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NĀRĀYAṆA PAṆḌITA'S ENUMERATION OF COMBINATIONS AND ASSOCIATED REPRESENTATION OF NUMBERS AS SUMS OF BINOMIAL COEFFICIENTS

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In his compendious work *Bṛhatsaṃhitā*, the great astronomer Varāhamihira (c.550 AD) discusses the number of possible combinations which can be formed by choosing 4 perfumes from a class of 16 basic perfumes and describes the construction of *meru* to compute this number. Varāhamihira also indicates a method of exhibiting all the 1,820 possible combinations in the form of a *prastāra* (array). Bhaṭṭotpala, the tenth century commentator of *Bṛhatsaṃhitā*, has explained in detail the construction of this *prastāra*.

A general mathematical treatment of most of the combinatorial problems considered in the earlier literature has been given by Nārāyaṇa Paṇḍita in his work *Gaṇitakaumudī* (c.1356 AD). In particular, Nārāyaṇa Paṇḍita has considered the problem of enumeration of combinations and has given explicit rules for the construction of the *prastāra* of combinations both from "above" and "below". The questions of *naṣṭa* and *uddiṣṭa* which consist of finding the row number associated with a given combination, and conversely, in relation to this *prastāra*, naturally lead to representations of number as sums of binomial coefficients. In this paper, we shall show that there are indeed two different representations of numbers as unique sums of binomial coefficients which are naturally associated with the *prastāra* described by Nārāyaṇa Paṇḍita.

Key words: Binomial coefficients, Combinatorial representation of numbers, *Meru*, *Naṣṭa*, *Prastāra* or enumeration, *Uddiṣṭa*, *Vikalpa* or combinations.

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Introduction

The history of combinatorics in India dates back to very ancient times beginning with the work of Piṅgala on Sanskrit prosody and the work of Bharata on music. This study of combinatorics has had a continuing tradition with a view to applications in very practical problems in music, architecture, perfumery and garland making. The work *Bṛhatsaṃhitā* of the great astronomer Varāhamihira (c.550 AD) has a chapter titled *Gandhayukti* (perfumery), which deals with preparation of a large class of perfumes by combining a set of basic perfumes in various ways. In this context, Varāhamihira remarks that there are C(16,4) = 1,820 combinations[1] which can be formed by choosing 4 perfumes from a class of 16 basic perfumes. Later, he goes on to construct a *Meru* (which is a tilted version of Piïgala's *Varṇameru*, or the 'Pascal Triangle' in modern parlance) with a view to compute the number of such combinations. Varāhamihira also indicates a method of exhibiting all the 1,820 possible combinations in the form of a *prastāra* (array). Bhaṭṭotpala, the tenth century commentator of *Bṛhatsaṃhitā*, has explained in detail the construction of this *prastāra*. The work of Varāhamihira can indeed be regarded as pioneering in the study of enumeration of combinations of *r* objects chosen out of a collection of *n* objects. Incidentally it may be noted that Śārṅgadeva (c.1250 AD), in his famous treatise on music *Saṅgītaratnākara*, enumerated all the permutations of *n* objects in the form of a *prastāra*, which leads to the so-called factorial representation of integers, as has been explained in [7].

A general mathematical treatment of most of the combinatorial problems considered in the earlier literature has been given by Nārāyaṇa Paṇḍita in his work *Gaṇitakaumudī* (c.1356 AD). In particular, Nārāyaṇa Paṇḍita has considered the problem of enumeration of combinations and has given explicit rules for the construction of the *prastāra* of combinations both from "above" and "below". The questions of *naṣṭa* and *uddiṣṭa* which consist of finding the row number associated with a given combination, and conversely, in relation to this *prastāra*, naturally lead to representations of number as sums of binomial coefficients. In this paper, we shall show that there are indeed two different representations of numbers as unique sums of binomial coefficients which are naturally associated with the *prastāra* described by Nārāyaṇa Paṇḍita.

I. Combinatorics in Varāhamihira's *Bṛhatsaṃhitā* (c. 550 AD)

Varāhamihira the great astronomer wrote a compilation entitled *Bṛhatsaṃhitā* around 550 AD. The work, consisting of 106 chapters, deals mainly with various aspects of divination, but also discusses many other topics. In particular, Chapter 76 on *Gandhayukti* (perfumery) deals with the composition of a various classes of perfumes and presents some very interesting results on combinatorics as well as magic squares.

In verse 20 of the chapter given below, Varāhamihira mentions that there are 1,820 combinations which can be formed by choosing 4 perfumes from a set of 16 basic perfumes.

षोडशके द्रव्यगणे चतुर्विकल्पेन भिद्यमानानाम्।
अष्टादश जायन्ते शतानि सहितानि विंशत्या॥

In a collection of sixteen substances, the number of variations which arise when we choose four out of them is eighteen hundred and twenty.

In verse 22, Varāhamihira gives a method of construction of a *Meru* (a tabular figure) which may be used to calculate the number of combinations. This verse, given below, also very briefly indicates a way of arranging these combinations in the form of an array or a *prastāra*.

पूर्वेण पूर्वेण गतेन युक्तं स्थानं विनान्त्यं प्रवदन्ति सङ्ख्याम्।
इच्छाविकल्पैः क्रमशोऽभिनीय नीते निवृत्तिः पुनरन्यनीतिः ॥

The [number above] is to be added to the earlier and the earlier one passed by [and so on] without the last place is said to be the *saṅkhyā* [or the number of combinations (*bheda* or *vikalpa*)]. From the desired combinations [displayed in a *loṣṭaka-prastāra* (enumeration displayed by moving clay-balls or *loṣṭakas*)], by systematically moving [the last *loṣṭaka*], when the end is reached, there is a return. Again, there is a moving of the other [or the previous *loṣṭaka*].

The method of construction of the *Meru* and the method of sequentially arranging the combinations in the form of an array by means of *loṣṭaka-prastāra* (movement of clay-balls) have both been explained by Bhaṭṭotpala (c.950 AD) in his famous commentary on *Bṛhatsaṃhitā*. Bhaṭṭotpala first explains the construction of the tabular figure or Meru for obtaining the *saṅkhyā* or the desired number of combinations.

एकाद्यानङ्कान् षोडशान्तानुपर्युपरि संस्थाप्योपरितनमङ्कं प्रत्येकेन पूर्वेण पूर्वेणाङ्कगतेनाङ्केन युक्तं कार्यम्। य एव पूर्वः स गतः। यथैकस्योपरि द्वौ स्थितौ तत्र द्वयोरपेक्षया एकः पूर्वो गतश्च तेन पूर्वेण पूर्वेण गतेनोपरितनमङ्कं युक्तं कार्यं तेनापि तदुपरितनं यावत्पञ्चदशं स्थानम्, यत उक्तम् – स्थानं विनान्त्यं प्रवदन्ति सङ्ख्यामिति। अन्त्यं स्थानं विनाऽन्त्यं वर्जयित्वा सङ्ख्यां प्रवदन्ति कथयन्ति। ततो भूयोऽधः प्रभृति पुनः पूर्वेण गतेन युक्तं कार्यम्। यावच्चतुर्दशं स्थानम्। ततोऽपि पुनरधः प्रभृति पूर्वेन गतेन युक्तं कार्यम्, यावत्त्रयोदशं स्थानम्। एवं तत्राष्टादशशतानि विंशत्यधिकानि भवन्ति। उदाहरणार्थमेकादीनां षोडशान्तानां न्यासः।

अत्र पूर्वेण पूर्वेण गतेनाङ्केनैकादिनोत्तरमङ्कं द्वितीयादिकमुपरिस्थितं युक्तं कार्यम्। तेनानुप्रविष्टेनान्त्यं विनोपरि प्रथमे स्थाने षोडश जाताः। द्वितीये विंशत्यधिकं शतम्। तृतीये स्थाने पञ्चशतानि षष्ट्यधिकानि। चतुर्थे चाष्टादशशतानि विंशत्यधिकानि जातानि। एवं चतुर्विकल्पानां सङ्ख्या जाता।

Numbers starting from 1 and ending with 16 are placed one above the other [in a column] and the number above is to be added to each of the earlier ones, the numbers passed by. The earlier number is the number that is passed by; just as when 2 is placed above 1, then 1 is both the earlier number and the one passed by. With that is added the number above, [and so on] till we reach 15, because it is said "without the last place, is said to be the *saṅkhyā*". The *saṅkhyā* is obtained when we leave out the last place. [Having obtained the second column of numbers this way], then again from the bottom we have to add the number above with the earlier, the earlier one passed by. Till we reach the fourteenth place. After that, [in the third column of numbers], again from the bottom we have to add the number with the earlier one passed by, till we reach the thirteenth place. This way, we get [in the end] 1820. For illustration, we display the numbers starting from 1 ending with 16 [in the first column].

