Indian Journal of History of Science, 47.4 (2012) 633–677

NĀRĀYAṆA'S TREATMENT OF VARGA-PRAKṚTI

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(Received 28 July 2011; revised 31 October 2012)

One of the most glorious chapters in the history of ancient Indian mathematics is the analysis of the hard and subtle number-theoretic problem of finding all integer solutions of the quadratic indeterminate equation $Dx^2 + 1 = y^2$ (called *varga-prakṛti*). One of the mathematicians who discussed this equation was Nārāyaṇa Paṇḍita (c. 1350 AD), whose contributions in this area tend to get overshadowed by those of his brilliant predecessors Brahmagupta, Jayadeva and Bhāskara II.

In this article, we shall try to highlight some subtle touches in the exposition of Nārāyaṇa on *varga-prakṛti* in the light of the theory of the simple continued fraction expansion of $\sqrt{D}$. In this connection, we shall recall the relevant notation and results from the theory. For the convenience of the reader, we shall also recall the history of the equation in India, state the main results and algorithms developed by Indian algebraists for solving the equation, and then discuss some noteworthy features in the treatment of Nārāyaṇa.

Key words: Nārāyaṇa Paṇḍita, Cakravāla, Simple continued fraction, Rational approximation.

1. Introduction

The main aim of the paper is to present and discuss Nārāyaṇa's work on the quadratic indeterminate equation $Dx^2 + 1 = y^2$ in the light of the theory of simple continued fractions (which we shall abbreviate

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as SCF). In a paper in IJHS ([Sig1]), Parmanand Singh has shown that the ancient Indian *cakravāla* method for solving the problem can be interpreted in the framework of the SCF expansion of $\sqrt{D}$; certain steps from the latter are to be omitted. We shall try to give a systematic and detailed mathematical exposition of Singh's observations, with emphasis on Nārāyaṇa's version of the *cakravāla*.

The *cakravāla* algorithm prescribes an inductive construction of a sequence of triples $(\alpha_n, \beta_n, \gamma_n)$, where $D{\alpha_n}^2 + \gamma_n = {\beta_n}^2$, and asserts (correctly) that, after finitely many iterations, one will arrive at a triple $(\alpha_j, \beta_j, \gamma_j)$ where $\gamma_j \in \{\pm 1, \pm 2, \pm 4\}$, whence one gets an integer solution of $Dx^2 + 1 = y^2$ by Brahmagupta's formulae (Theorem 3.2 and Corollary 3.3).

From the viewpoint of the theory of continued fractions, any *cakravāla* pair $(|\beta_n|, |\alpha_n|)$ corresponds to (p_ℓ, q_ℓ) for some $\ell (\geq n)$, where $\frac{p_\ell}{q_\ell}$ is the ℓ-th convergent in the simple continued fraction expansion of $\sqrt{D}$. Each *cakravāla* step from $(\alpha_n, \beta_n, \gamma_n)$ to $(\alpha_{n+1}, \beta_{n+1}, \gamma_{n+1})$ corresponds to a move from the ℓ-th convergent to either the $(\ell+1)$-th or the $(\ell+2)$-th convergent. We shall highlight a necessary and sufficient mathematical condition when the jump from ℓ-th to $(\ell+2)$-th convergent becomes possible (Proposition 4.13 and Corollary 6.2).

One of the noteworthy features of Nārāyaṇa's treatment of the specific example $97x^2 + 1 = y^2$ is that, at a certain stage $(n = 3)$, he is able to make a jump from the 6-th convergent to the 8-th one, whereas the earlier prescription by Jayadeva and Bhāskara II at the same stage would be a move from the 6-th to the 7-th convergent. For constructing α_{n+1}, the methods of Jayadeva and Bhāskara II involve choosing an integer x, from among the integer solutions of $\alpha_n x + \beta_n = \gamma_n y$, which minimises $|D - x^2|$; whereas Nārāyaṇa implicitly uses the principle of minimising $|\sqrt{D} - x|$ instead of $|D - x^2|$. In the light of the theory of continued fractions, one can see that it is Nārāyaṇa's approach that ensures that one takes the jump from ℓ to $\ell + 2$ (instead of $\ell + 1$) whenever there is a scope for it.

The paper has been organised as follows. Section 2 provides general information about Nārāyaṇa Paṇḍita. The first part briefly recalls the conditions in various parts of India during the 14th century, when Nārāyaṇa lived; the second part gives a brief account of the occurrence of algebraic topics in Nārāyaṇa's texts.

The mathematical discussions in the paper begin from section 3. This section makes a brief survey on ancient Indian research on the the equation $Dx^2 + 1 = y^2$ up to the time of Nārāyaṇa, with special emphasis on the results of Brahmagupta which are crucial to some of our subsequent discussions.[1]

Section 4 is a compilation of results in the theory of continued fractions which will be needed in our analysis of the methods of Nārāyaṇa and his predecessors. The first part of this section (till Theorem 4.10) recalls standard results on continued fractions without proofs. The other part (Lemma 4.11 – Proposition 4.20) presents a few technical results, with proofs, which will arise during the later discussions on the mathematical significance of the various steps in Nārāyaṇa's algorithm. This part may be omitted on first reading; each of the results could be looked at as and when they are actually referred to in the subsequent sections.

Sections 5–7 form the main sections of the paper. Section 5 presents the various versions of the *cakravāla* algorithm for finding the integer solutions of the equation $Dx^2 + 1 = y^2$. Section 6 relates the quantities occurring in the *cakravāla* algorithm with quantities arising in the standard simple continued fraction expansion of $\sqrt{D}$ that were recalled in section 4. This section brings out the mathematical subtleties of Nārāyaṇa's version of the *cakravāla*. Section 7 is a sequel to section 6. It recalls some of the steps in Nārāyaṇa's solution of the specific example $D = 97$ and highlights the special features of the example.

Section 8 gives Nārāyaṇa's applications of the above theory to finding rational approximations for $\sqrt{D}$. In view of results mentioned in chapter 4, Nārāyaṇa's approximations turn out to be the "best" possible in a precise sense. Section 9 makes concluding remarks on the achievements of Nārāyaṇa in his treatment of the equation $Dx^2+1 = y^2$. Finally an Appendix gives Nārāyaṇa's original verses (in Roman script) on the *cakravāla* algorithm.

2. On The Algebraist Nārāyaṇa Paṇḍita

The Time of Nārāyaṇa

Nārāyaṇa Paṇḍita, the son of Nṛisiṁha (or Narasiṁha), lived in the 14th century.[2] He is the author of two works: (1) *Bījagaṇitāvataṁsa* ("Ornament of Algebra"[3]) and (2) *Gaṇita Kaumudī* ("Moonlight on Mathematics"[4]). The *Gaṇita Kaumudī*, a general treatise on mathematics, was composed in the Śaka year 1278 which corresponds to 1356 AD. A cross-reference shows that the algebra treatise *Bījagaṇitāvataṁsa* was composed earlier but the exact date has not been determined ([D2]). For convenience, we assign the tentative date (c. 1350 AD). We do not know which part of India Nārāyaṇa belonged to.

During the two centuries preceding Nārāyaṇa's time, the socio-political condition, especially in north India, was not favourable for the flourish of advanced learning and research. The centres of learning at Nalanda and Vikramashila had been destroyed by Turkish invasions around the end of the 12th century. The intellectual traditions were generally on the decline. However, the second half of the 14th century witnessed a period of stability from which savants like Nārāyaṇa might have benefited. In the area under the Delhi Sultanate, the reign of the gifted but whimsical Muhammad-bin-Tughlaq (1325–51) was followed by that of Firuz Shah (1351–88), a liberal administrator and a patron of learning who is remembered for his undertaking of various works of public utility. In south India, the Vijayanagara kingdom was founded in 1336 by Harihara and Bukka, disciples of the medieval saint Vidyāraṇya. The prosperous Vijayanagara kingdom was to play a dominant role in the political and cultural life of south India for the next two hundred years.

Algebra in Nārāyaṇa's Treatises

Nārāyaṇa Paṇḍita was perhaps one of the last stalwarts in the algebra-tradition of ancient India. His special love for the subject comes out in the very name of his algebra treatise *Bījagaṇitāvataṁsa*: "Ornament of Algebra".

Like his illustrious predecessors Brahmagupta and Bhāskara II, Nārāyaṇa repeatedly emphasises the importance of algebra ([DS, p 5]). As in the *Bījagaṇita* of Bhāskara II, Part I of the *Bījagaṇitāvataṁsa* begins with an invocation to the Lord as well as to Algebra:

> I adore that Brahma, also that science of calculation with the unknown, which is the one invisible root-cause of the visible and multiple-qualitied universe, also of multitude of rules of the science of calculation with the known.

A similar mystic analogy is again used at the beginning of Part II:

> As out of Him is derived the entire universe, visible and endless, so out of algebra follows the whole of arithmetic with its endless varieties (of rules). Therefore, I always make obeisance to Śiva and also to (*avyakta-*) *gaṇita* (algebra).

In Part I (5–6), Nārāyaṇa explains the indispensability and charm of the subject in the following words:

> People ask questions whose solutions are not to be found by arithmetic; but their solutions can generally be found by algebra. Since less intelligent men do not succeed in solving questions by the rules of arithmetic, I shall speak of the lucid and easily intelligible rules of algebra.

The *Bījagaṇitāvataṁsa* has a structure similar to what we see in the *Bījagaṇita* of Bhāskara II. Part I discusses notation and the laws of signs, arithmetic of zero and infinity, operations with unknowns, arithmetic of surds, methods of indeterminate analysis (pulverisation method for the linear indeterminate equation and cyclic method for the so-called Pellian equation) and rational approximations of surds. Part II discusses solutions of equations. For more details, see [D2].

The exposition on indeterminate analysis occurs again in Nārāyaṇa's general mathematics text *Gaṇita Kaumudī* (chapters 9 and 10). There are other algebraic topics in this treatise. Chapter 11 gives rules for finding the divisors of a number; chapter 13 discusses topics in combinatorics (permutations and combinations, partitions, binomial and multinomial coefficients, etc), series and related topics; chapter 14 deals with magic squares in great detail. The Sanskrit verses, edited by Padmakara Dvivedi, was published from Varanasi in two parts in 1936 and 1942; the verses on indeterminate analysis occur in Part II ([N]). An English translation of the *Gaṇita Kaumudī* by Parmanand Singh, with Notes, has been published serially in the Gaṇita Bhāratī (1998–2002); the portion on indeterminate analysis occurs in [Sig2].

Nārāyaṇa is mentioned with great respect by later Indian mathematicians like Jñānarāja (1503), Sūryadāsa (1540), Gaṇeśa (1545) and others. Along with the *Bījagaṇita* of Bhāskara II, the *Bījagaṇitāvataṁsa* of Nārāyaṇa is referred to by them for advanced topics in algebra.

3. A Brief History of varga-prakṛti

A polynomial equation in more than one variable, whose coefficients are integers, and whose integer roots (or sometimes rational roots) are sought, is called an *indeterminate* equation. The term is suggestive of the fact that an indeterminate equation may have infinitely many solutions. Indeterminate equations are also called Diophantine equations in honour of Diophantus of Alexandria (c. 250 AD) who investigated rational roots of indeterminate equations.

There are three great landmarks in the study of indeterminate equations in ancient India: (1) the *kuṭṭaka*, a method for solving the linear indeterminate equation $ay - bx = c$, (2) the *bhāvanā*, a composition law on the roots of the quadratic indeterminate equation $Dx^2 + m = y^2$, and (3) the *cakravāla*, an algorithm for solving the quadratic indeterminate equation $Dx^2 + 1 = y^2$.

