the product sequence will be 0, 8, 16, 24, 32, 40, 48, 56.

Now the *chādya* and *chādaka* squares are given in Fig. 4. The process of folding leads to the pan-diagonal magic square as shown in Fig. 5.

4	5	4	5	4	5	4	5
3	6	3	6	3	6	3	6
2	7	2	7	2	7	2	7
1	8	1	8	1	8	1	8
5	4	5	4	5	4	5	4
6	3	6	3	6	3	6	3
7	2	7	2	7	2	7	2
8	1	8	1	8	1	8	1

24	16	8	0	32	40	48	56
32	40	48	56	24	16	8	0
24	16	8	0	32	40	48	56
32	40	48	56	24	16	8	0
24	16	8	0	32	40	48	56
32	40	48	56	24	16	8	0
24	16	8	0	32	40	48	56
32	40	48	56	24	16	8	0

Fig. 4. The *Chādya* and *Chādaka* Squares

60	53	44	37	4	13	20	29
3	14	19	30	59	54	43	38
58	55	42	39	2	15	18	31
1	16	17	32	57	56	41	40
61	52	45	36	5	12	21	28
6	11	22	27	62	51	46	35
63	50	47	34	7	10	23	26
8	9	24	25	64	49	48	33

Fig. 5. 8x8 Magic Square with Sum 260

We shall now show that, Nārāyaṇa's folding method is indeed a general procedure, which can be used to construct a large class of pan-diagonal *samagarbha* magic squares (magic squares of order 4m).

Theorem 1:[2] Let n=4m be a number divisible by 4. Let $p(i)$, for $1 \leq i \leq n$, denote an arbitrary permutation of 1, 2, ..., n and let $q(i)$, for $0 \leq i \leq n$-1, denote an arbitrary permutation of 0, 1, ..., n-1. We can define $p(i)$ and $q(i)$ for all integers i by assuming that

$$p(i) = p(k) \text{ and } q(i) = q(k) \text{ whenever } ia\text{''}k \pmod{n} \qquad (4)$$

Now consider the square arrays S and T defined by

$$S(i, j) = p(j+2mi) \qquad (5a)$$

$$T(i, j) = q(i+2mj) \qquad (5b)$$

Then, S and T will be pan-diagonal magic squares whenever

$$p(i) + p(i+2m) = 4m+1 \qquad (6a)$$

$$q(i) + q(i+2m) = 4m-1 \qquad (6b)$$

Further, the array S+*r*T will be a pan-diagonal magic square for any number *r*; in particular S+*n*T will be a pan-diagonal magic square with entries 1, 2, ...n^2.

Proof: We first show that the sums of all the rows and principal diagonals of T are the same. The sum of the numbers in the *j*-th row is

$$\sum_{i=0}^{4m-1} q(i+2mj) = \sum_{i=0}^{4m-1} q(i) = (4m)(4m-1)/2 \qquad \ldots(7)$$

The sum of the numbers in the diagonal *i*=*j* are given by

$$\sum_{i=j=0}^{4m-1} q(i+2mj) = \sum_{i=0}^{4m-1} q[(2m+1)i] = \sum_{i=0}^{4m-1} q(i) = (4m)(4m-1)/2 \qquad \ldots(8)$$

where we have used the fact that (2*m*+1) is co-prime to 4*m*. The sum of the numbers in the other principal diagonal *i*+*j* = 4*m*-1 is given by

$$\sum_{i+j=4m-1} q(i+2mj) = \sum_{i=0}^{4m-1} q(i+2m(4m-1-i)) = \sum_{i=0}^{4m-1} q[i(1-2m) - 2m] = (4m)(4m-1)/2 \qquad (9)$$

where we have used the fact that (1-2*m*) is co-prime to 4*m*.

We shall now see that the condition 6(b) is needed in order to ensure that each column of T also has the same magic sum (4*m*)(4*m*-1)/2. The sum of the numbers in the *i*-th column is given by

$$\sum_{j=0}^{4m-1} q(i+2mj) = \sum_{j=0}^{2m-1} [q(i+4mj) + q(i+4mj+2m)] = (4m)(4m-1)/2 \tag{10}$$

where we have made use of the condition (6b).