Here, from the earlier, that is the earlier number passed by, namely 1 etc., should be added the number above, 2 etc. Adding thus, without the last one, in the top of the first column we have 16; in the second [column] 120; in the third [column] 560; and in the fourth [column] 1,820. Thus is obtained the number of combinations of 4 objects (*caturvikalpa*) [selected from among 16 objects].

Thus the first column of the *Meru* is filled, from below, with numbers 1, 2,... 16. The next column also has the entry 1 at the bottom. The entry above that is 1+2 = 3; the next is 1+2+3 = 6; and so on and the fifteenth entry is 1+ 2+ ... + 15 = 120, which is the *dvivikalpa*, or the number of combinations of 2 objects selected from among 16 objects, the binomial co-efficient C(16,2). The third column starts from below with 1, and then has the entries 1+ 3 = 4, 1+3+ 6 = 10, and so on, and the fourteenth entry is 1+ 3+ 6+ ... + 120 = 560, which is the *trivikalpa*, or the number of combinations of 3 objects selected from among 16 objects, the binomial co-efficient C(16,3). The fourth column also starts from below with 1, and then has the entries 1+4 = 5, 1+ 4+ 10 = 16, and so on, and the thirteenth entry is 1+ 4+10+ ...+560 = 1820, which is *caturvikalpa*, or the number of combinations of 4 objects selected from among 16 objects, the binomial co-efficient C(16,4).

In Table 1 we give the *Meru* generated by the above procedure. Clearly, this *Meru* is a tilted version of the *Varṇa-meru*, described by Piṅgala in his *Chandaḥsūtra* (and is known in the mathematical literature as "Pascal Triangl≥). Piṅgala's rule for the construction of the *Varṇa-meru* is based on the recursion relation:

$$C(n,r) = C(n-1,r-1) + C(n-1,r) \qquad \ldots(1)$$

The basic rule given by Varāhamihira for the construction of the Meru, given in Table 1, is the following well-known property of the binomial co-efficients, which is obtained by iterating relation (1):

$$C(n,r) = C(n-1,r-1) + C(n-2,r-1) + \ldots\ldots + C(r-1,r-1) \qquad \ldots(2)$$

This relation was later explicitly stated by Nārāyaṇa Paṇḍita in his discussion of *vārasaṅkalita* (repeated summation) of the sequence of natural numbers.

After explaining the construction of the above *Meru*, Bhaṭṭotpala goes on to explain the method of *loṣṭaka-prastāra*, which makes use of the movement of clay balls (*loṣṭaka*) to generate all the desired combinations, sequentially, in the form of an array or *prastāra*. We shall briefly discuss Bhaṭṭotpala's *loṣṭaka-prastāra* in the Appendix of this paper.

II. *Prastāra* of Combinations in *Gaṇitakaumudī* (c 1356)

Nārāyaṇa Paṇḍita gives a very general mathematical treatment of combinatorics in the 13th Chapter, entitled *Aṅkapāśa* (net of numbers), of his

Table 1. Varāhamihira' *Meru* for Calculating C(16,4)

16			
15	120		
14	105	560	
13	91	455	1820
12	78	364	1365
11	66	286	1001
10	55	220	715
9	45	165	495
8	36	120	330
7	28	84	210
6	21	56	126
5	15	35	70
4	10	20	35
3	6	10	15
2	3	4	5
1	1	1	1

celebrated work *Gaṇitakaumudī*. Verses 92-93 of the chapter deal with the *prastāra* of combinations (*mūlakrama-bheda-prastāra*):

न्यस्याल्पमाद्यान् महतोऽधस्ताच्छेषं यथोपरि।
ऊने तदुत्क्रमादङ्कानेकैकोनान्समालिखेत्॥
चयपङ्क्तिर्भवेद्यावत्तावत् प्रस्तारजो विधिः।

> [Starting from] the beginning (i.e, from the left), place the [next] lesser digit below the greater one; the remaining digits [to the right] are to be placed as in the row above. If there are gaps [to the left], then place in reverse order, digits which are successively one less than the previous. This method of generating *prastāra* is to be followed till the [smallest] arithmetic sequence is obtained.

In order to illustrate the above rule of enumeration, Nārāyaṇa gives the examples of the enumeration of combinations of three digits selected from the eight digits {1,2,..., 8} and the enumeration of the combinations of six digits selected from the nine digits {1,2,...,9}. We display them in Tables 2 & 3.

In general, Nārāyaṇa's rule for generating the *prastāra* of the C(n,r) combinations of r symbols selected from among the (ordered) set of symbols

{1, 2, ... , *n*} may be stated as follows:

Table 2. Nārāyaṇa's *prastāra* of C(8,3) Combinations of 3 Digits Chosen from {1,2,..., 8}

1	**678**	15	**158**	29	**257**	43	**146**
2	**578**	16	**348**	30	**157**	44	**236**
3	**478**	17	**248**	31	**347**	45	**136**
4	**378**	18	**148**	32	**247**	46	**126**
5	**278**	19	**238**	33	**147**	47	**345**
6	**178**	20	**138**	34	**237**	48	**245**
7	**568**	21	**128**	35	**137**	49	**145**
8	**468**	22	**567**	36	**127**	50	**235**
9	**368**	23	**467**	37	**456**	51	**135**
10	**268**	24	**367**	38	**356**	52	**125**
11	**168**	25	**267**	39	**256**	53	**234**
12	**458**	26	**167**	40	**156**	54	**134**
13	**358**	27	**457**	41	**346**	55	**124**
14	**258**	28	**357**	42	**246**	56	**123**

Table 3. Nārāyaṇa's Prastàra of C(9,6) Combinations of 6 Digits Chosen from {1,2,...,9}

1	**456789**	15	**135789**	29	**124689**	43	**134679**	57	**345678**	71	**123478**
2	**356789**	16	**125789**	30	**123689**	44	**124679**	58	**245678**	72	**234568**
3	**256789**	17	**234789**	31	**234589**	45	**123679**	59	**145678**	73	**134568**
4	**156789**	18	**134789**	32	**134589**	46	**234579**	60	**235678**	74	**124568**
5	**346789**	19	**124789**	33	**124589**	47	**134579**	61	**135678**	75	**123568**
6	**246789**	20	**123789**	34	**123589**	48	**124579**	62	**125678**	76	**123468**
7	**146789**	21	**345689**	35	**123489**	49	**123579**	63	**234678**	77	**123458**
8	**236789**	22	**245689**	36	**345679**	50	**123479**	64	**134678**	78	**234567**
9	**136789**	23	**145689**	37	**245679**	51	**234569**	65	**124678**	79	**134567**
10	**126789**	24	**235689**	38	**145679**	52	**134569**	66	**123678**	80	**124567**
11	**345789**	25	**135689**	39	**235679**	53	**124569**	67	**234578**	81	**123567**
12	**245789**	26	**125689**	40	**135679**	54	**123569**	68	**134578**	82	**123467**
13	**145789**	27	**234689**	41	**125679**	55	**123469**	69	**124578**	83	**123457**
14	**235789**	28	**134689**	42	**234679**	56	**123459**	70	**123578**	84	**123456**

- The first row of the *prastāra* is given by the sequence of symbols n-r+1, n-r+2, ..., n.
- To go from any row to the next row, scan the row from the left and below the first entry i >2, such that *i-1* does not appear earlier in the row, place the symbol i-1.