The linear indeterminate equation $ay - bx = \pm c$ (a, b, c positive integers) has infinitely many integer solutions when the greatest common divisor of a and b divides c (otherwise the equation does not have any integer solution). This fact was known to ancient Indian mathematicians and astronomers. They evolved an efficient method called *kuṭṭaka* (pulverisation) for finding the solutions and applied the method to solve problems in astronomy. The earliest description of the *kuṭṭaka* occurs in a highly condensed form in the *Āryabhaṭīya* (499 AD) of Āryabhaṭa.

Bhāskara I (c. 600 AD) gives a detailed exposition, with several examples, in his commentary on the *Āryabhaṭīya* and his own astronomy texts. The solution was discussed, with variations and refinements, by Brahmagupta (628 AD), Mahāvīra (850 AD), Govindasvāmin (c. 860 AD), Pṛthūdakasvāmin (c. 860 AD), Āryabhaṭa II (950 AD), Śrīpati (1039 AD), Bhāskara II (1150 AD), Nārāyaṇa (c. 1350 AD) and others.

Having successfully dealt with the linear indeterminate equation, Indian algebraists took up the harder problem of investigating the quadratic indeterminate equation $Dx^2 + m = y^2$, where D is a positive integer which is not a perfect square. The equation was called *varga-prakṛti* (square-natured) in ancient India. Special emphasis was laid on solving the important case $Dx^2 + 1 = y^2$.

Before discussing its history, we recall a few facts about the equation. If D is negative, or if D is a square of a positive integer, then $Dx^2 + m = y^2$ has only finitely many integer solutions. In fact, it is easy to see that $(\pm 1, 0); (0, \pm 1)$ are the only solutions of $Dx^2 + 1 = y^2$ when $D = -1$; and that $(0, \pm 1)$ are the only solutions of $Dx^2 + 1 = y^2$ when D is a negative integer less than -1 or when D is a positive integer which is a perfect square. Thus the problem becomes mathematically interesting only when D is a positive integer which is not a perfect square; and we will assume this throughout the rest of the paper. In this case the equation $Dx^2 + 1 = y^2$ has infinitely many integer solutions. The general equation $Dx^2 + m = y^2$ need not have any integral solution. For instance, $3x^2 + 2 = y^2$ does not have any integer solution. However, when it does have an integral solution, it has infinitely many.

The first major breakthrough on the *varga-prakṛti* was achieved by Brahmagupta in 628 AD through a brilliant innovation — a law of composition called *samāsabhāvanā*. In modern language and notation, his principle can be formulated as follows:

Theorem 3.1. *[Brahmagupta's Bhāvanā]*
For a fixed positive integer D, the solution space of the equation $Dx^2 + m = y^2$ admits the binary operation

$$(x_1, y_1, m_1) \odot (x_2, y_2, m_2) = (x_1 y_2 + x_2 y_1, D x_1 x_2 + y_1 y_2, m_1 m_2).$$

In other words, if (x_1, y_1, m_1) and (x_2, y_2, m_2) are solutions of $Dx^2 + m = y^2$, then so is $(x_1 y_2 + x_2 y_1, D x_1 x_2 + y_1 y_2, m_1 m_2)$.

The result occurs at the very beginning of his section on *Varga-Prakṛti* ([Br, chapter 18; verses 64–65]). It is also stated by Ācārya Jayadeva, Bhāskara II (1150), Nārāyaṇa (1350), Jñanarāja (1503) and Kamalākara (1658); a proof is described in the *Bījapallava* (1602) of Kṛṣṇa Bhaṭṭa (born c. 1548).

This composition principle of Brahmagupta is of paramount significance in modern algebra and number theory, a *theorema eximium* in the words of Euler ([W, p 204–205]). It is a statement of the multiplicativity of a "norm function", a very important concept in modern mathematics. It is also a statement on the composition of binary quadratic forms, another important and rich topic which is still an active area of research. The very idea of constructing a binary composition on an abstractly defined unknown set is the quintessence of modern "abstract algebra". The discovery of algebraic structures like Theorem 3.1, on a set of significance, is now an important theme in mathematics research. For further discussion, see [Du1, pp 81–89] and [Du2, pp 180–183].

After stating his composition law (Theorem 1), Brahmagupta immediately derives infinitely many rational solutions of $Dx^2+1=y^2$ ([Br, Ch. 18; verses 64–65]). Next he uses the composition rule to generate an infinite number of integer solutions from a given integer solution of $Dx^2+1=y^2$ ([Br, Ch. 18, verse 66]). This result was applied later by Nārāyaṇa to generate a sequence of progressively refined rational approximations of $\sqrt{D}$, as will be discussed in section 8. More generally, when an equation $Dx^2+m=y^2$ has an integral solution, Brahmagupta uses that solution and a non-trivial integral solution of $Dx^2+1=y^2$ to generate infinitely many integral solutions of $Dx^2+m=y^2$.

Brahmagupta then gives the following consequences of Theorem 1 ([Br, Ch. 18, verses 67–68]):

Theorem 3.2. *[Brahmagupta]*

(i) *If* $Dp^2+4=q^2$*, then* $(\frac{1}{2}p(q^2-1), \frac{1}{2}q(q^2-3))$ *is a solution of* $Dx^2+1=y^2$.
(ii) *If* $Dp^2-4=q^2$*, and* $r=\frac{1}{2}(q^2+3)(q^2+1)$*, then* $(pqr, (q^2+2)(r-1))$ *is a solution of* $Dx^2+1=y^2$.

Since $(\pm1)^2=1$ and $(\pm2)^2=4$, by Theorems 3.1 and 3.2, we have

Corollary 3.3. *From any positive integer solution of* $Dx^2 + m = y^2$*, where* $m \in \{-1, \pm 2, \pm 4\}$*, one can derive a positive integer solution of* $Dx^2 + 1 = y^2$ *by repeated use of Theorem 1.*

The above consequence was explicitly recorded by Śrīpati ([Sih, p 40]; [DS, p 157]), but already illustrated by Brahmagupta in his examples. Brahmagupta's methods enable one to find integer solutions in a wide variety of cases, for instance, the solutions of the difficult equations $83x^2 + 1 = y^2$ and $92x^2 + 1 = y^2$ ([Br, Ch. 18, verses 71–72]). A more elaborate description of his results is given in [Du1, pp 77–79; 90–95] and [Du2, pp 176–180].

Brahmagupta had thus given a partial solution to the problem of solving $Dx^2 + 1 = y^2$. Sometime during the 7th–11th century, Indian algebraists discovered a method for determining the complete integer solution of $Dx^2 + 1 = y^2$ by a method called *cakravāla* (cyclic method; *cakra*: wheel or disc). The *cakravāla* algorithm had been described by Jayadeva (who lived prior to 1073 AD)[5], Bhāskara II (1150) and Nārāyaṇa (c. 1350). However, the true originator is not known. We shall describe the *cakravāla* in section 5.

The composition laws of Brahmagupta play a crucial role in the working of the *cakravāla* algorithm. Jayadeva, Bhāskara II and Nārāyaṇa first state the composition laws of Brahmagupta and then describe the *cakravāla* ([Sh], [Bh], [N]). Bhāskara II illustrates the *cakravāla* with difficult numerical cases like $D = 61$ and $D = 67$. For the equation $61x^2 + 1 = y^2$, the smallest solution in positive integers is $(x, y) = (226153980,\ 1766319049)$, indicating the unexpected intricacy of the problem[6]; for $D = 67$, the smallest positive integral solution is $(x, y) = (5967,\ 48842)$ (cf. [DS, pp 166–171]).

Nārāyaṇa (c. 1350 AD) discusses the solutions of the equation $Dx^2 + 1 = y^2$, illustrates the method with the cases $D = 103$ and $D = 97$, and shows how to use the solutions to give rational approximations to $\sqrt{D}$. The minimum positive integral solution for $103x^2 + 1 = y^2$ is $(x, y) = (22419,\ 227528)$; for $97x^2 + 1 = y^2$, it is $(x, y) = (6377352,\ 62809633)$. Nārāyaṇa also mentions solutions of a few other quadratic indeterminate equations with numerical examples [DS, pp 173–181]; [Sig1, pp 65–67].

As his methods for these cases do not differ from those of his predecessors Brahmagupta and Bhāskara II, we shall confine our discussions only to Nārāyaṇa's treatment of the equation $Dx^2 + 1 = y^2$ where we see considerable originality.

In Europe, Fermat posed the problem of finding integer solutions of $Dx^2 + 1 = y^2$ in 1657. This problem was intended to inspire contemporary mathematicians to pursue research in number theory. A general method for solving Fermat's problem was discovered by Brouncker in 1657, within a few months of Fermat's challenge.[7] This solution was described by Wallis in his book on algebra published in 1685 (and again in 1693). The problem was again taken up in the next century by the two greatest figures of 18th century mathematics: L. Euler and J.L. Lagrange. They developed the theory of continued fractions, investigated the problem in the framework of this theory and established results like Theorem 4.3 (in section 4). While the initial discoveries were made by Euler, Lagrange first published formal proofs of the results during 1768–69. Due to an error of Euler, the equation $Dx^2 + 1 = y^2$ got designated as "Pell's equation".

The study of indeterminate equations, especially the study of the equation $Dx^2+1 = y^2$, played an important role in the evolution of classical algebra in ancient India as well as in modern Europe. The equation $Dx^2 + 1 = y^2$ has several applications. Integer solutions of $Dx^2 + 1 = y^2$ yield the "best" rational approximations to $\sqrt{D}$ in a precise sense (see section 8); in modern algebra and number theory, they yield units in the domain of integers of the quadratic field $\mathbb{Q}(\sqrt{D})$. The equation is also closely related to the study of binary quadratic forms. The solution of the equation is a key step in the solution of the general quadratic Diophantine equation in two variables. It also played a role in the solution of Hilbert's 10th problem on the non-existence of an algorithm for solving arbitrary Diophantine equations (completed in 1970). Efficient generation of solutions of the equation $Dx^2+1 = y^2$ is an active area of research in algorithmic number theory and computer science. A few references are mentioned in [Du2, pp 154–155]. The algorithmic efficiency of various methods are discussed in the article of H.W. Lenstra [L]. As Lenstra remarks [L, p 192]:

The last word on algorithms for solving Pell's equation has not been spoken yet.

4. Theory of Continued Fractions

In this section, we record results on continued fractions needed for our subsequent discussions. We shall refer to the book of Barnard and Child [BC] for proofs of standard results. Another nice introduction to the theory of continued fractions and the solution of the equation $Dx^2 + 1 = y^2$ is given in chapters XXV and XVII–XXVIII of the book by Hall and Knight [HK]. Silverman's recent book [Si] gives a beautiful student-friendly account of the simple continued fraction and the solution of the equation $Dx^2 + 1 = y^2$ (chapters 30, 39 and 40). An alternative approach to solving the equation, due to Dirichlet, by a masterly use of the Pigeonhole Principle of Combinatorics, is also presented in Silverman's book (chapters 31 and 32). An instructive discussion on the *cakravāla* and the Brouncker-Wallis methods for solving the equation $Dx^2 + 1 = y^2$ has been made in Section 1.9 (pp 25–36) of the book [E] by Edwards.

We first recall some notation and terminology. For any real number x, the symbol $[x]$ denotes the largest integer which is $\leq x$. For instance, $[\sqrt{97}] = 9$ and $\left[\dfrac{\sqrt{97}+7}{3}\right] = 5$.

Any expression of the form

$$F = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}},$$

where a_1 is a non-negative integer and each a_i is a positive integer for $i \geq 2$, is called a *simple continued fraction* (SCF, in short). For convenience, we shall use the notation $[a_1, a_2, a_3, \cdots]$ for the above expression. The fraction obtained by terminating the expression F at the nth stage, i.e., the fraction $[a_1, a_2, \cdots, a_n]$, is called the *nth convergent* of F. For instance, $\sqrt{97}$ can be expressed as a SCF (Theorem 4.3) and one can see

(Example 4.6) that

$$\sqrt{97} = [9, 1, 5, 1, \cdots] = 9 + \cfrac{1}{1 + \cfrac{1}{5 + \cfrac{1}{1 + \cdots}}}.$$

In this example, the first convergent is $\frac{9}{1}$, the second convergent is $9+\frac{1}{1} = \frac{10}{1}$, the third convergent is $9+\cfrac{1}{1+\cfrac{1}{5}} = 9+\frac{5}{6} = \frac{59}{6}$, the fourth convergent is $9 + \cfrac{1}{1+\cfrac{1}{5+\cfrac{1}{1}}} = \frac{69}{7}$, and so on.