Thus we have shown that T is a magic square. To show that it is pan-diagonal, we consider the sums along the diagonals $i+j = c$, for $0 \le c \le 4m-1$, and obtain

$$\sum_{i+j=c} q(i+2mj) = \sum_{i=0}^{4m-1} q[i+2m(c-i)] = \sum_{i=0}^{4m-1} q[i(1-2m)+2mc] = (4m)(4m-1)/2 \tag{11}$$

where we have used the fact that $(1-2m)$ is co-prime to $4m$. That the other set of diagonals $i-j = c$, for $0 \le c \le 4m-1$, also add to the same sum, can be proved along the same lines.

Thus we have shown that T is a pan-diagonal magic square. In the same way, it can be shown that S is also a pan-diagonal magic square, whenever (6a) is satisfied. Hence, it follows that S+rT will be a pan-diagonal magic square for any number r. The fact that S+nT has entries 1, 2, ...n^2 can be shown by a simple argument which demonstrates that no two elements of the array S+nT are the same. This completes the proof Theorem 1.

We shall now see how the examples given by Nārāyaṇa are particular instances of the above result. If we set $n = 4$ and choose $p(0) = 2$, $p(1) = 1$, $p(2) = 3$, $p(3) = 4$, $q(0) = 3$, $q(1) = 2$, $q(2) = 0$ and $q(3) = 1$, we see that the conditions (6a) and (6b) are satisfied. The resultant 4x4 array S as defined by (5a) will be

2	3	2	3
1	4	1	4
3	2	3	2
4	1	4	1

which is same as the *chādya* square considered by Nārāyaṇa as shown in Fig. 1. The array 4x4 array 4T as defined by (5b) is given by

15	10	0	5
0	5	15	10
15	10	0	5
0	5	15	10

which is nothing but the *chādaka* square considered by Nārāyaṇa as shown in Fig. 1, except that the order of columns is reversed. Thus the composition S+4T is nothing but the folding process of Nārāyaṇa as shown in Fig. 2 and leads to the pan-diagonal 4x4 magic square as shown in the Figure.

In the same way, if we set n = 8 and choose $p(0) = 4$, $p(1) = 3$, $p(2) = 2$, $p(3) = 1$, $p(4) = 5$, $p(5) = 6$, $p(6) = 7$, $p(7) = 8$, $q(0) = 7$, $q(1) = 6$, $q(2) = 5$ $q(3) = 4$, $q(4) = 0$, $q(5) = 1$, $q(6) = 2$, and $q(7) = 3$, we see that the conditions (6a) and (6b) are satisfied. The 8x8 array S will be the same as the *chādya* square considered by Nārāyaṇa as shown in Fig. 4. The array 8T will be the same as the chàdaka square considered by Nārāyaṇa as shown in Fig. 4, except that the order of columns is reversed. The composition S+8T is nothing but the folding process of Nārāyaṇa as shown in Fig. 5 and leads to the pan-diagonal 8x8 magic square as shown in the Figure.

Nārāyaṇa's Folding Method for the Construction of *Viṣama* Magic Squares

Nārāyaṇa has also outlined a method for constructing magic squares of odd orders by using the technique of folding two squares which are constructed suitably. His description of the procedure is as follows:

पङ्क्ती मूलगुणाख्ये स्तः प्राग्वत्साध्ये तदादिमम्।
आदिमायामूर्ध्वपङ्क्तौ मध्यमे कोष्ठके लिखेत्॥
तदधः क्रमं पङ्क्त्याङ्काञ्छिष्टाङ्कानूर्ध्वतः क्रमात्।
द्वितीयाद्यास्तु तद्वच्च द्वितीयाद्यांश्च संलिखेत्॥
छाद्यच्छादकयोः प्राग्वद्विधिःसंपुटने भवेत्।

Two sequences referred to as the *mūlapaṅkti* and the *guõapaṅkti* are to be determined as earlier. The first number should be written in the middle cell of the top row and below this the numbers of the sequence in order.