- The symbols to the right of *i* are brought down to the next row and placed in the same order to the right of *i*-1.
- To the left of *i*-1, place the symbols *i*-2, *i*-3, and so on in order, till the next row also has *r* symbols.
- The process is repeated till we reach the last row of the *prastāra*, given by the sequence 1, 2, ..., *r*.

In verses 94-99, Nārāyaṇa discusses another way of generating the *prastāra* in the inverse order by moving *laḍḍukas* (sweetmeats) and also indicates that this could be used for *naṣṭa* and *uddiṣṭa*. We shall discuss the method of *laḍḍuka-cālana* in the Appendix.

III. The *Naṣṭa* and *Uddiṣṭa* Processes and the Associated Combinatorial Representation of Numbers

As we have seen, Nārāyaṇa's rule gives a systematic enumeration of the C(*n*,*r*) combinations of *r* symbols selected from the *n* symbols {1, 2, ..., *n*} in the form of a *prastāra* or an array. Associated with such a *prastāra* are two process, namely the *naṣṭa* process which involves writing down the combination occurring in the *k*-th row of the *prastāra* for any number $k <$ C(*n*,*r*); the *uddiṣṭa* process solves the converse problem of finding out the row-number corresponding to any given combination of *r* symbols selected from the *n* symbols {1,2,...,*n*}. We shall now show that, both the *naṣṭa* and *uddiṣṭa* processes, of associating the *k*-th row of the *prastāra* with a certain combination, are related to a certain decomposition of the number *k* as a sum of binomial co-efficients. In fact, depending on the way we number the rows of the *prastāra*, from the top or from the bottom, there arise two different combinatorial decompositions of each number $k <$ C(*n*, *r*), which we shall refer to as the combinatorial decomposition of the first and second kind respectively.

The Combinatorial Decomposition of the First Kind

We first consider the case where the rows are numbered from below, with the last row being taken as the zero-th row and the top row being numbered as the [C(*n*, *r*) -1]-th row. We refer to the associated decomposition of each integer as the combinatorial decomposition of the first kind. We shall illustrate this first by considering the example of the *prastāra* of the C(8,3) combinations of 3 symbols chosen from {1,2,...,8} given in Table 4, which is the same as Table 2 with the rows being numbered from below, starting from number zero. We notice that the last row of the *prastāra* ends with 3, the next three rows end with 4 and the next six rows end with 5. In fact, we can easily see that the last symbol of the *k*-th row of the *prastāra* is 3 if $k = 0$, that is $0 \le k <$C(3,3); it is 4 if C(3,3) $\le k<$ C(4, 3); and it is 5 if C(4,3) $\le k<$ C(5,3), and so on. In other words, the last symbol of the *k*-th row is n_3+1, if C(n_3,3) $\le k<$ C(n_3+1,3).

Indeed it follows more generally that the last element in the *k*-th row of the C(n, r) combinations obtained by selecting r symbols from among {1, 2,..., n} is given by the symbol n_r+1, where n_r is the number for which C(n_r, r) $\le k <$ C(n_r+1, r).

Returning to our example of the *prastāra* of the C(8,3) combinations of 3 symbols chosen from amongst {1, 2, ...,8}, if the last (namely the third) element of the *k*-th row has been determined to be n_3+1, then we can determine the next symbol to the left as follows. Clearly this symbols is less than n_3+1 and it can also be thought of as the last symbol in the [k- C(n_3,3)]-th row in the *prastāra* of the C(n_3,2) combinations obtained by choosing two symbols from among the symbols {1, 2, .., n_3}. If n_2 is such that $0 \le n_2< n_3$ and C(n_2,2) $\le k$- C(n_3,3)< C(n_2+1,2), then from what we have remarked earlier, we can see that the penultimate symbol of the *k*-th row will be n_2+1.

Having determined the third and second elements of the *k*-th row as n_3+1 and n_2+1, we can determine the first symbol in the *k*-th row by noting that it can be viewed as the symbol in the [k- C(n_3,3)- C(n_2,2)]-th row of the *prastāra* of the C(n_2,1) combinations obtained by choosing one of the symbols amongst {1, 2,..., n_2}. This is the number n_1+1, where n_1 is equal to k-C(n_3,3)-C(n_2,2). Thus, we have also arrived at the decomposition

$$k = C(n_3,3)+ C(n_2,2) + C(n_1,1) \tag{3}$$

where the successive symbols of the *k*-th row are given by n_1+1, n_2+1 and n_3+1. Since we have $n_3 \ge 2$, $n_2 \ge 1$ and $n_1 \ge 0$, we can see that the above relation is generally valid, once we adopt the convention that C(n_r, r) = 0 whenever $n_r< r$.

Table 4. *Prastāra* of the C(8,3) Combinations of 3 Digits Chosen from {1,2,..., 8} with the Rows Numbered from Below

55	**678**	41	**158**	27	**257**	13	**146**
54	**578**	40	**348**	26	**157**	12	**236**
53	**478**	39	**248**	25	**347**	11	**136**
52	**378**	38	**148**	24	**247**	10	**126**
51	**278**	37	**238**	23	**147**	9	**345**
50	**178**	36	**138**	22	**237**	8	**245**
49	**568**	35	**128**	21	**137**	7	**145**
48	**468**	34	**567**	20	**127**	6	**235**
47	**368**	33	**467**	19	**456**	5	**135**
46	**268**	32	**367**	18	**356**	4	**125**
45	**168**	31	**267**	17	**256**	3	**234**
44	**458**	30	**167**	16	**156**	2	**134**
43	**358**	29	**457**	15	**346**	1	**124**
42	**258**	28	**357**	14	**246**	0	**123**

We can illustrate the above combinatorial decomposition by considering a couple of examples. For instance, according to Table 4, the row number 8 is given by the combination 245. In other words we have $n_1 = 1$, $n_2 = 3$ and $n_3 = 4$. Now, if we compute C(4,3) + C(3,2) + C(1,1), we get 4+3+1 = 8. Again, the row-number 25 is associated with the combination 347, so that $n_1 = 2$, $n_2 = 3$ and $n_3 = 6$. Now, if we compute C(6,3) + C(3,2) + C(2,1), we get 20+3+2 = 25. Again, the row-number 35 is associated with the combination 128 so that $n_1 = 0$, $n_2 = 1$ and $n_3 = 7$. Now, if we compute C(7,3) + C(1,2) + C(0,1), we get 35+0+0 = 35.

More generally we have the following theorem which gives a unique combinatorial decomposition of every number $k <$ C(n,r), using which one can read of the entries n_1, n_2,..., n_r of the k-th row in the *prastāra* of the C(n,r) combinations obtained by selecting r symbols from amongst {1, 2, ..., n}.

Theorem: For any natural number k with $1 \leq k < C(n,r)$, we have the following decomposition (combinatorial representation of the first kind):[2]

$$k = C(n_r,r) + C(n_{r-1},r\text{-}1) + ... + C(n_j,j) \qquad (4)$$

with uniquely defined n_r, n_{r-1},..., n_j, satisfying $n\text{-}1 \geq n_r > n_{r-1} > ... > n_j \geq j \geq 1$.