With each SCF $[a_1, a_2, a_3, \cdots]$, we define, inductively, a sequence of non-negative integers as follows:

$p_0 = 1$
$p_1 = a_1$
$p_n = a_n p_{n-1} + p_{n-2}$ for $n \geq 2$,

$q_0 = 0$
$q_1 = 1$
$q_n = a_n q_{n-1} + q_{n-2}$ for $n \geq 2$.

We record two properties of the p_n and q_n [BC, p 397 (B) and (C)]:

Lemma 4.1. *The following identities hold:*

(i) $p_n q_{n-1} - p_{n-1} q_n = (-1)^n$ *for every* $n \geq 1$.
(ii) $p_n q_{n-2} - p_{n-2} q_n = (-1)^{n-1} a_n$ *for every* $n \geq 2$.

In particular, p_n and q_n are coprime. Moreover, a basic result [BC, p 389] shows that $\frac{p_n}{q_n}$ is the nth convergent of the simple continued fraction $[a_1, a_2, a_3, \cdots]$, i.e., $\frac{p_n}{q_n} = [a_1, a_2, a_3, \cdots, a_n]$. In fact, as p_n, q_n are coprime, we have:

Proposition 4.2. *p_n is the numerator and q_n the denominator of the nth convergent $[a_1, a_2, a_3, \cdots, a_n]$.*

We have seen that in the SCF expansion of $\sqrt{97}$, the first four convergents are $\frac{9}{1}, \frac{10}{1}, \frac{59}{6}, \frac{69}{7}$ respectively. By Proposition 4.2, we get that $p_1 = 9$, $q_1 = 1$; $p_2 = 10$, $q_2 = 1$; $p_3 = 59$, $q_3 = 6$; $p_4 = 69$, $q_4 = 7$.

It can be shown that any positive irrational number can be expressed uniquely as a simple continued fraction (see [BC, pp 400-401]); in particular, any irrational surd $\sqrt{D}$ has a SCF expansion. The following theorem, due to Euler and Lagrange, gives the relationship between certain convergents of the SCF expansion of $\sqrt{D}$ and the positive integral solutions of the equation $Dx^2 + 1 = y^2$ (see [BC, p 531, 535, 538] for proofs). The notation $[a_1, \overline{a_2, a_3, \cdots, a_n}]$ will denote the SCF expansion $[a_1, a_2, a_3, \cdots, a_n, a_2, a_3, \cdots, a_n, a_2, a_3, \cdots, a_n, \cdots]$.

Theorem 4.3. *Let D be a positive integer which is not a perfect square. Then:*

(i) $\sqrt{D}$ can be expressed uniquely as a simple continued fraction. In fact, there exists a positive integer c such that $\sqrt{D}$ has a simple continued fraction expansion of the form

$$\sqrt{D} = [a_1, \overline{a_2, a_3, \cdots, a_{c+1}}]$$

with the properties $a_{c+1} = 2a_1$ and $a_i = a_{c+2-i}\ \forall\ i,\ 2 \le i \le c$.

(ii) The positive integral solutions of the equation $Dx^2 + 1 = y^2$ are precisely $(x, y) = (q_\ell, p_\ell)$ where $\frac{p_\ell}{q_\ell}$ is the ℓth convergent in the SCF expansion of $\sqrt{D}$ and $\ell = kc$ if c is even while $\ell = 2kc$ if c is odd $(k = 1, 2, \cdots)$.

(iii) If (x_1, y_1) is the minimum positive integral solution of the equation $Dx^2 + 1 = y^2$, then the other solutions are precisely (x_n, y_n), where x_n and y_n are given by the relation

$$y_n + x_n\sqrt{D} = (y_1 + x_1\sqrt{D})^n.$$

Remark 4.4. Note that, once (x_1, y_1) is determined, the other solutions (x_n, y_n) of $Dx^2 + 1 = y^2$ can be determined by Brahmagupta's composition law (Theorem 3.1).

Definitions. (1) The positive integer c (in Theorem 4.3 (i)) is called the *cycle* of the SCF expansion of $\sqrt{D}$.

(2) In view of Theorem 4.3 (iii), the minimum positive integral solution of $Dx^2+1=y^2$ is called the *fundamental solution* of the equation.

Example 4.5. Consider $D=2$.

$$\sqrt{2}=1+(\sqrt{2}-1)=1+\frac{1}{\sqrt{2}+1}=1+\frac{1}{2+(\sqrt{2}-1)}=1+\frac{1}{2+\frac{1}{2+\cdots}}=[1,\overline{2}].$$

Thus, in this case, the cycle $c=1$. By Theorem 4.3, the solutions of $2x^2+1=y^2$ are given by $x=q_{2k}$, $y=p_{2k}$ $(k=1,2,\cdots)$. The second convergent $\frac{p_2}{q_2}=1+\frac{1}{2}=\frac{3}{2}$, so that the fundamental solution of $2x^2+1=y^2$ is $(x_1,y_1)=(2,3)$. Using Brahmagupta's rule Theorem 3.1, one can see that $x_2=12$, $y_2=17$ and $x_4=408$, $y_4=577$. Thus the fourth convergent $\frac{p_4}{q_4}=\frac{17}{12}$ and the eighth convergent $\frac{p_8}{q_8}=\frac{577}{408}$, which can also be verified directly.

Throughout the rest of the paper, D will denote a positive integer which is not a perfect square and $a_1,a_2,a_3,\cdots$ will denote positive integers such that $\sqrt{D}=[a_1,a_2,a_3\cdots]$. p_n and q_n are defined inductively as before.

We now define certain quantities b_n, r_n associated with the SCF expansion of $\sqrt{D}$ which will be crucial to our discussions on the *cakravāla* method. Set

$b_1 := 0$
$r_1 := 1$
$b_{n+1} := a_n r_n - b_n$ for $n \geq 1$
$r_{n+1} := \dfrac{D-{b_{n+1}}^2}{r_n}$ for $n \geq 1$.

One can then see ([BC, pp 530–531]) that, for every $n \geq 1$,

$$a_n=\left[\frac{\sqrt{D}+b_n}{r_n}\right], \text{ i.e., } a_n<\frac{\sqrt{D}+b_n}{r_n}<a_n+1,$$

and that

$$\frac{\sqrt{D}+b_n}{r_n} = a_n + \frac{\sqrt{D}-b_{n+1}}{r_n} = a_n + \frac{r_{n+1}}{\sqrt{D}+b_{n+1}} = a_n + \cfrac{1}{\cfrac{\sqrt{D}+b_{n+1}}{r_{n+1}}}.$$

Thus, $a_1 = [\sqrt{D}]$ is the integral part of $\sqrt{D} = \dfrac{\sqrt{D}+0}{1}$ (and we have $b_1 = 0,\ r_1 = 1$) while $\dfrac{\sqrt{D}-[\sqrt{D}]}{1}$ is the fractional part (which gives $b_2 = [\sqrt{D}],\ r_1 = 1$); a_2 is the integral part of the reciprocal of the above fraction $\dfrac{\sqrt{D}-b_2}{r_1}\left(=\dfrac{r_2}{\sqrt{D}+b_2}\right)$, and so on.

To get familiar with the quantities b_n, r_n, we compute a few initial terms of the SCF expansion of $\sqrt{97}$.

Example 4.6. We have

$$\sqrt{97} = \frac{\sqrt{97}+0}{1} = 9 + \frac{\sqrt{97}-9}{1} = 9 + \frac{16}{\sqrt{97}+9}.$$

$$\frac{\sqrt{97}+9}{16} = 1 + \frac{\sqrt{97}-7}{16} = 1 + \frac{3}{\sqrt{97}+7}.$$

$$\frac{\sqrt{97}+7}{3} = 5 + \frac{\sqrt{97}-8}{3} = 5 + \frac{11}{\sqrt{97}+8}.$$

$$\frac{\sqrt{97}+8}{11} = 1 + \frac{\sqrt{97}-3}{11} = 1 + \frac{8}{\sqrt{97}+3}.$$

This gives

$$b_1 = 0,\ b_2 = 9,\ b_3 = 7,\ b_4 = 8,\ b_5 = 3, \cdots$$

and

$$r_1 = 1,\ r_2 = 16,\ r_3 = 3,\ r_4 = 11,\ r_5 = 8, \cdots .$$

Thus

$$\sqrt{97} = 9 + \cfrac{1}{1 + \cfrac{1}{5 + \cfrac{1}{1 + \cdots}}}.$$

Continuing the process, it can be seen that

$$\sqrt{97} = [9, \overline{1, 5, 1, 1, 1, 1, 1, 1, 5, 1, 18}],$$

with the cycle $c = 11$.

In general, we have initial relations like

$$b_1 = 0,\ r_1 = 1,\ a_1 = [\sqrt{D}]$$

$$b_2 = a_1,\ r_2 = D - {a_1}^2,\ a_2 = \left[\frac{\sqrt{D} + a_1}{D - {a_1}^2}\right]$$

and so on. Similarly, the r_n are related to the p_n and q_n:

$$p_1 = a_1,\ q_1 = 1,\ r_2 = D - {a_1}^2 = D{q_1}^2 - {p_1}^2$$

$$p_2 = a_2 a_1 + 1,\ q_2 = a_2,$$

$$r_3 = \frac{D - {b_3}^2}{r_2} = \frac{D - (a_2 r_2 - a_1)^2}{r_2} = \frac{D - {a_1}^2 - {a_2}^2{r_2}^2 + 2a_2 a_1 r_2}{r_2}$$

$$= 1 - {a_2}^2(D - {a_1}^2) + 2a_2 a_1 = (a_2 a_1 + 1)^2 - D{a_2}^2 = {p_2}^2 - D{q_2}^2,$$

and so on.

In general, one has the following result [BC, p 534]:

Theorem 4.7. $D{q_n}^2 + (-1)^n r_{n+1} = {p_n}^2$.

As illustration, we see that in the SCF of $\sqrt{97}$, we have:

$$p_1 = 9,\ q_1 = 1,\ r_2 = 16 \text{ and } 97 \times 1^2 - 16 = 9^2.$$

$$p_2 = 10,\ q_2 = 1,\ r_3 = 3 \text{ and } 97 \times 1^2 + 3 = 10^2.$$

$$p_3 = 59,\ q_3 = 6,\ r_4 = 11 \text{ and } 97 \times 6^2 - 11 = 59^2.$$

$$p_4 = 69,\ q_4 = 7,\ r_5 = 8 \text{ and } 97 \times 7^2 + 8 = 69^2.$$

We record another observation [BC, pp 398–400].

Proposition 4.8. *The odd convergents form an increasing sequence bounded above by $\sqrt{D}$, the even convergents form a decreasing sequence bounded below by $\sqrt{D}$ (in particular, every odd convergent is less than every even convergent) and each convergent is a closer approximation than the preceding. In symbols,*

$$\frac{p_1}{q_1} < \frac{p_3}{q_3} < \frac{p_5}{q_5} \cdots < \sqrt{D} < \cdots < \frac{p_6}{q_6} < \frac{p_4}{q_4} < \frac{p_2}{q_2}$$

and

$$\left|\sqrt{D} - \frac{p_n}{q_n}\right| < \left|\sqrt{D} - \frac{p_{n-1}}{q_{n-1}}\right| \ \forall\ n \geq 2.$$

Moreover,

$$|\sqrt{D} - \frac{p_n}{q_n}| < \frac{1}{q_n q_{n+1}} \ \forall\ n \geq 1.$$

One can directly verify the inequalities for the first few convergents of the SCF expansion of $\sqrt{97}$, viz.,

$$\frac{9}{1} < \frac{59}{6} < \cdots < \sqrt{97} < \cdots < \frac{69}{7} < \frac{10}{1}$$

and

$$\cdots \frac{69}{7} - \sqrt{97} < \sqrt{97} - \frac{59}{6} < \frac{10}{1} - \sqrt{97} < \sqrt{97} - \frac{9}{1}.$$

We now mention a few useful identities [BC, p 535, relations (C) and (D)].