> The rest of the numbers are to be entered in order from above. The first number of the second sequence is to be written in the same way [in the middle cell of the top row]; the second etc. numbers are also to be written in the same way. The rule of combining the covered and the coverer is the same as before.

The details of this method are best illustrated by considering the following example discussed by Nārāyaṇa.

Nārāyaṇa's Example: 7x7 Magic Square with Sum 238

Here 1, 2, 3, 4, 5, 6, 7 is taken as the base sequence and 0, 1, 2, 3, 4, 5, 6 as the second sequence. The sum of the base sequence is 28. When this is reduced from the desired magic sum of 238, we get 210. Dividing this by 21, the sum of the second sequence, we get the factor 10. Hence the product sequence is 0, 10, 20, 30, 40, 50, 60. Nārāyaṇa then suggests that the elements of the base sequence may be used to fill the central column of the *chādya* square and the rest of the columns are to be filled by successive cyclic permutations of this sequence as shown in Fig. 6. The *chādaka* square is to be filled by the elements of the product sequence in a similar manner, again as shown in Fig. 6. The method of folding is displayed in Fig. 7 and leads to the desired 7x7 magic square with sum 238. As we can see from Fig.7, the magic square so obtained is not a pan-diagonal magic square.

5	6	7	1	2	3	4
6	7	1	2	3	4	5
7	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	1
3	4	5	6	7	1	2
4	5	6	7	1	2	3

40	50	60	0	10	20	30
50	60	0	10	20	30	40
60	0	10	20	30	40	50
0	10	20	30	40	50	60
10	20	30	40	50	60	0
20	30	40	50	60	0	10
30	40	50	60	0	10	20

Fig. 6. The *Chādya* and *Chādaka* Squares

We shall now show that, Nārāyaṇa's folding method is indeed a general procedure, which can be used to construct a large class of pan-diagonal *Viṣama* magic squares (magic squares of odd order).

5+30	6+20	7+10	1+0	2+60	3+50	4+40
6+40	7+30	1+20	2+10	3+0	4+60	5+50
7+50	1+40	2+30	3+20	4+10	5+0	6+60
1+60	2+50	3+40	4+30	5+20	6+10	7+0
2+0	3+60	4+50	5+40	6+30	7+20	1+10
3+10	4+0	5+60	6+50	7+40	1+30	2+20
4+20	5+10	6+0	7+60	1+50	2+40	3+30

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35	26	17	1	62	53	44
46	37	21	12	3	64	55
57	41	32	23	14	5	66
61	52	43	34	25	16	7
2	63	54	45	36	27	11
13	4	65	56	47	31	22
24	15	6	67	51	42	33

Fig. 7. Folding Process for Construction of 7x7 Magic Square with Sum 238

Theorem 2:[3] Let n be an odd number and let $p(i)$ for $1 \le i \le n$ denote an arbitrary permutation of 1, 2, ..., n and let $q(i)$ for $0 \le i \le n$-1 denote an arbitrary permutation of 0, 1, ..., n-1. We can define $p(i)$ and $q(i)$ for all integers i by assuming that

$$p(i) = p(k) \text{ and } q(i) = q(k) \text{ whenever } ia\text{”}k \pmod{n} \qquad (12)$$

Now consider the square arrays S and T defined by

$$S(i, j) = p(i+j) \qquad (13a)$$

$$T(i, j) = q(i-j) \qquad (13b)$$

Then, S and T will be magic squares whenever

$$p(n-1) = (n+1)/2 \qquad (14a)$$

$$q(0) = (n-1)/2 \qquad (14b)$$

Further, the array S+rT will be a magic square for any number r.