Proof: We first prove the existence of the decomposition (4). Suppose, $k = C(n_r,r)$ for some n_r satisfying $n\text{-}1 \geq n_r \geq r \geq 1$, then we are through. Otherwise, choose n_r, with $n\text{-}1 \geq n_r \geq r \geq 1$, such that

$$C(n_r,r) < k < C(n_r+1,r) \qquad (5)$$

Since,

$$C(n_r+1,r) = C(n_r,r) + C(n_r,r\text{-}1) \qquad (6)$$

we have

$$k\text{-}\ C(n_r,r) < C(n_r,r\text{-}1) \qquad (7)$$

Again, the number k- $C(n_r,r)$ may be of the form $C(n_{r-1},r\text{-}1)$ for some n_{r-1} such that $n_r > n_{r-1} \geq r\text{-}1 \geq 1$, in which case we are through; or else, we choose n_{r-1} such that $C(n_{r-1},r\text{-}1) < k\text{-}\ C(n_r,r) < C(n_{r-1}+1,r\text{-}1)$ and so on. In any case, by induction, it follows that $k\text{-}C(n_r,r)$ can be written in the form

$$C(n_{r-1},r\text{-}1) + C(n_{r-2},r\text{-}2) + ... + C(n_j,j)$$

with $n_r > n_{r-1} > ... > n_j \geq j \geq 1$, and hence we arrive at the result

$$k = C(n_r,r) + C(n_{r-1},r\text{-}1) + ... + C(n_j,j)$$

To prove the uniqueness of the decomposition (4), we start with the following property of the binomial coefficients which [like (2)] can be read off from the Piṅgala's *varṇa-meru* (Pascal triangle):

$$C(n,r) = C(n\text{-}1,r) + C(n\text{-}1,r\text{-}1)$$

$$= C(n\text{-}1,r) + C(n\text{-}2,r\text{-}1) + ... + C(n\text{-}r,0) \qquad (8)$$

The above equation is actually the combinatorial decomposition of the number $C(n,r)$ -1:

$$C(n,r)\ \text{-}1 = C(n\text{-}1,r) + C(n\text{-}2,r\text{-}1) + ... + C[n\text{-}(r\text{-}1),1] \qquad (9)$$

Now suppose

$$k = C(n_r,r) + C(n_{r-1},r\text{-}1) + ... + C(n_j,j) \qquad (10a)$$

and

$$k = C(m_r,r) + C(m_{r-1},r-1) + ... + C(m_s,s) \quad (10b)$$

are two decompositions of k satisfying the conditions of the theorem. We assume that $n_i = m_i$, $t+1 \le i \le r$, and $n_t > m_t$, and show that this leads to a contradiction. Since, $m_t > m_{t-1} > m_{t-2} > ...$, we have $m_t - i \ge m_{t-i}$. Hence,

$$k \le C(m_r,r) + C(m_{r-1},r-1) + ... + C(m_{t+1},t+1) + C(m_t,t) + C(m_t-1, t-1) + ... + C[m_t-(t-1), 1]$$

Using the property (9), the above inequality reduces to

$$k \le C(m_r,r) + C(m_{r-1},r-1) + ... + C(m_{t+1},t+1) + C(m_t+1, t) -1$$

Since $n_i = m_i$, $t+1 \le i \le r$, and $n_t \ge m_t+1$, we get

$$k \le C(n_r,r) + C(n_{r-1},r-1) + ... + C(n_{t+1},t+1) + C(n_t, t)-1 < k$$

The above contradiction completes the proof of the uniqueness of the decomposition (4).

We shall now show how the processes of *naṣṭa* and *uddiṣṭa* can be easily carried out by means of the above combinatorial representation. Let us first consider the process of *naṣṭa* which seeks to determine the particular combination which is in the k-th row of the *prastāra* (counted from below as per our enumeration), without any explicit knowledge of the other rows. For this purpose, we write down the combinatorial decomposition of the row-number k as given in Theorem 1:

$$k = C(n_r,r) + C(n_{r-1},r-1) + ... + C(n_j,j)$$

with uniquely defined $n_r, n_{r-1},..., n_j$, satisfying $n-1 \ge n_r > n_{r-1} > ... > n_j \ge j \ge 1$. Then the k-th row of the *prastāra* is given by $1, 2, 3,...,j-1, n_j+1, ..., n_r+1$, when $j>1$ and by $n_j+1, ..., n_r+1$, when $j=1$. Incidentally, the top row of the *prastāra* has the row-number $C(n,r)-1$, and the associated combinatorial representation is given by the identity (9) that we derived earlier:

$$C(n,r) -1= C(n-1,r) + C(n-2,r-1) + ... + C[n-(r-1),1]$$

As an illustration of the process of *naṣṭa*, let us obtain the 28th row of the *prastāra* (given in Table 4) of the $C(8,3)$ combinations obtained by selecting 3 symbols from amongst {1, 2, ...,8}. For this purpose, we need to obtain the combinatorial decomposition of $28<C(8,3)$. We first have,

C(6,3)<28< C(7,3), which gives us n_3=6; then, 28-C(6,3)= 8 and C(4,2)<8<C(5,2), which gives us n_2 = 4; and finally 8-C(4,2) = 2 = C(2,1), which gives us n_1 = 2. Thus the combinatorial decomposition of 28 is given by

$$28 = C(6,3) + C(4,2) + C(2,1)$$

and, hence, the 28th row of the *prastāra* is given by 357.

Now for the process of *uddiṣṭa*: Suppose we are given that $p_1, p_2, ..., p_r$ is a combination among the C(n,r) combinations of r symbols selected from {1, 2, ...,n}. Then the number of the row in which this combination occurs in the *prastāra* is given by

$$C(p_r\text{-}1,r) + C(p_{r-1}\text{-}1,r\text{-}1) +... + C(p_1\text{-}1,1),$$

where it is understood that C(p,q) = 0 if $p<q$.

As an illustration we may find the row-number of the combination 127 in the *prastāra* (given in Table 4) of the C(8,3) combinations of 3 symbols selected from {1,2,...,8}. The row-number k is given by

$$k = C(7\text{-}1, 3) + C(2\text{-}1, 2) + C(1\text{-}1,3) = C(6,3) = 20.$$

Combinatorial Decomposition of the Second Kind

We shall now enumerate the rows of the *prastāra* of the C(n, r) combinations obtained by selecting r symbols from amongst {1, 2, ...n} in a different way, by counting them form top downwards, with the top row being assigned the row number 0. The bottom row will then be the [C(n, r)-1]-th row. We shall now briefly discuss another representation of each of these row-numbers as sums of binomial co-efficients, which we refer to as combinatorial representation of the second kind.

Given any natural number $0 \le k \le C(n,r)-1$, either $k \ge C(n-1,r-1)$, or $k \le C(n-1,r-1)-1$. If $k \ge C(n-1,r-1)$, then we take C(n-1,r-1) as the first term in the decomposition of k. Now,

$$k- C(n-1,r-1) \le C(n, r)-1- C(n-1,r-1) = C(n-1,r)-1.$$

Then again, we check whether $k- C(n-1,r-1) \ge C(n-1,r-2)$, and so on. In any case, we have by induction that k-C(n-1,r-1) has a combinatorial decomposition depending on n and r, thereby yielding such a representation for k itself.

On the other hand, if $k \leq C(n\text{-}1,r\text{-}1)\text{-}1$, then again check whether $k \geq C(n\text{-}2,r\text{-}2)$, and so on. In any case, we again have by induction that k has a combinatorial decomposition depending on n and r.

In this way, it can be shown that for each natural number k such that $0 \leq k \leq C(n,r)\text{-}1$, there is a combinatorial representation of the second kind

$$k = C(n_1,k_1) + C(n_2,k_2) + ... + C(n_j,k_j) \tag{11}$$

where $n > n_1 > n_2 > ... > n_j > 0$ and $k_i < n_i$, are unique, with the following properties:

(i) The first term $C(n_1,k_1)$ is either $C(n\text{-}1,r\text{-}1)$ or $C(n\text{-}2,r\text{-}2)$, or ..., or $C(n\text{-}r,0)$.

(ii) The differences $n_1\text{-}k_1$, $n_2\text{-}k_2$, ... keep decreasing by 1.