Lemma 4.9. *The following identities hold:*

(i) $p_{n+1} = r_{n+2}q_n + b_{n+2}q_{n+1}$.
(ii) $p_n = r_{n+1}q_{n+1} - b_{n+2}q_n$.
(iii) $Dq_n = r_{n+1}p_{n+1} - b_{n+2}p_n$.

The next result follows from basic theorems in continued fractions [BC, pp 399–401; 535]:

Theorem 4.10. *Let F be a simple continued fraction. Each convergent $\frac{p_n}{q_n}$ is a closer approximation to F than the preceding convergent $\frac{p_{n-1}}{q_{n-1}}$.*

Any convergent $\frac{p}{q}$ *of* F *is a closer approximation to* F *than any fraction* $\frac{r}{s}$ *with* $s < q$.

In particular, if $\{(x_n, y_n) \mid n \geq 1\}$ *is the sequence of positive integral solutions of* $Dx^2 + 1 = y^2$ *(arranged in increasing order, as in Theorem 4.3), then:*

(i) *The sequence gives progressively better approximations to* $\sqrt{D}$ *and converges to* $\sqrt{D}$ *in the limit.*
(ii) *Among all fractions with denominator* x_n *or less,* $\frac{y_n}{x_n}$ *is the best approximation to* $\sqrt{D}$.

We now mention a few results on the quantities p_n, q_n, r_n and b_n which will help us appreciate Nārāyaṇa's treatment of *cakravāla*. The notation $\mathbb{Z}$ denotes the set of integers.

Lemma 4.11. *Let* J_n *be the set of all integers* z *for which* $\frac{q_n z + p_n}{r_{n+1}}$ *is an integer. Then*

$$J_n = \{b_{n+2} + tr_{n+1} \mid t \in \mathbb{Z}\}.$$

Proof. Let $z = b_{n+2} + tr_{n+1}$ for some integer t and set $s := \frac{q_n z + p_n}{r_{n+1}}$. Then

$$r_{n+1}s = q_n(b_{n+2}+tr_{n+1})+p_n = (q_n b_{n+2}+p_n)+q_n t r_{n+1} = r_{n+1}(q_{n+1}+tq_n),$$

the last equality follows from Lemma 4.9 (ii). Thus, $s = q_{n+1} + tq_n$ is an integer, i.e., $z \in J_n$.

For the converse, choose $z \in J_n$ so that $s = \frac{q_n z + p_n}{r_{n+1}}$ is an integer. From this relation and Lemma 4.9 (ii), we have

$$q_n z + p_n = r_{n+1}s \text{ and } q_n b_{n+2} + p_n = r_{n+1}q_{n+1}.$$

On subtraction, we get the equality of integers

$$q_n(z - b_{n+2}) = r_{n+1}(s - q_{n+1}).$$

Since p_n and q_n are coprime, it follows from Lemma 4.9 (ii) that q_n and r_{n+1} are coprime. Hence we see that r_{n+1} divides $z - b_{n+2}$, i.e., $z = b_{n+2} + tr_{n+1}$ for some integer t. □

Remark 4.12. Note that the integers in J_n are in arithmetic progression and that $\sqrt{D}$ lies between the following two integers in J_n: b_{n+2} and $b_{n+2} + r_{n+1}$ [BC, p 531, relation (C)], i.e.,

$$b_{n+2} < \sqrt{D} < b_{n+2} + r_{n+1}.$$

Thus, among the integers z in J_n, it is either $z = b_{n+2}$ or $z = b_{n+2} + r_{n+1}$ that will minimise quantities like $|z^2 - D|$ and $|z - \sqrt{D}|$.

The next result gives necessary and sufficient conditions for $\sqrt{D}$ to be closer to $b_{n+2} + r_{n+1}$ than to b_{n+2}. The set J_n below is as defined in Lemma 4.11.

Proposition 4.13. *For $n \geq 0$, the following are equivalent:*

(i) $a_{n+2} = 1$.
(ii) *$\sqrt{D}$ is closer to $b_{n+2} + r_{n+1}$ than to b_{n+2}, i.e.,*
$b_{n+2} + \frac{r_{n+1}}{2} < \sqrt{D} < b_{n+2} + r_{n+1}$.
(iii) *Among all elements z in J_n, $|z - \sqrt{D}|$ is the least for* $z = b_{n+2} + r_{n+1}$.

Proof. Since $a_{n+2} = \left[\frac{\sqrt{D} + b_{n+2}}{r_{n+2}}\right]$, we have

$$a_{n+2} = 1 \Leftrightarrow 1 < \frac{\sqrt{D} + b_{n+2}}{r_{n+2}} < 2 \Leftrightarrow \frac{1}{2} < \frac{r_{n+2}}{\sqrt{D} + b_{n+2}} < 1.$$

Therefore, as $r_{n+1}r_{n+2} = D - {b_{n+2}}^2$, we have:

$$a_{n+2} = 1 \Leftrightarrow \frac{1}{2} < \frac{\sqrt{D} - b_{n+2}}{r_{n+1}} < 1 \Leftrightarrow b_{n+2} + \frac{r_{n+1}}{2} < \sqrt{D} < b_{n+2} + r_{n+1}.$$

This proves the equivalence (i) $\Leftrightarrow$ (ii).

Now note that

$$b_{n+2} + \frac{r_{n+1}}{2} < \sqrt{D} \Leftrightarrow b_{n+2} + r_{n+1} - \sqrt{D} < \sqrt{D} - b_{n+2}.$$

Since $b_{n+2} < \sqrt{D} < b_{n+2} + r_{n+1}$, the equivalence of (ii) and (iii) follows. □

Since $b_2 = [\sqrt{D}]$ and $r_1 = 1$, putting $n = 0$ in the above result, we have

Corollary 4.14. *The following are equivalent:*

(i) $a_2 = 1$.
(ii) $\sqrt{D}$ *is closer to* $[\sqrt{D}] + 1$ *than to* $[\sqrt{D}]$, *i.e.,* $[\sqrt{D}] + \frac{1}{2} < \sqrt{D} < [\sqrt{D}] + 1$.

In the context of the above result, we also note the following:

Lemma 4.15. *Let* $b = [\sqrt{D}]$. *Then the following are equivalent:*

(i) $\sqrt{D}$ *is closer to* b *than to* $b + 1$.
(ii) D *is closer to* b^2 *than to* $(b + 1)^2$.

Proof. Note that, since D and b are both positive integers, we have $D < b^2 + b + \frac{1}{4}$ if and only if $D < b^2 + b + \frac{1}{2}$, i.e., $D < (b + \frac{1}{2})^2$ if and only if $D < \dfrac{b^2 + (b + 1)^2}{2}$.

Now (i) holds $\Leftrightarrow \sqrt{D} < b + \frac{1}{2} \Leftrightarrow D < (b + \frac{1}{2})^2 \Leftrightarrow D < \dfrac{b^2 + (b + 1)^2}{2}$ $\Leftrightarrow$ (ii) holds. Hence the result. □

We now give a necessary condition for D to be closer to $(b_{n+2} + r_{n+1})^2$ than to ${b_{n+2}}^2$.

Lemma 4.16. *Suppose that among all elements* z *in* J_n, $|z^2 - D|$ *is the least for* $z = b_{n+2} + r_{n+1}$. *Then* $|z - \sqrt{D}|$ *is also the least for* $z = b_{n+2} + r_{n+1}$ *and* $a_{n+2} = 1$.

Proof. By hypothesis,

$(b_{n+2} + r_{n+1})^2 - D < D - {b_{n+2}}^2$, i.e., $2{b_{n+2}}^2 + 2b_{n+2}r_{n+1} + {r_{n+1}}^2 < 2D$.

Hence

$$(b_{n+2} + \frac{r_{n+1}}{2})^2 < {b_{n+2}}^2 + b_{n+2}r_{n+1} + \frac{{r_{n+1}}^2}{2} < D.$$

Thus

$$b_{n+2} + \frac{r_{n+1}}{2} < \sqrt{D}\ (< b_{n+2} + r_{n+1}).$$

Now the result follows from Proposition 4.13. □

Remark 4.17. The converse of Lemma 4.16 is not true. For $D = 97$, $a_8 = 1$ but we will see (Example 7.2) that among all elements z in J_6, $|z^2 - 97|$ is the least for $z = b_8$ $(= 5)$ and not for $z = b_8 + r_7$ $(= 14)$.

Below, we observe an identity:

Lemma 4.18. $\dfrac{(b_{n+2} + r_{n+1})^2 - D}{r_{n+1}} = \dfrac{D - (r_{n+2} - b_{n+2})^2}{r_{n+2}}.$

Proof. The result follows from the equalities

$$\frac{(b_{n+2} + r_{n+1})^2 - D}{r_{n+1}} = \frac{{r_{n+1}}^2 + 2b_{n+2}r_{n+1} - (D - {b_{n+2}}^2)}{r_{n+1}} = r_{n+1} + 2b_{n+2} - r_{n+2}$$

and

$$\frac{D - (r_{n+2} - b_{n+2})^2}{r_{n+2}} = \frac{(D - {b_{n+2}}^2) + 2b_{n+2}r_{n+2} - {r_{n+2}}^2}{r_{n+2}} = r_{n+1} + 2b_{n+2} - r_{n+2}.$$

□

Corollary 4.19. *If* $a_{n+2} = 1$*, then* $\dfrac{(b_{n+2} + r_{n+1})^2 - D}{r_{n+1}} = r_{n+3}.$

Proof. If $a_{n+2} = 1$, then $r_{n+2} - b_{n+2} = a_{n+2}r_{n+2} - b_{n+2} = b_{n+3}$. Hence, by Lemma 4.18,

$$\frac{(b_{n+2} + r_{n+1})^2 - D}{r_{n+1}} = \frac{D - {b_{n+3}}^2}{r_{n+2}} = \frac{r_{n+3}r_{n+2}}{r_{n+2}} = r_{n+3}.$$

□

The following result records two technical consequences of the condition $a_{n+2} = 1$, which will be used in the discussion on the *cakravāla*.

Proposition 4.20. *If* $a_{n+2} = 1$, *then*

(i) $q_n(b_{n+2} + r_{n+1}) + p_n = r_{n+1}q_{n+2}$.
(ii) $p_n(b_{n+2} + r_{n+1}) + Dq_n = r_{n+1}p_{n+2}$.

Proof. (i)

$$q_n(b_{n+2} + r_{n+1}) + p_n = (b_{n+2}q_n + p_n) + q_nr_{n+1} = q_{n+1}r_{n+1} + q_nr_{n+1}$$

by Lemma 4.9 (ii). Hence, as $a_{n+2} = 1$, we have

$$q_n(b_{n+2} + r_{n+1}) + p_n = r_{n+1}(q_{n+1} + q_n) = r_{n+1}(a_{n+2}q_{n+1} + q_n) = r_{n+1}q_{n+2}.$$

(ii)

$$p_n(b_{n+2} + r_{n+1}) + Dq_n = (b_{n+2}p_n + Dq_n) + p_nr_{n+1} = p_{n+1}r_{n+1} + p_nr_{n+1}$$

by Lemma 4.9 (iii). Hence, as $a_{n+2} = 1$, we have

$$p_n(b_{n+2} + r_{n+1}) + Dq_n = r_{n+1}(p_{n+1} + p_n) = r_{n+1}(a_{n+2}p_{n+1} + p_n) = r_{n+1}p_{n+2}.$$

□

5. The Cakravāla algorithm

In this section we shall describe the versions of the *cakravāla* algorithm, for solving the indeterminate equation $Dx^2 + 1 = y^2$, due to Jayadeva, Bhāskara II and Nārāyaṇa. For convenience, we shall present them using modern language and notation.