Proof: We first show that the square S is semi-magic. The sum of the numbers in the i-th column is given by

$$\sum_{j=0}^{n-1} p(i+j) = \sum_{j=0}^{n-1} p(j) = n(n-1)/2 \qquad (15)$$

The sum of the numbers in the j-th row is similarly found to be the same. Now, the sum of the numbers along the diagonal i=j is given by

$$\sum_{i=j=0}^{n-1} p(i+j) = \sum_{i=0}^{n-1} p(2i) = \sum_{i=0}^{n-1} p(i) = n(n-1)/2 \qquad (16)$$

where the second equality follows from the fact that n is odd (2 is co-prime to n). The sum of the numbers, along the other principal diagonal $i+j = n-1$, is given by

$$\sum_{i+j=n-1} p(i+j) = \sum_{i=0}^{n-1} p(n-1) = n(n-1)/2 \qquad (17)$$

where the second equality follows from the condition (14a). Thus we have shown that S is an nxn magic square. Similarly, by making use of the condition (14b), we can show that T is also an nxn magic square. It follows that S+*r*T will be an nxn magic square for any number *r*, thereby completing the proof of Theorem 2.

We shall now see how the example given by Nārāyaṇa is a particular instance of the above result. If we set $n = 7$ and choose $p(0) = 5$, $p(1) = 6$, $p(2) = 7$, $p(3) = 1$, $p(4) = 2$, $p(5) = 3$, $p(6) = 4$, $q(0) = 3$, $q(1) = 2$, $q(2) = 1$, $q(3) = 0$, $q(4) = 6$, $q(5) = 5$ and $q(6) = 4$, we see that the conditions (14a) and (14b) are satisfied. The resultant 7x7 array S as defined by (13a) will be

5	6	7	1	2	3	4
6	7	1	2	3	4	5
7	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	1
3	4	5	6	7	1	2
4	5	6	7	1	2	3

which is same as the *chādya* square considered by Nārāyaṇa as shown in Fig. 6. The array 7x7 array 10T as defined by (13b) is given by

30	20	10	0	60	50	40
40	30	20	10	0	60	50
50	40	30	20	10	0	60
60	50	40	30	20	10	0
0	60	50	40	30	20	10
10	0	60	50	40	30	20
20	10	0	60	50	40	30

which is nothing but the *chādaka* square considered by Nārāyaṇa as shown in Figure 6, except that the order of columns is reversed. Thus the composition S+10T is nothing but the folding process of Nārāyaṇa as shown in Fig.7 and leads to the 7x7 magic square as shown in the Figure.

As we noted earlier, the 7x7 magic square in Fig. 7 obtained by Nārāyaṇa's folding method is not pan-diagonal. We shall now show that, for the case of odd-numbers n which are not multiples of 3 (that is, numbers of the form $6m\pm1$), a simple modification of Nārāyaṇa's folding method can be used to construct pan-diagonal magic squares.

Theorem 3: Let n be an odd number not divisible by 3, and let $p(i)$ for $1 \leq i \leq n$ and $q(i)$ for $0 \leq i \leq n$-1 be as defined in Theorem 2, satisfying (12), (14a) and (14b). Then, the square arrays S and T defined by

$$S(i, j) = p(i+2j) \quad (18a)$$

$$T(i, j) = q(i-2j) \quad (18b)$$

will be nxn pan-diagonal magic squares and the same is true of the array S+rT for any number r.

Proof: We first consider the array S defined by (18a). It is easy to see that S is a semi-magic square, following the same line of argument used in Theorem 2. To show that it is pan-diagonal, we consider the sums along the diagonals $i+j = c$, for $0 \leq c \leq n$-1, and obtain

$$\sum_{i+j=c} p(i+2j) = \sum_{j=0}^{n-1} p(j+c) = n(n-1)/2 \qquad (19)$$

We now consider the sum along the diagonals i-j = c, for $0 \le c \le n$-1, and obtain

$$\sum_{i-j=c} p(i+2j) = \sum_{j=0}^{n-1} p(3j+c) = n(n-1)/2 \qquad (20)$$

where the second equality follows from the fact that 3 is co-prime to n. The fact that T is an nxn pan-diagonal magic square can be proved along the same lines. It then follows that that S+rT will be an nxn pan-diagonal magic square for any number r, thereby completing the proof of Theorem 3.