We shall illustrate the combinatorial representation of the second kind by a few examples. First we note that the decomposition of $C(n,r)\text{-}1$, the last row of the *prastāra*, is essentially the well-known relation (2) used in the construction of Varāhamihira's *Meru*:

$$C(n,r)\text{-}1 = C(n\text{-}1,r\text{--}1) + C(n\text{-}2,r\text{-}1) + ... + C(r,r\text{-}1) \tag{12}$$

As a second illustration, we shall now consider the specific case of the *prastāra* of the C(8,3) combinations of three symbols selected from among {1,2,...8} which was given in Table 2. We shall renumber the rows from the top, with top row being assigned the number 0, as displayed in Table 5. Let us first consider the combinatorial representation of the second kind of say 48≤C(8,3)-1. Since, 48≥C(7,2), the first term in the representation will be C(7,2). Now, 48-C(7,2) = 48-21=27. Again 21≥C(6,2), and the second term in the representation will be C(6,2). Now, 21-C(6,2) = 27-15 = 12. Again, 12≥C(5,2), and C(5,2) will be the next term in the decomposition. Now, 12-C(5,2) = 12-10 = 2. Now 2<C(4,2). Further, 2<C(3,1). However, 2≥C(2,0), and hence C(2,0) is the next term in the decomposition. Finally we have 2-C(2,0) = C(1,0).Thus we arrive at the desired decomposition for 48:

$$48 = C(7,2)+C(6,2)+C(5,2)+C(2,0)+C(1,0) \tag{13}$$

In Table 6 we have displayed the combinatorial representations of both the first and the second kind for all numbers $0 \leq k \leq C(8,3)\text{-}1$.

We shall finally indicate how the *naṣṭa* process can be carried out by means of the combinatorial representation of the second kind. For this purpose, let us see how the form of the combination in row 48 can be determined from the combinatorial representation given by (13). In this connection we may note that the first C(7,2) rows of the *prastāra* (as shown in Table 5) end with 8, the next C(6,2) rows end with 7 and the next C(5,2) rows end with 6. The presence of these three terms in the combinatorial representation of 48 (and the absence of C(4,2)) shows that the last symbol of the combination in row 48 is 5. The combination in the row whose number is given by C(7,2)+C(6,2)+C(5,2) (i.e., row-number 46) is 345. To find out the penultimate symbol of row 48, we need to consider the distribution of the last elements in the *prastāra* of C(4,2) combinations of two symbols selected from among {1,2,3,4}. The first C(3,1) rows in this *prastāra* end with 4 and since 48-th row is the third from 46th, we can conclude that the penultimate element of the combination in the 48th row is 4. The first symbol is determined by noting that it will be the last row in the *prastāra* of C(3,1) combinations selecting one element from among {1,2,3}. Hence we conclude that the combination in row 48 is given by 145.

Table 5. *Prastāra* of the C(8,3) Combinations of 3 Digits Chosen from {1,2,..., 8} with the Rows Numbered from Top

0	**678**	14	**158**	28	**257**	42	**146**
1	**578**	15	**348**	29	**157**	43	**236**
2	**478**	16	**248**	30	**347**	44	**136**
3	**378**	17	**148**	31	**247**	45	**126**
4	**278**	18	**238**	32	**147**	46	**345**
5	**178**	19	**138**	33	**237**	47	**245**
6	**568**	20	**128**	34	**137**	48	**145**
7	**468**	21	**567**	35	**127**	49	**235**
8	**368**	22	**467**	36	**456**	50	**135**
9	**268**	23	**367**	37	**356**	51	**125**
10	**168**	24	**267**	38	**256**	52	**234**
11	**458**	25	**167**	39	**156**	53	**134**
12	**358**	26	**457**	40	**346**	54	**124**
13	**258**	27	**357**	41	**246**	55	**123**

Table 6. Combinatorial Representation of the First and Second Kind for Numbers *k*< C(8,3)

No	Comb Rep of the First Kind	Comb Rep of the Second Kind	No	Comb Rep of the First Kind	Comb Rep of the Second Kind
0	0	0	28	C(6,3)+C4,2)+C(2,1)	C(7,2)+C(5,1)+C(3,0)+C(2,0)
1	C(3,3)	C(5,0)	29	C(6,3)+C4,2)+C(3,1)	C(7,2)+C(5,1)+C(3,0)+C(2,0)+C(1,0)
2	C(3,3)+C(2,2)	C(5,0)+C(4,0)	30	C(6,3)+C5,2)	C(7,2)+C(5,1)+C(4,1)
3	C(3,3)+C(2,2)+C(1,1)	C(5,0)+C(4,0)+C(3,0)	31	C(6,3)+C5,2)+C(1,1)	C(7,2)+C(5,1)+C(4,1)+C(2,0)
4	C(4,3)	C(5,0)+C(4,0)+C(3,0)+C(2,0)	32	C(6,3)+C5,2)+C(2,1)	C(7,2)+C(5,1)+C(4,1)+C(2,0)+C(1,0)
5	C(4,3)+C(2,2)	C(5,0)+C(4,0)+C(3,0)+C(2,0)+C(1,0)	33	C(6,3)+C5,2)+C(3,1)	C(7,2)+C(5,1)+C(4,1)+C(3,1)
6	C(4,3)+C(2,2)+C(1,1)	C(6,1)	34	C(6,3)+C5,2)+C(4,1)	C(7,2)+C(5,1)+C(4,1)+C(3,1)+C(1,0)
7	C(4,3)+C(3,2)	C(6,1)+C(4,0)	35	C(7,3)	C(7,2)+C(5,1)+C(4,1)+C(3,1)+C(2,1)
8	C(4,3)+C(3,2)+C(1,1)	C(6,1)+C(4,0)+C(3,0)	36	C(7,3)+C(2,2)	C(7,2)+C(6,2)
9	C(5,3)+C(3,2)+C(2,1)	C(6,1)+C(4,0)+C(3,0)+C(2,0)	37	C(7,3)+C(2,2)+C(1,1)	C(7,2)+C(6,2)+C(3,0)
10	C(5,3)	C(6,1)+C(4,0)+C(3,0)+C(2,0)+C(1,0)	38	C(7,3)+C(3,2)	C(7,2)+C(6,2)+C(3,0)+C(2,0)
11	C(5,3)+C(2,2)	C(6,1)+C(5,1)	39	C(7,3)+C(3,2)+C(1,1)	C(7,2)+C(6,2)+C(3,0)+C(2,0)+C(1,0)
12	C(5,3)+C(2,2)+C(1,1)	C(6,1)+C(5,1)+C(3,0)	40	C(7,3)+C(3,2)+C(2,1)	C(7,2)+C(6,2)+C(4,1)
13	C(5,3)+C(3,2)	C(6,1)+C(5,1)+C(3,0)+C(2,0)	41	C(7,3)+C(4,2)	C(7,2)+C(6,2)+C(4,1)+C(2,0)
14	C(5,3)+C(3,2)+C(1,1)	C(6,1)+C(5,1)+C(3,0)+C(2,0)+C1,0)	42	C(7,3)+C(4,2)+C(1,1)	C(7,2)+C(6,2)+C(4,1)+C(2,0)+C(1,0)
15	C(5,3)+C(3,2)+C(2,1)	C(6,1)+C(5,1)+C(4,1)	43	C(7,3)+C(4,2)+C(2,1)	C(7,2)+C(6,2)+C(4,1)+C(3,1)
16	C(5,3)+C(4,2)	C(6,1)+C(5,1)+C(4,1)+C(2,0)	44	C(7,3)+C(4,2)+C(3,1)	C(7,2)+C(6,2)+C(4,1)+C(3,1)+C(1,0)
17	C(5,3)+C(4,2)+C(1,1)	C(6,1)+C(5,1)+C(4,1)+C(2,0)+C1,0)	45	C(7,3)+C(5,2)	C(7,2)+C(6,2)+C(4,1)+C(3,1)+C(2,1)
18	C(5,3)+C(4,2)+C(2,1)	C(6,1)+C(5,1)+C(4,1)+C(3,1)	46	C(7,3)+C(5,2)+C(1,1)	C(7,2)+C(6,2)+C(5,2)
19	C(5,3)+C(4,2)+C(3,1)	C(6,1)+C(5,1)+C(4,1)+C(3,1)+C1,0)	47	C(7,3)+C(5,2)+C(2,1)	C(7,2)+C(6,2)+C(5,2)+C(2,0)
20	C(6,3)	C(6,1)+C(5,1)+C(4,1)+C(3,1)+C2,1)	48	C(7,3)+C(5,2)+C(3,1)	C(7,2)+C(6,2)+C(5,2)+C(2,0)+C(1,0)
21	C(6,3)+C(2,2)	C(7,2)	49	C(7,3)+C(5,2)+C(4,1)	C(7,2)+C(6,2)+C(5,2)+C(3,1)
22	C(6,3)+C(2,2)+C(1,1)	C(7,2)+C(4,0)	50	C(7,3)+C(6,2)	C(7,2)+C(6,2)+C(5,2)+C(3,1)+C(1,0)
23	C(6,3)+C(3,2)	C(7,2)+C(4,0)+C(3,0)	51	C(7,3)+C(6,2)+C(1,1)	C(7,2)+C(6,2)+C(5,2)+C(3,1)+C(2,1)
24	C(6,3)+C(3,2)+C(1,1)	C(7,2)+C(4,0)+C(3,0)+C(2,0)	52	C(7,3)+C(6,2)+C(2,1)	C(7,2)+C(6,2)+C(5,2)+C(4,2)
25	C(6,3)+C(3,2)+C(2,1)	C(7,2)+C(4,0)+C(3,0)+C(2,0)+C(1,0)	53	C(7,3)+C(6,2)+C(3,1)	C(7,2)+C(6,2)+C(5,2)+C(4,2)+C(1,0)
26	C(6,3)+C(4,2)	C(7,2)+C(5,1)	54	C(7,3)+C(6,2)+C(4,1)	C(7,2)+C(6,2)+C(5,2)+C(4,2)+C(2,1)
27	C(6,3)+C(4,2)+C(1,1)	C(7,2)+C(5,1)+C(3,0)	55	C(7,3)+C(6,2)+C(5,1)	C(7,2)+C(6,2)+C(5,2)+C(4,2)+C(3,2)