As before, D will denote a positive integer which is not a perfect square. We shall use the notation $(\alpha, \beta; m)$ for a triple satisfying $D\alpha^2 + m = \beta^2$. One way of understanding the underlying idea of the *cakravāla* algorithm can then be put as follows: from an integer triple $(\alpha_n, \beta_n; m_n)$ such that $|m_n|$ is "small", one has to construct an integer triple $(\alpha_{n+1}, \beta_{n+1}; m_{n+1})$ with $|m_{n+1}|$ "small" (not necessarily smaller than $|m_n|$) eventually arriving at an integer triple of the form $(\alpha, \beta; 1)$. At each stage, the integers α_n, β_n and m_n are mutually coprime.

The authors of *cakravāla* invariably begin with an initial triple $(\alpha_1, \beta_1; m_1)$ as follows:

$\alpha_1 := 1,$

$\beta_1 := [\sqrt{D}]$ or $[\sqrt{D}]+1$, depending on whether D is closer to $[\sqrt{D}]^2$,

or closer to $([\sqrt{D}]+1)^2$; and

$m_1 := {\beta_1}^2 - D$.

Thus,

$$D{\alpha_1}^2 + m_1 = {\beta_1}^2.$$

Note that β_1 is the unique integer β for which $|\beta^2 - D|$ is minimum. By Lemma 4.15, it may also be interpreted as the unique integer β for which $|\beta - \sqrt{D}|$ is minimum. For instance, if $D = 103$, then $\beta_1 = 10$ and $m_1 = -3$; whereas if $D = 97$, then $\beta_1 = 10$ and $m_1 = 3$.

We now explain the main inductive step of the *cakravāla* algorithm as described by Jayadeva and Bhāskara II. Suppose that one has completed n steps and arrived at an integer triple $(\alpha_n, \beta_n; m_n)$ with α_n, β_n, m_n pairwise coprime, viz., one has the identity

$$D{\alpha_n}^2 + m_n = {\beta_n}^2.$$

Then one determines an integer z_n for which

(1B) $\dfrac{\alpha_n z_n + \beta_n}{m_n}$ is an integer, and

(2B) $|{z_n}^2 - D|$ is the least among all z satisfying (1B).

(Step 2B occurs clearly in the verses of Bhāskara II. Jayadeva's verses mention ${z_n}^2 - D$ rather than $|{z_n}^2 - D|$.)

Note that the step (1B) is equivalent to finding integer solutions of the equation $m_n w - \alpha_n z = \beta_n$ (an equation in two variables z and w) for which ancient Indians had already evolved the *kuṭṭaka* algorithm.

Now, using z_n, define

$$m_{n+1} := \frac{{z_n}^2 - D}{m_n}.$$

$$\alpha_{n+1} := \frac{\alpha_n z_n + \beta_n}{m_n}.$$

$\beta_{n+1} := \alpha_{n+1}z_n - \alpha_n m_{n+1}$. (This is Jayadeva's rule. The corresponding statement of Bhāskara is given in Variation II below.)

By construction, α_{n+1} is an integer. We shall show (Proposition 5.2) that β_{n+1} and m_{n+1} are also integers. We first give an alternative expression for β_{n+1}.

Lemma 5.1. $\beta_{n+1}\ (= \alpha_{n+1}z_n - \alpha_n m_{n+1}) = \dfrac{\beta_n z_n + D\alpha_n}{m_n}$.

Proof.

$$\alpha_{n+1}z_n - \alpha_n m_{n+1} = \frac{(\alpha_n z_n + \beta_n)z_n - \alpha_n m_{n+1} m_n}{m_n}$$
$$= \frac{\beta_n z_n + \alpha_n ({z_n}^2 - m_n m_{n+1})}{m_n} = \frac{\beta_n z_n + D\alpha_n}{m_n}.$$

□

Proposition 5.2. *The following conditions hold:*

(i) β_{n+1} *and* m_{n+1} *are integers.*
(ii) $D{\alpha_{n+1}}^2 + m_{n+1} = {\beta_{n+1}}^2$.
(iii) $\alpha_{n+1}\beta_n - \beta_{n+1}\alpha_n = 1$.

Proof. We have

$$m_n\alpha_{n+1}\beta_n = (\alpha_n z_n + \beta_n)\beta_n = \alpha_n z_n \beta_n + D{\alpha_n}^2 + m_n = \alpha_n(\beta_n z_n + D\alpha_n) + m_n.$$

Since α_{n+1} is an integer, and α_n, m_n are coprime, it follows that m_n divides $\beta_n z_n + D\alpha_n$ and hence, by Lemma 5.1, β_{n+1} is an integer. Moreover,

$$m_n(\alpha_{n+1}z_n - \beta_{n+1}) = m_n\alpha_n m_{n+1} = \alpha_n m_n m_{n+1} = \alpha_n({z_n}^2 - D).$$

Since $\alpha_{n+1}, \beta_{n+1}$ are integers and α_n, m_n are coprime, it follows that m_n divides ${z_n}^2 - D$, i.e., m_{n+1} is an integer. Thus (i) holds.

Applying Brahmagupta's composition law (Theorem 3.1) on the two triples $(\alpha_n, \beta_n; m_n)$ and $(1, z_n; {z_n}^2 - D)$, we get

$$(\alpha_n, \beta_n; m_n) \odot (1, z_n; {z_n}^2 - D) = (\alpha_n z_n + \beta_n, D\alpha_n + \beta_n z_n; m_n({z_n}^2 - D)),$$

i.e., we get the identity

$$D(\alpha_n z_n + \beta_n)^2 + m_n({z_n}^2 - D) = (D\alpha_n + \beta_n z_n)^2$$

and hence the identity

$$D\left(\frac{\alpha_n z_n + \beta_n}{m_n}\right)^2 + \frac{{z_n}^2 - D}{m_n} = \left(\frac{D\alpha_n + \beta_n z_n}{m_n}\right)^2.$$

Thus, by Lemma 5.1, condition (ii) holds.

From the earlier relation

$$m_n\alpha_{n+1}\beta_n = \alpha_n(\beta_n z_n + D\alpha_n) + m_n,$$

and Lemma 5.1, we get

$$m_n\alpha_{n+1}\beta_n = m_n\alpha_n\beta_{n+1} + m_n$$

and hence condition (iii). □

Conditions (i) and (ii) show that we have obtained *an integer triple* $(\alpha_{n+1}, \beta_{n+1}; m_{n+1})$ and condition (iii) shows that α_{n+1} and β_{n+1} are coprime from which it follows that $\alpha_{n+1}, \beta_{n+1}, m_{n+1}$ are pairwise coprime.

Note that even if α_n, β_n are positive, the quantities $\alpha_{n+1}, \beta_{n+1}$ (defined as above) may be negative. However, if $(\alpha_{n+1}, \beta_{n+1}; m_{n+1})$ is an integer triple, then so is $(|\alpha_{n+1}|, |\beta_{n+1}|; m_{n+1})$ and therefore, at any stage, from any integer solution (x, y) of $Dx^2 + m_n = y^2$, one can always get a positive integral solution. Nārāyaṇa explicitly makes this observation while working out the case $D = 103$ ([N, p 237]; [Sig2, p 63]).

To get a positive integral solution of $Dx^2 + 1 = y^2$, it is enough to arrive at a stage n in which $m_n \in \{\pm 1, \pm 2, \pm 4\}$; for, in that case, one can use Brahmagupta's formulae (Theorem 3.2 and Corollary 3.3) to obtain a solution. Jayadeva and other ancient Indian authors assert that one will always arrive at such a stage.

We shall show in the next section (Theorem 6.1) that, for each n, there exists ℓ $(\geq n)$ for which $\alpha_n = \pm q_\ell$, $\beta_n = \pm p_\ell$ (with α_n and β_n having the same sign) and $m_n = (-1)^\ell r_{\ell+1}$. This correspondence between the pair $(|\alpha_n|, |\beta_n|)$ and the pair (q_ℓ, p_ℓ) will confirm the assertion that the process terminates after a finite number of steps (Remark 6.5).

We now list the major variations in different expositions regarding the conditions (1B) and (2B) for z_n and the constructions $\alpha_{n+1}, \beta_{n+1}, m_{n+1}$.

Variation I.

While Bhāskara II mentions (1B) and (2B), Nārāyaṇa is silent about condition (2B) as the reader can see from his verses quoted in our Appendix. In practice, Nārāyaṇa follows Bhāskara's steps in his first major example $D = 103$ (Example 7.4). But while solving the case $D = 97$ (Example 7.2), Nārāyaṇa tacitly chooses z_n for which

(1N) $\dfrac{\alpha_n z_n + \beta_n}{m_n}$ is an integer, and

(2N) $|z_n - \sqrt{D}|$ is the least among all z satisfying (1N).

Remark 5.3. We have the correspondence $\alpha_n = (-1)^j q_\ell$ and $\beta_n = (-1)^j p_\ell$ for some $\ell \geq n$ (Theorem 6.1). It then follows from Lemma 4.11 that the condition (1B) or (1N) is satisfied by $J_\ell = \{b_{\ell+2} + tr_{\ell+1} \mid t \in \mathbb{Z}\}$ and hence, by Remark 4.12, condition (2B) or (2N) is satisfied either by $b_{\ell+2}$ or by $b_{\ell+2} + r_{\ell+1}$.

Remark 5.4. The conditions (2B) and (2N) are not equivalent. (1) It is possible that, at some stage $n > 1$, the conditions (1B) and (2B) hold for two integers u and v satisfying $u^2 < D < v^2$ and $v^2 - D = D - u^2$, i.e., among integers z in $J_\ell = \{b_{\ell+2} + tr_{\ell+1} \mid t \in \mathbb{Z}\}$, $|z^2 - D|$ is minimum for both $u(= b_{\ell+2})$ and $v(= b_{\ell+2} + r_{\ell+1})$. Jayadeva and Bhāskara do not say which integer is to be chosen in such a case; such a situation does not occur in the examples they have given. As will be seen in section 7, Nārāyaṇa's example $D = 97$ illustrates such a situation: among integers z of the form $5 + 8t, t \in \mathbb{Z}$, $|z^2 - 97|$ is minimum for both $u = 5$ and $v = 13$. However, in such a case, we automatically have $u < \sqrt{D} < v$ with $v - \sqrt{D} < \sqrt{D} - u$, i.e., $|z - \sqrt{D}|$ is minimum for $z = v$, so that, v would be the unambiguous choice by version (2N). In fact, a tie can never occur if (2N) is adopted instead of (2B), as $\sqrt{D}$ is irrational and hence cannot be the arithmetic mean of two integers.

(2) More significantly, it is possible that, at some stage $n > 1$, one gets two *distinct* integers u and v satisfying (1B) or (1N) such that u uniquely satisfies (2B) while v uniquely satisfies (2N). For, it is possible that D is closer to u^2 than to v^2 but $\sqrt{D}$ is closer to v than to u. For instance, 97 is closer to $25(= 5^2)$ than to $196(= 14^2)$ but $\sqrt{97}$ is closer to 14 than to 5; and, in Nārāyaṇa's example $D = 97$ (Example 7.2), there comes a stage ($n = 3$) when the condition (1B) or (1N) is satisfied by integers of the form $5 + 9t, t \in \mathbb{Z}$, so that $u = 5$ satisfies (2B) but

$v = 14$ satisfies (2N). Again, when $D = 67$, a stage comes (see [DS, p 167]) when the condition (1B) or (1N) gets fulfilled by integers of the form $5 + 6t, t \in \mathbb{Z}$. Now $u = 5$ satisfies (2B) as 67 is closer to $25 (= 5^2)$ than to $121 (= 11^2)$, but $v = 11$ satisfies (2N) as $\sqrt{67}$ is closer to 11 than to 5.