We can elucidate the above result by using it to construct a pan-diagonal 7x7 square with magic sum of 238. As before, we again choose p(0) = 5, p(1) = 6, p(2) = 7, p(3) = 1, p(4) = 2, p(5) = 3, p(6) = 4, q(0) = 3, q(1) = 2, q(2) = 1, q(3) = 0, q(4) = 6, q(5) = 5 and q(6) = 4. The 7x7 array S as defined by (18a) is displayed as the *chādya* square in Fig. 8. This differs from the *chādya* square in Figure 6 in that, though the elements of the base sequence are used to fill the central column of the square, the rest of the columns are to be filled by the second successive cyclic permutation of this sequence at each step. The 7x7 array 10T can be obtained using (18b) and the resulting array, with the order of its reversed, is displayed as the *chādaka* square in Fig. 8. Again we see that the *chādaka* square is formed from the product sequence following the same rule as in the case of the

2	4	6	1	3	5	7
3	5	7	2	4	6	1
4	6	1	3	5	7	2
5	7	2	4	6	1	3
6	1	3	5	7	2	4
7	2	4	6	1	3	5
1	3	5	7	2	4	6

10	30	50	0	20	40	60
20	40	60	10	30	50	0
30	50	0	20	40	60	10
40	60	10	30	50	0	20
50	0	20	40	60	10	30
60	10	30	50	0	20	40
0	20	40	60	10	30	50

Fig. 8. The *Chādya* and *Chādaka* Squares in Modified Nārāyaṇa Method

chādya square. The resulting 7x7 magic square S+10T is displayed in Figure 9 as the result of the process of folding of the *chādya* and *chādaka* squares. As we can easily check, the 7x7 square in Fig. 9 is indeed a pan-diagonal magic square with sum 238.

2+60	4+40	6+20	1+0	3+50	5+30	7+10
3+0	5+50	7+30	2+10	4+60	6+40	1+20
4+10	6+60	1+40	3+20	5+0	7+50	2+30
5+20	7+0	2+50	4+30	6+10	1+60	3+40
6+30	1+10	3+60	5+40	7+20	2+0	4+50
3+10	2+20	4+0	6+50	1+30	3+10	5+60
4+20	3+30	5+10	7+60	2+40	4+20	6+0

=

62	44	26	1	53	35	17
3	55	37	12	64	46	21
14	66	41	23	5	57	32
25	7	52	34	16	61	43
36	11	63	45	27	2	54
47	22	4	56	31	13	65
51	33	15	67	42	24	6

Fig. 9. Folding Process for Construction of 7x7 Pan-diagonal Magic Square with Sum 238

Notes & References

1. Nārāyaṇa seems to have been a pioneer in the development of this method which was investigated much later in the 18th century by de la Hire, Euler and other European mathematicians.

2. Both the statement and proof of Theorem 1 closely follow the treatment in J.V. Uspensky and M.A. Heaslet, *Elementary Number Theory*, McGraw-Hill, New York 1939, pp.164-66. This result can be trivially extended to the general case considered by Nārāyaṇa, where $p(i)$ and $q(i)$ are arbitrary permutations of arithmetical sequences (*mūlapaṅkti* and *guṇapaṅkti*), by suitably modifying the conditions (6a) and (6b).

3. Both the statement and proof of Theorem 2 closely follow the treatment in J.V. Uspensky and M.A. Heaslet, *Elementary Number Theory*, McGraw-Hill, New York 1939, pp.161-4. This result can be trivially extended to the general case considered by Nārāyaṇa, where $p(i)$ and $q(i)$ are arbitrary permutations of arithmetical sequences (*mūlapaṅkti* and *guṇapaṅkti*), by suitably modifying the conditions (14a) and (14b).

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