Appendix

The Method of *Loṣṭaka-Prastāra* or *Laḍḍukka -Cālana*

Varahamihira's *Loṣṭaka-prastāra*

In Section I of the paper we cited the cryptic verse by Varāhamihira in the *Gandhayukti* chapter of *Bṛhatsaṃhitā*, where he gives a method for constructing a *Meru* (a tabular figure) which could be used to calculate the number combinations C(16,4), which can be formed by choosing 4 perfumes from a set of 16 basic perfumes, and also indicates a way of arranging these combinations in the form of an array or *prastāra*. The method of sequentially arranging the 1,620 combinations in the form of an array by means of *loṣṭaka-prastāra* (movement of clay-balls) has been explained in detail by Bhaṭṭotpala in his commentary as follows:

अथैतेषां लोष्टकप्रस्तारार्थमाह – इच्छाविकल्पैः क्रमशोऽभिनीय नीते निवृत्तिः पुनरन्त्यनीतिः। तत्र तावद् घनादीनां द्रव्याणां षोडशानां नामाद्यक्षराणि लिखेत्। तत्राद्यानि त्रीणि स्थिरलोष्टचिह्नितानि कृत्वा चतुर्थं चरलोष्टं चिह्नितं कृत्वा तदपि क्रमेणान्यस्मिन् द्रव्ये नीत्वा तस्मिन्निवृत्तिः। पञ्चमे प्रत्यानयनम्। एवमनेन क्रमेण सर्वेषां कार्यम्। यत उक्तमिच्छाविकल्पैरिति। अभीष्टैर्विकल्पैश्चरलोष्टं क्रमशः क्रमेणाभिनीय अन्यत्र सञ्चार्य नीते निवृत्तिः कार्या। पुनरन्यनीतिः। अन्यस्मिन् स्थाने चरलोष्टकेऽन्त्यवर्णे प्राप्तस्थिरलोष्टकमन्यत्स्थानान्तरं नयेत्। तत्र तं स्थिरं कृत्वा तदग्रतः स्थितेन चरलोष्टकेन सह स्थानान्तरं नीयमानेन भेदाः प्रदर्श्याः। ततः पुनरपि स्थानान्तरस्थं चरलोष्टकमन्यत्स्थानान्तरं नीत्वा तत्र तं स्थिरं विनिधाय तदग्रतः स्थितेन चरलोष्टकेन स्थानान्तरनीयमानेन भेदाः प्रदर्श्याः। एवमन्त्ये स्थिरलोष्टके उपान्तप्राप्तेऽन्यत् स्थिरलोष्टकं स्थानान्तरं नयेत्। तत उपान्त्यान्त्यौ तदग्रस्थितौ कृत्वान्त्ये सञ्चार्यमाणे भेदाः प्राग्वत् प्रदर्श्याः। एवं सर्वेषां क्रमेण कार्यम्। यावत्सर्वं एवान्त्यस्थानमाश्रिता भवन्ति।

तद्यथा – आद्यं त्रिकं स्थिरलोष्टचिह्नितं कृत्वा चतुर्थे संचार्यमाणे त्रयोदश उत्पाद्यन्ते। ततस्तृतीयेन लोष्टकेन चतुर्थं द्रव्यं नीते चतुर्थे लोष्टके संचार्यमाणे द्वादश उत्पद्यन्ते। ततस्तृतीये पञ्चमस्थानं नीते चतुर्थे लोष्टके संचार्यमाणे एकादश उत्पद्यन्ते। ततस्तृतीये षष्ठस्थानं नीते दश। एवं सप्तमे नव। अष्टमेऽष्टौ। एवं नवमे सप्त। दशमे षट्। एकादशे पञ्च। द्वादशे चत्वारः। त्रयोदशे त्रयः। चतुर्दशे द्वौ। पञ्चदशे एकः। एवमेकनवतिः ९१। ततः प्रथमद्वितीयचतुर्थस्थानानि स्थिरलोष्टचिह्नितानि कृत्वा चतुर्थे संचार्यमाणे

द्वादश। ततस्तृतीये प्राग्वत् स्थानान्तरेषु नीयमाने चतुर्थे संचार्यमाणे एकादश दश नवाष्ट सप्त षट् पञ्च चत्वारस्त्रयो द्वावेक इत्युत्पद्यन्ते। एवमष्टसप्ततिः ७८। पूर्वैः सहैकोनसप्तत्यधिकं शतम् १६९। ततः प्रथम द्वितीयपञ्चमेषु स्थानेषु स्थिरलोष्टचिह्नितेषु चतुर्थे संचार्यमाणे एकादश दश नवाष्ट सप्त षट् पञ्च चत्वारस्त्रयो द्वावेक उत्पद्यन्ते। एवं षट्षष्टिः ६६। पूर्वैः सह शतद्वयं पञ्चत्रिंशदधिकम् २३५। ततः प्रथमद्वितीय-षष्टस्थानेषु स्थिरलोष्टचिह्नितेषु चतुर्थे संचार्यमाणे दश नवाष्ट सप्त षट् पञ्च चत्वारस्त्रयो द्वावेक उत्पद्यन्ते। एवं पञ्चपञ्चाशत् ५५। पूर्वैः सह शतद्वयं नवत्यधिकम् २९०। ततः प्रथमद्वितीयसप्तमेषु स्थानेषु स्थिरलोष्टचिह्नितेषु चतुर्थे संचार्यमाणे नवाष्ट सप्त षट् पञ्च चत्वारस्त्रयो द्वावेकश्च भवन्ति। एवं पञ्चचत्वारिंशत् ४५ । पूर्वैः सह शतत्रयं पञ्चत्रिंशदधिकम् ३३५ । ततः प्रथम-द्वितीयाष्टमेषु स्थानेषु स्थिरलोष्टचिह्नितेषु चतुर्थे संचार्यमाणे अष्ट सप्त षट् पञ्च चत्वारस्त्रयो द्वावेकश्च भवन्ति। एवं षट्त्रिंशत् ३६। पूर्वैः सह शतत्रयमेकसप्तत्यधिकम् ३७१। ततः प्रथम-द्वितीयनवमस्थानेषु स्थिरलोष्टचिह्नितेषु चतुर्थे संचार्यमाणे सप्त षट् पञ्च चत्वारस्त्रयो द्वावेकश्च भवन्ति । एवमष्टविंशतिः २८। एवमनेन क्रमेणान्त्ये स्थानान्तरं नीयमाने चतुर्थे संचार्यमाणे एकविंशतिः २१। पुनरपि पञ्चदश १५। पुनरपि दश १०। पुनः षट् ६। पुनस्त्रयः ३। पुनरेकः। एवं चतुरशितिः ८४। फुर्वैः सह चत्वारि शतानि पञपञ्चाशदधिकानि ४५५। एवमनेन प्रकारेण यथा यथा प्रथमाद्या लोष्टका द्वितीयादिषु संचरन्ति तथा तथान्यानि त्रयोदशशतानि प'ज्च-षष्ट्यधिकानि १३६५ उत्प्द्यन्ते। एकीकृतान्यष्टादशशतानि विंशत्यधिकानि भवन्ति १८२०। अस्माभिर्ग्रन्थगौरवभयात्सुप्रसिद्धत्वाच्चोद्देशमात्र एव प्रदर्शितः।