A mathematical significance of condition (2N) will be discussed in the next section.

Variation II.
Bhāskara II does not explicitly define β_{n+1} as $\alpha_{n+1}z_n - \alpha_n m_{n+1}$. After defining α_{n+1}, Bhāskara II says: "thence the greater root". This could mean that β_{n+1} can now be determined from the relation $D{\alpha_{n+1}}^2 + m_{n+1} = {\beta_{n+1}}^2$. Or it could mean, as interpreted by the commentator Kṛṣṇa (c. 1548), that β_{n+1} could be determined from the relation $\beta_{n+1} = \dfrac{D\alpha_n + \beta_n z_n}{m_n}$. For, as shown in the proof of Proposition 5.2, both the expressions $\dfrac{\alpha_n z_n + \beta_n}{m_n}$ and $\dfrac{D\alpha_n + \beta_n z_n}{m_n}$ arise naturally by applying Brahmagupta's composition law on the given triple $(\alpha_n, \beta_n; m_n)$ and the natural triple $(1, z_n; {z_n}^2 - D)$. Lemma 5.1 gives the equality of the two expressions for β_{n+1} given by Jayadeva and Kṛṣṇa respectively.

Remark 5.5. Among the three expressions for β_{n+1}, Kṛṣṇa's formula $\beta_{n+1} = \dfrac{\beta_n z_n + D\alpha_n}{m_n}$ involves computations with smaller numbers than the formula $\beta_{n+1} = \sqrt{D{\alpha_{n+1}}^2 + m_{n+1}}$; but Jayadeva's formula $\beta_{n+1} = \alpha_{n+1}z_n - \alpha_n m_{n+1}$ is still simpler, computationally. Nārāyaṇa prescribes Jayadeva's expression.

As a quick illustration of the *cakravāla*, we recall below the famous example of Bhāskara II.

Example 5.6. *[Bhāskara II] Solve, in positive integers,* $61x^2 + 1 = y^2$.
As 64 is the perfect square nearest to 61, we have the initial triple $(1, 8; 3)$. Now one finds z satisfying (1B) and (2B), i.e., a z for which $|z^2 - 61|$ is minimised subject to the condition that $\dfrac{1 \times z + 8}{3}$ is an integer. Clearly

$z = 7$. Now

$$\alpha_2 = \frac{1 \times 7 + 8}{3} = 5;\ \beta_2 = \frac{1 \times 61 + 8 \times 7}{3} = 39;\ m_2 = \frac{7^2 - 61}{3} = -4.$$

Thus we have the second triple $(5, 39; -4)$. Applying Brahmagupta's formula (Theorem 3.2) on the triple $(5, 39; -4)$, one gets the fundamental solution $(226153980, 1766319049)$ of $61x^2 + 1 = y^2$.

In the next section, we shall discuss the conditions (2B) and (2N) in the framework of SCF. For a few other perspectives on the *cakravāla*, see section 7 of [Du2].

6. The conditions (2B) and (2N): SCF interpretation

We now relate α_n, β_n, m_n and z_n occurring in the *cakravāla* described in section 5 with the SCF quantities p_n, q_n, r_n, b_n defined in section 4. We show that:

Theorem 6.1. *For each $n \geq 1$, there exists ℓ ($\geq n$) such that*

$$|\alpha_n| = q_\ell,\ |\beta_n| = p_\ell \text{ and } m_n = (-1)^\ell r_{\ell+1},$$

α_n and β_n having the same sign, and z_n is either $b_{\ell+2}$ or $b_{\ell+2} + r_{\ell+1}$. Moreover, if the nth cakravāla step corresponds to the ℓ-th step of the SCF expansion of $\sqrt{D}$, i.e., if $|\alpha_n| = q_\ell$ and $|\beta_n| = p_\ell$, then the $(n+1)$-th cakravāla step corresponds to either the $(\ell + 1)$-th or the $(\ell + 2)$-th SCF step.

Proof. We first consider $n = 1$.

Case I. $\sqrt{D}$ is closer to $[\sqrt{D}]$ than to $[\sqrt{D}]$ +1, i.e.,

$$[\sqrt{D}] < \sqrt{D} < [\sqrt{D}] + \frac{1}{2},$$

equivalently (by Lemma 4.15), D is closer to $[\sqrt{D}]^2$ than to $([\sqrt{D}]+1)^2$. In this case,

$$\begin{aligned}\alpha_1 &= 1 = q_1,\\ \beta_1 &= [\sqrt{D}] = a_1 = p_1,\\ m_1 &= {\beta_1}^2 - D = -r_2.\end{aligned}$$

Case II. $\sqrt{D}$ is closer to $[\sqrt{D}]+1$ than to $[\sqrt{D}]$, i.e.,

$$[\sqrt{D}] + \frac{1}{2} < \sqrt{D} < [\sqrt{D}] + 1,$$

equivalently (by Lemma 4.15), D is closer to $([\sqrt{D}]+1)^2$ than to $[\sqrt{D}]^2$. Note that, by Corollary 4.14, this case occurs if and only if $a_2 = 1$. Hence, in this case,

$$\begin{aligned}\alpha_1 &= 1 = a_2 = q_2,\\ \beta_1 &= [\sqrt{D}] + 1 = a_1 + 1 = a_1 a_2 + 1 = p_2,\\ m_1 &= {\beta_1}^2 - D = {p_2}^2 - D{q_2}^2 = r_3.\end{aligned}$$

Before proving the general case, we give examples (from Nārāyaṇa himself) to illustrate the distinction between Cases I and II from the viewpoint of SCF. For $D = 103$, we have

$$\sqrt{103} = 10 + \frac{3}{\sqrt{103}+10} = 10 + \frac{1}{6+\cdots}.$$

In this case, $a_2 = 6 > 1$, so there is no scope for skipping the first SCF step $\ell = 1$. We have to settle for $\alpha_1(=1) = q_1$ and $\beta_1(=10) = p_1$. By contrast, for $D = 97$, we have

$$\sqrt{97} = 9 + \frac{16}{\sqrt{97}+9} = 9 + \cfrac{1}{1 + \cfrac{3}{\sqrt{97}+7}}.$$

In this example, $a_2 = 1$, so that the *cakravāla* can begin from the stage $\ell = 2$. Here $\frac{p_2}{q_2} = 9 + \frac{1}{1} = \frac{10}{1}$ and we can start with $\alpha_1(=1) = q_2$ and $\beta_1(=10) = p_2$.

We now prove the result for $n > 1$ by induction. Suppose that

$$\alpha_n = (-1)^j q_\ell,\ \beta_n = (-1)^j p_\ell \text{ and } m_n = (-1)^\ell r_{\ell+1}.$$

By Remark 4.12, the value of z_n has to be either $b_{\ell+2}$ or $b_{\ell+2} + r_{\ell+1}$, no matter whether we adopt the rule (2B) or the rule (2N). We show that, depending on the value of z_n that we choose, either

$$\alpha_{n+1} = (-1)^{j+\ell} q_{\ell+1},\ \beta_{n+1} = (-1)^{j+\ell} p_{\ell+1} \text{ and } m_{n+1} = (-1)^{\ell+1} r_{\ell+2}$$

or

$$\alpha_{n+1} = (-1)^{j+\ell} q_{\ell+2},\ \beta_{n+1} = (-1)^{j+\ell} p_{\ell+2} \text{ and } m_{n+1} = (-1)^{\ell+2} r_{\ell+3}.$$

Case A. $z_n = b_{\ell+2}$.
In this case,

$$m_{n+1} = \frac{{z_n}^2 - D}{m_n} = \frac{{b_{\ell+2}}^2 - D}{(-1)^\ell r_{\ell+1}} = (-1)^{\ell+1} r_{\ell+2};$$

$$\alpha_{n+1} = \frac{\alpha_n z_n + \beta_n}{m_n} = \frac{(-1)^j (q_\ell b_{\ell+2} + p_\ell)}{(-1)^\ell r_{\ell+1}} = (-1)^{j+\ell} q_{\ell+1}$$

by Lemma 4.9 (ii); and

$$\begin{aligned}\beta_{n+1} = \alpha_{n+1} z_n - \alpha_n m_{n+1} &= (-1)^{j+\ell} q_{\ell+1} b_{\ell+2} - (-1)^j (-1)^{\ell+1} r_{\ell+2} q_\ell \\ &= (-1)^{j+\ell} (r_{\ell+2} q_\ell + b_{\ell+2} q_{\ell+1}) = (-1)^{j+\ell} p_{\ell+1}\end{aligned}$$

by Lemma 4.9 (i).

Thus, in Case A, the *cakravāla* step $n \to n+1$ corresponds to a SCF step from ℓ to $\ell + 1$.

Case B. $z_n = b_{\ell+2} + r_{\ell+1}$.
By Proposition 4.13 (for rule (2N)) or Lemma 4.16 (for rule (2B)), this case will occur only when $a_{\ell+2} = 1$. Hence, substituting $a_{\ell+2} = 1$, we have

$$m_{n+1} = \frac{{z_n}^2 - D}{m_n} = \frac{(b_{\ell+2} + r_{\ell+1})^2 - D}{(-1)^\ell r_{\ell+1}} = (-1)^\ell r_{\ell+3}$$

by Corollary 4.19;

$$\alpha_{n+1} = \frac{\alpha_n z_n + \beta_n}{m_n} = \frac{(-1)^j \{q_\ell (b_{\ell+2} + r_{\ell+1}) + p_\ell\}}{(-1)^\ell r_{\ell+1}} = (-1)^{j+\ell} q_{\ell+2}$$

by Proposition 4.20 (i); and, by Lemma 5.1,

$$\beta_{n+1} = \frac{\beta_n z_n + D\alpha_n}{m_n} = \frac{(-1)^j p_\ell (b_{\ell+2} + r_{\ell+1}) + (-1)^j D q_\ell}{(-1)^\ell r_{\ell+1}} = (-1)^{j+\ell} p_{\ell+2}$$

by Proposition 4.20 (ii).

Thus, in Case B, the *cakravāla* step $n \to n+1$ corresponds to a SCF step from ℓ to $\ell+2$. □

Suppose a stage n has been reached in the *cakravāla* (any version (2B) or (2N)) and it corresponds to stage ℓ in the SCF expansion. In view of Proposition 4.13 and the above theorem, we now have a precise necessary and sufficient condition when any *cakravāla* step from the stage n to $n+1$ could correspond to a jump from stage ℓ to $\ell+2$ of the SCF expansion (skipping $\ell+1$).

Corollary 6.2. *With notation as in Theorem 6.1, the following conditions are equivalent:*

(i) $a_{\ell+2} = 1$.
(ii) *Among all elements z satisfying (1B) or (1N), $|z - \sqrt{D}|$ is the least for $z = b_{\ell+2} + r_{\ell+1}$.*
(iii) *A leap from ℓ to $\ell+2$ is possible (in a suitable version of the cakravāla).*

Remark 6.3. This remark is a continuation of Remark 5.4. Corollary 6.1 shows that a leap from ℓ to $\ell+2$ is possible if and only if $a_{\ell+2} = 1$. We can now see that it is the version (2N), used implicitly by Nārāyaṇa, that ensures that the leap is indeed made by the *cakravāla* whenever there is the opportunity, i.e., whenever $a_{\ell+2} = 1$. For, the case-by-case analysis in the proof of Theorem 6.1 shows that the leap takes place only when z_n is chosen to be $b_{\ell+2} + r_{\ell+1}$ and Corollary 6.2 shows that this is precisely the choice of (2N) whenever $a_{\ell+2} = 1$.