Then he states the method to [enumerate] these combinations in the form of a *loṣṭaka-prastāra*: "*icchāvikalpaiḥ*... ". There, [in the first row], the first *akṣara* of the sixteen substances (perfumes) starting with *ghana* may be written. There mark the first three substances with a fixed *loṣṭaka* and the fourth with a moving *loṣṭaka* and then move that [fourth] through other substances and then finally bring it back. [After the moving *loṣṭaka* goes through all the substances from fourth to sixteenth], it is brought back to the fifth. Thus, following this procedure, all the [*loṣṭakas*] are to be moved. That is why it is said "From the desired combinations". From the desired combinations, the moving *loṣṭaka* is sequentially moved through and then it has to be returned. "*punaranyanītiḥ*". In the next [row], after the moving *loṣṭaka* has reached the last substance, the [previous] fixed *loṣṭaka* has to be moved to the next place. Fixing it at that place, the moving *loṣṭaka*, next to it, is again moved through the [subsequent] places and various combinations may thus be exhibited. Then again, the fixed *loṣṭaka* which has already been displaced [once] is displaced further and fixed there, and the the moving *loṣṭaka* next to it is again moved through the [subsequent] places and [this way] the various combinations are to be exhibited. Thus in the end, if the fixed [third] *loṣṭaka* attains the penultimate place, then another fixed *loṣṭaka* is moved from its place. Next to it are placed the *loṣṭakas* which were at the penultimate place and the last place; again, by moving the last, the combinations have to be generated as before. In this way all have to be moved in order, till all the *loṣṭakas* reach the last [possible] places.

This is how it is done [in the given example]. When the first three places are marked with fixed *loṣṭakas* and the fourth is moved [from the fourth to the sixteenth place], thirteen [different combinations] are produced. Then, while the third *loṣṭaka* is placed at the fourth substance, and the fourth *loṣṭaka* is moved twelve are produced. Then, while the third [*loṣṭaka*] is placed at the fifth place and the fourth [*loṣṭaka*] is moved, eleven are produced. When the third [*loṣṭaka*] is fixed in the sixth place, ten. In this way [if it is fixed] at the seventh [place], nine. [If it is fixed] at eight, eight. [If it is fixed] at ninth, seven. [If it is fixed] at tenth, six. [If it is fixed] at eleventh, five. [If it is fixed] at twelfth, four. [If it is fixed] at thirteenth, three. [If it is fixed] at fourteenth, two. [If it is fixed] at fifteenth, one. Thus 91 [combinations are produced]. Then if the first, third (*)[3] and fourth places are marked with fixed *loṣṭakas* and the fourth [*loṣṭaka*] is moved, 12 [combinations are produced]. Then if the third [*loṣṭaka*] is like before fixed at successive places and the fourth [*loṣṭaka*] is moved, then 11, 10, 9, 8, 7, 6, 5, 4, 3, 2 and 1 [combinations] are produced. In this way [we get in all] 78. Combined with what we had obtained earlier, [we get in all] 169. Then if the first, fourth (*) and fifth places are marked with fixed *loṣṭakas* and the fourth [*loṣṭaka*] is moved, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2 and 1 [combinations are produced]. In this way [we get] 66. Added to the earlier ones, [we get in all] 235. Then if the first, fifth (*) and sixth places are marked with fixed *loùñakas* and the fourth [*loṣṭaka*] is moved, 10, 9, 8, 7, 6, 5, 4, 3, 2 and 1 [combinations are produced]. In this way [we get] 55. Combined with the earlier ones, [we get in all] 290. Then if the first, sixth (*) and seventh places are marked with fixed *loṣṭakas* and the fourth [*loṣṭaka*] is moved, 9, 8, 7, 6, 5, 4, 3, 2 and 1 [combinations are produced]. In this way [we get] 45. Combined with the earlier ones, [we get in all] 335. Then if the first, seventh (*) and eighth places are marked with fixed *loṣṭakas* and the fourth [*loṣṭaka*] is moved, 8, 7, 6, 5, 4, 3, 2 and 1 [combinations are produced]. In this way [we get] 36. Combined with the earlier ones, [we get in all] 371. Then if the first, eighth (*) and ninth places are marked with fixed *loṣṭakas* and the fourth [*loṣṭaka*] is moved, 7, 6, 5, 4, 3, 2 and 1 [combinations are produced]. In this way [we get] 28. In this way, following the above procedure, [if the third is moved to ninth place] and the fourth is moved to various places, 21. Then again 15; then again 10; then 6; then 3; then 1. In this way [we get] 84. Combined with the earlier ones, [we get in all] 455. In this way

> following the same procedure, when the first and the other *loṣṭakas*, move [entirely among the] the second, third, etc., places, we obtain the other 1335 [combinations]. Taking them all together, [the total number of combinations] will be 1820. In order to avoid prolixity, and since all this is well known, we have only demonstrated an example here.

The *loṣṭaka-prastāra* discussed above is dsiplayed in Table 6. In the first 13 rows, the first three *loṣṭakas* are fixed at the first three places and the fourth is moved through the 4th to 16th places. Then, in the 14th row, the fourth *loṣṭaka* is brought back to the 5th place and the third *loṣṭaka* is moved to the 4th place. In the 14th to the 25th rows, the third *loṣṭaka* stays in the 4th place and the fourth is moved through the 5th to 16th places. Then, in the 26th row, the third *loṣṭaka* is moved to the 5th place and the fourth is brought back to the 6th place. Then the fourth *loṣṭaka* is moved again, and so on, till we reach the 91st row when the third and the fourth *loṣṭakas* are in the 15th and 16th places. In the 92nd row, the third and fourth *loṣṭakas* are brought back to the 4th and 5th places the second *loṣṭaka* is moved to the third place (as we have noted above, there is an error in the published version of the commentary at this point). Then the fourth *loṣṭaka* is moved again, and so on, till we reach the 169th row, when the third and fourth *loṣṭakas* have reached the 15th and 16th places. Then in the 170th row, the third and the fourth *loṣṭakas* are brought to the 5th and 6th places, and the second *loṣṭaka* is moved to the 4th place (again, there is an error in the published version of the commentary at this point). Then the fourth *loṣṭaka* is moved again, and so on, till we reach the 455th row, where the second, third and the fourth *loṣṭakas* have reached the 14th, 15th and 16th places. Then in the 456th row, the second, third and fourth *loṣṭakas* are brought back to the 3rd, 4th and 5th places, and the first *loṣṭaka* is moved to the second place. After this, the four *loṣṭakas* move among the 2nd to 16th places only, and generate 1355 more combinations. In the 1820th row, the four *loṣṭakas* finally reach the 13th, 14th, 15th and 16th places.