To put the distinction between (2B) and (2N) in another way, we note that if $|{z_n}^2 - D|$ is least for $z_n = b_{\ell+2} + r_{\ell+1}$, then, by Lemma 4.16, $|z_n - \sqrt{D}|$ is also the least for $z_n = b_{\ell+2} + r_{\ell+1}$. But it may happen that $|{z_n}^2 - D|$ is the least for $z = b_{\ell+2}$, whereas $|z_n - \sqrt{D}|$ is the least for $z_n = b_{\ell+2} + r_{\ell+1}$. Such a case will be illustrated in the next section (Example 7.2).

Remark 6.4. One method for solving the equation $Dx^2 + 1 = y^2$ is to choose, at each stage n, the integer z_n such that z_n satisfies (1B), $z_n < \sqrt{D}$, and $D - {z_n}^2$ is minimised (subject to the two conditions); other prescriptions being as in section 5. This method is described in [V]. It is an algorithmic version of the usual SCF method for solving the equation $Dx^2 + 1 = y^2$, as presented in standard texts in algebra and number theory.

This method has been called *cakravāla* in [V] but, as we have seen, the ancient Indian *cakravāla* methods differ from it. In this method (i.e., the method described in [V]), there is no jump; ℓ is always n and z_n is invariably taken to be b_n. But an exciting aspect of the actual *cakravāla* is the contraction of certain steps of the SCF. *The possibility of the contraction occurs only when* $a_{\ell+2} = 1$ *for certain* ℓ.

Remark 6.5. The *cakravāla* steps (in any of the versions) invariably lead one to the fundamental solution of $Dx^2 + 1 = y^2$ and, as mentioned in Remark 4.4, repeated applications of Brahmagupta's law (Theorem 3.1) then yield *all* integer solutions. This follows from Theorem 4.3. Note that although the *cakravāla* jumps certain SCF steps, it jumps at most one step at a time and a leap from ℓ to $\ell+2$ can occur only when $a_{\ell+2} = 1$. Since $a_{c+1} = 2a_1 \neq 1$, it follows that the stage $\ell = c$ (or $\ell = 2c$) will not be skipped by the *cakravāla*. Thus the fundamental solution (x_1, y_1) will be reached by *cakravāla*.

It can be shown (cf. [Se, p 177]) that it is the fundamental solution that will be obtained if any m_n in any *cakravāla* step belongs to $\{-1, \pm 2, \pm 4\}$ and Brahmagupta's rule (Corollary 3.3) is applied to $(|\alpha_n|, |\beta_n|)$. We omit the proof and confine our discussions to Nārāyaṇa's original contributions.

Remark 6.6. The correspondence between the pair $(|\alpha_n|, |\beta_n|)$ and the pair (q_ℓ, p_ℓ) also gives an alternative proof of Proposition 5.2. Condition (i) of Proposition 5.2 is immediate; condition (ii) follows from Theorem 4.7 and condition (iii) follows from Lemma 4.1 using the case-by-case analysis in Theorem 6.1.

Remark 6.7. Note that the *cakravāla* method has been discussed earlier, with formal proofs, under other frameworks like the theory of "nearest square continued fraction" due to Krishnaswamy Ayyangar or the theory of "ideal expansion" due to Selenius (cf. [K1], [K2], [K3], [K4], [Se]). Following Singh [Sig1], we have tried to discuss the algorithm within the framework of the simple continued fraction since this approach is likely to be more convenient for readers who are already familiar with the notation and results of a widely read general algebra text like [BC] or [HK].

A note on the term *cakravāla*.

Sūryadāsa (c. 1540), a commentator on the Bījagaṇita of Bhāskara II, remarked that the *cakravāla* method is so called "because it proceeds as in a circle, the same set of operations being applied again and again in a continuous round." [DS, p 162]. The phrase is obscure; it appears that Sūryadāsa attributes the choice of the term to the inductive character of the main step. This explanation does not seem satisfactory. As elaborated by Divakaran in [Di], induction or recursion is a theme that occurs frequently in ancient Indian mathematics, the decimal system itself being a profound example. What was special about the induction in *cakravāla*? It seems more likely that the original inventor of the method coined the term *cakravāla* (cyclic) as an allusion to the periodicity of a continued fraction expansion of $\sqrt{D}$ (cf. [Sr, p 286]) or some equivalent concept. The word *cakra* (the wheel) has the nuance of being circular and yet forward-moving. And we know (Theorem 4.3) that the quantities a_i in the SCF associated to the *cakravāla* occur in cycles, and the convergents $\frac{p_n}{q_n}$ move towards $\sqrt{D}$, with some of them (a cyclic subset) yielding progressively larger solutions of $Dx^2 + 1 = y^2$. It can also be seen that the quantities b_n and r_n occur in cycles (cf. [V, p 32]).

The term *cakravāla* is used by Jayadeva, the earliest known expositor of the method:

iti cakravālakaraṇe'vasaraprāptāni yojyāni

However, the subtleties of the method might not have been realised by the 16th century commentators who wrote more than five centuries after the method was invented (and the term coined).

7. The Example $D = 97$

To illustrate the *cakravāla* algorithm, Nārāyaṇa gives two examples: $D = 103$ and $D = 97$ (in that order). As mentioned earlier, in his main verses describing the algorithm (quoted in the Appendix), Nārāyaṇa does not explicitly mention either (2B) or (2N); but, in his solution to his first example ($D = 103$), he follows the version (2B) (see

Example 7.4 at the end of this section) whereas in his solution to the second example ($D = 97$), he adopts the version (2N). It is as if, through the first example, Nārāyaṇa first puts his seal of approval for the method inherited from his predecessors (after all, there is no mathematical flaw in the Jayadeva-Bhāskara method (2B)); and then, through the second example, he presents his alternative method.

All the steps in Nārāyaṇa's solutions of the examples $D = 103$ and $D = 97$ have been described in [DS, pp 168–171] and [Sig2, pp 63–65] (also see [Sig1, pp 7–11]); the original verses occur in [N, p 237–239]. We recall below a few initial steps, along with SCF interpretations, for the example $D = 97$.

For clarity, we shall reserve the suffix n for the *cakravāla* steps of section 5 and the suffix ℓ for the SCF steps of section 4. For convenience of the reader, we state below a principle for deriving all integer solutions of a linear indeterminate equation $mw - \alpha z = \beta$ from a given integer solution (w_0, z_0) or, equivalently, to find all z such that $\frac{\alpha z + \beta}{m}$ is an integer [BC, p 415].

Lemma 7.1. *Suppose that α, β, m are integers. Let J denote the set of all integers z for which $\frac{\alpha z + \beta}{m}$ is an integer. If z_0 is an element of J, then the set J is given by $J = \{z_0 + mt \mid t \in \mathbb{Z}\}$.*

Example 7.2. *[Nārāyaṇa] To solve* $97x^2 + 1 = y^2$.
We have

$$(b_2 =)\ 9 < \sqrt{97} < 10\ (= b_2 + r_1).$$

Since $10\ (> \sqrt{97})$ is the integer closest to $\sqrt{97}$, the first step $n = 1$ corresponds to $\ell = 2$ and we have:

$$\begin{aligned} \alpha_1 &= 1\ (= q_2). \\ \beta_1 &= 10\ (= p_2). \\ m_1 &= 10^2 - 97 = 3\ (= r_3). \end{aligned}$$

These integers correspond to the first relation

$$97 \times 1^2 + 3 = 10^2.$$

Now the set of integers z satisfying (1N), i.e., such that $\dfrac{1 \times z + 10}{3}$ is an integer, is given by

$$J_2 = \{2 + 3t \mid t \in \mathbb{Z}\}.$$

Among these integers, $|z - \sqrt{97}|$ is minimised when $z = 11$. So $z_1 = 11$. Here

$$(b_4 =)\ 8 < \sqrt{97} < 11\ (= b_4 + r_3)\ = z_1.$$

So again there is a jump; $n = 2$ corresponds to $\ell = 4$:

$$\begin{aligned}
\alpha_2 &= \frac{1 \times 11 + 10}{3} = 7(= q_4).\\
m_2 &= \frac{11^2 - 97}{3} = 8(= r_5).\\
\beta_2 &= 7 \times 11 - 1 \times 8 = 69(= p_4).
\end{aligned}$$

These integers correspond to the next relation

$$97 \times 7^2 + 8 = 69^2.$$

The set of integers z satisfying (1N), i.e., such that $\dfrac{7z + 69}{8}$ is an integer, can be seen to be

$$J_4 = \{5 + 8t \mid t \in \mathbb{Z}\}.$$

Among these integers, $|z - \sqrt{97}|$ is minimised when $z = 13$. So $z_2 = 13$. However, $|z^2 - 97|$ is minimised when $z = 5$ as well as when $z = 13$. Here

$$(b_6 =)\ 5 < \sqrt{97} < 13\ (= b_6 + r_5)\ = z_2.$$

So, by rule (2N), there is again a jump with $n = 3$ corresponding to $\ell = 6$:

$$\begin{aligned}
\alpha_3 &= \frac{7 \times 13 + 69}{8} = 20(= q_6).\\
m_3 &= \frac{13^2 - 97}{8} = 9(= r_7).\\
\beta_3 &= 20 \times 31 - 7 \times 9 = 197(= p_6).
\end{aligned}$$

These integers correspond to the next relation

$$97 \times 20^2 + 9 = 197^2.$$

The set of integers z satisfying (1N), i.e., such that $\dfrac{20z + 197}{9}$ is an integer, can be seen to be

$$J_6 = \{5 + 9t \mid t \in \mathbb{Z}\}.$$

Among these integers, $|z - \sqrt{97}|$ is minimised when $z = 14$. So $z_3 = 14$ $(> \sqrt{97})$ and thus there is a jump if (2N) is followed. However, $|z^2 - 97|$ is minimised when $z = 5$ and the jump would not have taken place by rule (2B). Here

$$(b_8 =)\ 5 < \sqrt{97} < 14\ (= b_8 + r_7)\ = z_3.$$

Continuing the process by Nārāyaṇa's rule (2N), we will get the successive triples

$$\begin{aligned}(|\alpha_4|, |\beta_4|; m_4) &= (53, 522; 11) = (q_8, p_8; r_9),\\ (|\alpha_5|, |\beta_5|; m_5) &= (86, 847; -3) = (q_9, p_9; -r_{10}), \text{and then}\\ (|\alpha_6|, |\beta_6|; m_6) &= (569, 5604; -1) = (q_{11}, p_{11}; -r_{12}).\end{aligned}$$

At this stage, Brahmagupta's rule gives the fundamental solution $(x, y) = (6377352,\ 62809633) = (q_{22}, p_{22})$.

Thus, in the SCF expansion

$$\sqrt{97} = [9, \overline{1, 5, 1, 1, 1, 1, 1, 1, 5, 1, 18}],$$

rule (2N) ensures that a jump occurs from ℓ to $\ell + 2$ when $\ell + 1 = 1, 3, 5, 7, 10$, as in each of these cases, $a_{\ell+2} = 1$.

We summarise some of the salient features of Nārāyaṇa's example $D = 97$:

(i) The minimum positive integral solutions of $97x^2 + 1 = y^2$, viz., $x = 6377352$ and $y = 62809633$, are large, although for adjacent values of D, the solutions are relatively small.[8]

Among all examples of explicit equations $Dx^2 + 1 = y^2$ that are known to have been discussed by authors prior to the 17th century, the oft-quoted example $D = 61$ of Bhāskara II gives the largest minimum positive integral solution;[9] Nārāyaṇa's example $D = 97$ gives the next largest minimal positive integral solution.