In other words, Varāhamihira’s procedure for *loṣṭaka-prastāra* can be summarized as follows:

- Start with all *loṣṭakas* placed sequentially in the extreme left.
- At each stage, starting from the right, move the first *loṣṭaka* which can be moved to the right by one step. Leave the *loṣṭakas* to the left as they are. If there are *loṣṭakas* to the right, bring them back sequentially next to the *loṣṭaka* which has been moved.

- Repeat the operation till all the *loṣṭakas* get placed sequentially at the extreme right.

Varāhamihira’s procedure does provide a systematic way of enumerating all the C(n,r) combinations of selecting r objects from an ordered collection of n objects in the form of a *prastāra*. However, as we shall see below, the *prastāra* generated following Varàhamihira’s procedure is indeed very different from the one generated following the rule given (much later) by Nārāyaṇa Paṇḍita in *Gaṇitakaumudī*.

Table 7. Varāhamihira’s *Loṣṭaka-prastāra* for Generating the C(16,4) Combinations of 4 Perfumes Selected From a Collection of 16

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	
1	O	O	O	O												
2	O	O	O		O											
3	O	O	O			O										
4	O	O	O				O									
5	O	O	O					O								
6	O	O	O						O							
7	O	O	O							O						
8	O	O	O								O					
9	O	O	O									O				
10	O	O	O										O			
11	O	O	O											O		
12	O	O	O												O	
13	O	O	O													O
14	O	O		O	O											
15	O	O		O		O										
25	O	O		O												O
26	O	O			O	O										
27	O	O			O		O									
91	O	O													O	O
92	O		O	O	O											
93	O		O	O		O										
169	O		O												O	O
170	O			O	O	O										
171	O			O	O		O									
455	O													O	O	O
456		O	O	O	O											
457		O	O	O		O										
1818												O	O		O	O
1819												O		O	O	O
1820													O	O	O	O

Nārāyaṇa's Method of *laḍḍuka-cālana*

In Section II of the paper we discussed the procedure given by Nārāyaṇa Paṇḍita for enumerating the C(n,r) combinations of r digits selected from among n digits, in the form of a *prastāra*. Nārāyaṇa also presents an alternate way of generating the *prastāra* (in the reverse order or from below) by means of a sequence of movements of sweetmeats (*laḍḍuka-cālana*). Nārāyaṇa also indicates that this method can be used for carrying out the *naṣṭa* and *uddiṣṭa* processes. Nārāyaṇa's description of the method of *laḍḍuka-cālana* is as follows:

अन्तिमाङ्कमितैः कोष्ठैः कार्ये पङ्क्ती च तिर्यगे।
एकाद्येकोत्तरानङ्कानाद्यां विलिखेत्क्रमात्॥
आद्यकोष्ठेष्वधःस्थायां स्थानसम्मितलड्डुकान्।
नैरन्तर्येणविन्यस्याप्यथ लड्डुकचालनम् ॥
पुरःस्थितं पुरः कोष्ठे लड्डुकं चालयेत्पुनः।
पृष्ठस्थितं पुनर्नैरन्तर्यं यावद्भवेन्मुहुः ॥
पुरःस्थितं पुरो न्यस्य पृष्ठस्थान्सर्वलड्डुकान्।
नैरन्तर्येणादिमेषु तदग्रस्थं पुनः पुनः ॥
यावन्ति चालनानि स्युस्तैःसङ्ख्या च विवर्जिता।
लड्डुकोपरिगैरङ्कैर्नष्टभेदो भवेद्ध्रुवम् ॥
उद्दिष्टसङ्ख्याकोष्ठाङ्कैर्यत्सङ्ख्यं चलनं भवेत्।
तदूनसैकसङ्ख्यायां शेषमुद्दिष्टभिन्मितिः ॥

Two horizontal rows are to be made with the number of cells measured by the last number. Write numbers beginning with one, and increasing by one at each step, in the first row. In the row below, place as many *laḍḍukas* as the number of places in a sequence [from the left] without any gap. Now the movement of *laḍḍukas*. The *laḍḍuka* in the front is moved, then the one behind and so on till there is no gap. Then the *laḍḍuka* in the front is moved and the *laḍḍukas* behind it are all placed at the beginning so that there is no gap. Again the *laḍḍuka* in the front is moved and so on. From the *saṅkhyā* [the total number of combinations], the number of movements made is reduced. By means of the above numbers the lost combination (*naṣṭa-bheda*) [associated with a given row number] can be surely

> determined. Subtract the number of movements by the digits for the given combination from that [*saṅkhyā* or total number of combinations]. The remainder when added to one gives the *uddiṣṭa* [or the desired number of the row associated with the given combination].

The rule for movement of *laḍḍukas*, given in the above verses, is as follows:

- Start with *laḍḍukas* placed sequentially in the extreme left.
- At each stage, starting from the left, move the first *laḍḍuka* which can be moved to the right by one step. Leave the *laḍḍukas* to the right as they are.
- If there are *laḍḍukas* to the left, move them to the extreme left.

As an illustration of the process, Nārāyaṇa again considers the *prastāra* of the C(8,3) combinations obtained by selecting three digits from among the eight digits {1, 2,...,8}. The process of *laḍḍuka-cālana* generates the *prastāra*, as shown in Table 8, which is nothing but the *prastāra* displayed in Table 2, but enumerated in the reverse order or from the bottom.

Table 8. Nārāyaṇa's *Laḍḍuka-cālana* for generating the *prastāra* of C(8,3) combinations

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**
1	O	O	O					
2	O	O		O				
3	O		O	O				
4		O	O	O				
5	O	O			O			
6	O		O		O			
7		O	O		O			
8	O			O	O			
9		O		O	O			
10			O	O	O			
11	O	O				O		
12	O		O			O		
13		O	O			O		
14	O			O		O		
15		O		O		O		
16			O	O		O		
56						O	O	O

Nārāyaṇa suggests that the process of *laḍḍuka-cālana*, which generates the *prastāra* from below, can be used to simplify the *naṣṭa* process in some cases. For instance, in order to find the combination in the 40th row in the original *prastāra* (given in Table 2), we need to do C(8,3)-40 = 56-40 = 16 movements, in order to find the desired combination. From Table 8, we see that the desired combination is 346.

Finally, we may note that while the process *loṣṭaka-prastāra* of Varahamihira and the process of *laḍḍuka-cālana* of Nārāyaṇa, though of a similar nature, are formulated in terms of very different rules of movement. Consequently, they lead to very different *prastāras*. In Table 9, we display the first 14 rows of the *prastāras* of the C(8,3) combinations as generated by both the *loṣṭaka-prastàra* and the *laḍḍuka-cālana* processes.

Table 9: Prastāra of C(8,3) Combinations by *Loṣṭaka-prastāra* and *Laḍḍuka-cālana*

Loṣṭaka-prastāra		*Laḍḍuka-cālana*	
1	**123**	1	**123**
2	**124**	2	**124**
3	**125**	3	**134**
4	**126**	4	**234**
5	**127**	5	**125**
6	**128**	6	**135**
7	**134**	7	**235**
8	**135**	8	**145**
9	**136**	9	**245**
10	**137**	10	**345**
11	**138**	11	**126**
12	**145**	12	**136**
13	**146**	13	**236**
14	**147**	14	**146**

NOTES & REFERENCES

1. We employ the symbol C(n,r) to denote the binomial coefficient given by, C(n,r) = $n!/[(n-r)!r!]$ for nonnegative integers r, n such that $0 \le r \le n$ (where we take 0! = 1) and C(n,r) = 0 whenever $0 \le n < r$.
2. It may be noted that what we refer to as the combinatorial representation of the first kind is known in the literature as the Macaulay expansion. Our proof of this result follows the exposition in T. Hibi, *Algebraic Combinatorics on Convex Polytopes*, Carlslaw Publications, Glebe 1992.
3. Here, and in the subsequent places marked by a (*), the original wrongly has *dvitīya* (second) while it should actually be *tritīya* (third), *caturtha* (fourth) etc.

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