(ii) The celebrated case $D = 61$ terminates too quickly (Example 5.6). After the initial triple $(1, 8; 3)$, we get $(5, 39; -4)$ whence one can apply Brahmagupta's formula to arrive at the famous solution. In a sense, this example illustrates the power of Brahmagupta's *bhāvanā* more than the intricacy of the *cakravāla*. In fact, the other exciting example $D = 67$

of Bhāskara II ([DS, pp 166–168]) involves more steps (than $D = 61$) before the *bhāvanā* can be invoked: $(1, 8; -3)$, $(5, 41; 6)$, $(11, 90; -7)$ and then $(27, 221; -2)$.

In the example $D = 97$, the actual *cakravāla* process lingers much longer. This gives a more sustained training in the algorithm. For, in this case, the cycle $c = 11$ and none of the r_ℓ prior to $\ell = 11$ involves the numbers $\pm 1, \pm 2, \pm 4$ thus preventing an early short-cut through *bhāvanā*. And many technicalities are brought out into the open, as we discuss below, which enable one attain a deeper understanding.

(iii) This example ($D = 97$) illustrates a "tie" at a certain stage when, among all elements z in certain J_ℓ (cf. Lemma 4.11), viz., $J_4 = \{5 + 8t \mid t \in \mathbb{Z}\}$, there are two values ($z = 5, z = 13$) for which $|z^2 - D|$ is minimum. Such a situation does not occur in the examples (like $D = 61$, $D = 67$) cited in the preceding extant texts. None of the preceding authors on *cakravāla* like Jayadeva and Bhāskara II appears to give a clear guidance as to which z one ought to choose in the case of such a tie. Here $\ell = 4$ corresponds to the 2nd step in *cakravāla*, i.e., $n = 2$. The choice of $z = 13$ ensures that $n = 3$ corresponds to $\ell = 6$ rather than $\ell = 5$.

(iv) At another stage ($n = 3; \ell = 6$) of this same example, we have the situation: among all z in $J_6 = \{5 + 9t \mid t \in \mathbb{Z}\}$, $|z^2 - D|$ is minimised for a certain value of z (namely, $z = 5$) while $|z - \sqrt{D}|$ is minimised for a different value of z (viz., $z = 14$). It is only the choice $z = 14$ that ensures a jump from $\ell = 6$ to $\ell = 8$.

Remark 7.3. Selenius has pointed out that the method (2B), which is equivalent to his "ideal" semiregular continued fraction expansion, involves the least number of steps among all continued fraction expansions. Thus, though Nārāyaṇa's method (2N) sometimes takes a jump when (2B) does not, eventually the number of steps required will be the same for both (2N) and (2B).

For instance, if one follows rule (2B) for the example $D = 97$, then the triples that occur are:
$(\alpha_1, \beta_1; m_1) = (1, 10; 3) = (q_2, p_2; r_3)$,
$(\alpha_2, \beta_2; m_2) = (7, 69; 8) = (q_4, p_4; r_5)$,
$(\alpha_3, \beta_3; m_3) = (13, 128; -9) = (q_5, p_5; -r_6)$ or $(20, 197; 9) = (q_6, p_6; r_7)$

(depending on how the "tie" mentioned earlier is resolved),
$(|\alpha_4|, |\beta_4|; m_4) = (33, 325; -8) = (q_7, p_7; -r_8)$,
$(|\alpha_5|, |\beta_5|; m_5) = (86, 847; -3) = (q_9, p_9; -r_{10})$, and then
$(|\alpha_6|, |\beta_6|; m_6) = (569, 5604; -1) = (q_{11}, p_{11}; -r_{12})$.

So both (2B) and (2N) involve six steps. It will be interesting to get a formal proof for the minimality property of (2B) and (2N) in the framework and notation of the theory of simple continued fraction (section 4).

Example 7.4. *[Nārāyaṇa] To solve* $103x^2 + 1 = y^2$.

For $D = 103$, the starting triple is $(1, 10; -3)$. Now, among integers z such that $\frac{z+10}{-3}$ is an integer, $z = 11$ minimises $|z^2 - 103|$ as well as $|z - \sqrt{103}|$. Taking $z_1 = 11$, one gets the next triple $(7, 71; -6)$. Now, among the integers z such that $\frac{7z+71}{-6}$ is an integer, $|z^2 - 103|$ is minimum for $z = 7$ but $|z - \sqrt{103}|$ is minimum for $z = 13$, since $\sqrt{103}$ is closer to 13 than to 7 though 103 is closer to 49 than to 169. In accordance with (2B), Nārāyaṇa chooses $z_2 = 7$ to get the next triple $(20, 203; 9)$. Now, among the integers z such that $\frac{20z+203}{9}$ is an integer, $|z^2 - 103|$ (as also $|z - \sqrt{103}|$) is minimum for $z = 11$. Taking $z_3 = 11$, we get the next triple $(47, 477; 2)$ whence, by Brahmagupta's formulae, one obtains the fundamental solution (22419, 227528).

8. Application to Rational Approximation

Explorations of good rational approximations to an irrational surd $\sqrt{D}$ had been made in ancient India right from the Vedic times. For $D = 2$, the *Śulba Sūtra*s (c. 800 BC) describe an approximation equal to the fraction $\frac{577}{408}$. We have seen (Example 4.5) that $\frac{577}{408}$ is the eighth convergent $\frac{p_8}{q_8}$ in the SCF expansion of $\sqrt{2}$ and that the pair $(408, 577)$ satisfies $2 \times 408^2 + 1 = 577^2$.

Nārāyaṇa applied the *cakravāla-bhāvanā* method of generating arbitrarily large solutions of $Dx^2+1=y^2$ to determine progressively better rational approximations of $\sqrt{D}$. He says ([N, p 244]):

mūlaṁ grāhyaṁ yasya ca tadrūpakṣepaje pade tatra
jyeṣṭhaṁ hrasvapadena ca samuddharenmūlamāsannam

Obtain the roots (α, β of the *varga-prakṛti* equation) having unity (1) as the additive and the number (D) whose square-root is to be determined (as the multiplier). Then the greater root (β) divided by the lesser root (α) will be an approximate value of the square root ($\sqrt{D}$).

Note that, if $D\alpha^2+1=\beta^2$, then

$$\frac{\beta}{\alpha}-\sqrt{D}=\frac{\beta^2-D\alpha^2}{\alpha(\beta+\alpha\sqrt{D})}=\frac{1}{\alpha(\beta+\alpha\sqrt{D})}<\frac{1}{2\alpha^2}$$

(since $\beta>\sqrt{D}\alpha$) and thus, for a sufficiently large solution (α,β), $\dfrac{\beta}{\alpha}$ will be a good approximation for $\sqrt{D}$. Nārāyaṇa then invites the reader to find approximate solutions for two numerical examples: $\sqrt{10}$ and $\sqrt{\frac{1}{5}}$ in the following words ([N, p 244–245]):

daśānāmapi rūpāṇāṁ pañcamāṁśasya vā vada
āsannamūlaṁ jānāsi chet kriyāṁ prakṛteḥ sakhe

O friend, if you know the process of the *(varga) prakṛti*, tell me the approximate square root of 10 or $\frac{1}{5}$.

The choice of $\sqrt{D}$ is interesting; for the number $\sqrt{10}$ was widely used in the ancient world as an approximation for π. We discuss Nārāyaṇa's approximations for $\sqrt{10}$; the case of $\sqrt{\frac{1}{5}}$ is similar.

Nārāyaṇa mentions the rational approximations $\dfrac{19}{6}$, $\dfrac{721}{228}$ and $\dfrac{27379}{8658}$ for $\sqrt{10}$. To see how the fractions arise, consider the notation of section 5 for $D=10$. Since 9 is the perfect square nearest to 10, one has the initial triple $(1,3;-1)$. Now, using Brahmagupta's law (Theorem 3.1), we have

$$(1, 3; -1) \odot (1, 3; -1) = (6, 19; 1),$$
$$(6, 19; 1) \odot (6, 19; 1) = (228, 721; 1),$$
$$(6, 19; 1) \odot (228, 721; 1) = (8658, 27379; 1).$$

Thus one has the three successive fractions $\frac{19}{6}$, $\frac{721}{228}$ and $\frac{27379}{8658}$ as rational approximations to $\sqrt{10}$. To see the usefulness of the method, note that the third fraction $\frac{27379}{8658}$ $(= 3.162277662\dots)$ matches the value of $\sqrt{10}$ $(= 3.162277660\dots)$ up to nine decimal digits. This method of generating successively closer approximations was restated by Euler in 1732 ([D1, p 188]). By Theorem 4.10, Nārāyaṇa's approximations for $\sqrt{10}$ are the best possible in a very precise sense (viz, with the respective restrictions on the denominators).

The above method of Nārāyaṇa is not known to have been explicitly mentioned by any other ancient or medieval author. Jñānarāja (1503) gave a method of successive approximation ([D1, pp 193–194]) which may be expressed as follows: Let $a_0 = a = [\sqrt{D}]$ and $a_{i+1} = \frac{1}{2}(a_i + \frac{D}{a_i})$; then $a_i \approx \sqrt{D}$ and each a_i is a better approximation to $\sqrt{D}$ than the previous a_{i-1}. This method is perhaps computationally simpler; but Nārāyaṇa's method has a charm of its own, especially as it relates the approximation problem to a celebrated equation.

Even now, Nārāyaṇa's method is of pedagogic value: it gives one quick illustration of the usefulness of Pell's equation $Dx^2 + 1 = y^2$ to a student who may not have sufficient mathematical knowledge for comprehending deeper modern applications.

9. Concluding Remarks

As recalled in section 2, by the time of Nārāyaṇa, the intellectual traditions were already on the decline, at least in politically turbulent north India. One gets an impression that, leaving aside certain pockets of excellence in south India (like the school of Mādhava), there was a dearth of original scientific brilliance in medieval India — new treatises were mere repetitions or elaborations of earlier thoughts. But a consequence of this general impression is the risk of overlooking the contributions of even a first-rate mathematician like Nārāyaṇa Paṇḍita who happened to live during a not-so-glorious phase of India's intellectual history. Besides, one tends to miss the significance of a work like Nārāyaṇa's approach to Pell's equation due to the fact that the problem had got completely solved well before his time. In his earlier write-up on the *cakravāla* [Du2], the present author himself had not laid adequate emphasis on Nārāyaṇa's version (which is mentioned as a mere footnote in [Du2, p 187]).

In modern convention, a distinction is made between a new discovery (usually presented in a research paper) and existing knowledge (usually expounded in a book or a survey article). But ancient scientific traditions made no such distinction: a typical text recorded the accumulated knowledge on the topic along with the author's original contributions without any demarcation between them. Thus Nārāyaṇa's treatise too, inevitably, repeats the huge corpus of important results and methods of his great predecessors, often without any significant variation. But, as we have seen, a careful look at his presentation of the *varga-prakṛti* shows the touch of an accomplished mathematician taking a fresh look at an existing brilliant work and making a penetrating observation. As an original achievement, his contribution on the *varga-prakṛti* is not in the same league as that of Brahmagupta's composition law and applications or the invention of the *cakravāla*. But it is significant in its own right. And Nārāyaṇa's deep understanding of the intricacies of Pell's equation took place a full 300 years before Fermat and Brouncker took up this complex problem. More evidence of the richness of Nārāyaṇa's mathematical thought can be seen from his work on combinatorics, magic squares and other topics.

One is reminded of a remark of Sri Aurobindo, made in another context, about the period of decline of ancient Indian civilization [A, p 30]:

> There was a descent from the heights to the lower levels, but a descent that gathered riches on its way ...

Appendix: Nārāyaṇa's verses on the cakravāla

We quote below Nārāyaṇa's verses on the main algorithm for solving the equation $Dx^2 + 1 = y^2$ in integers by the *cakravāla* method. The Sanskrit verses are given in [N, p 236].

hrasvavṛhatprakṣepān bhājyaprakṣepabhājakān kṛtvā
kalpyo guṇo yathā tadvargāt saṁśodhayet prakṛtim prakṛter-
guṇavarge vā viśodhite jāyate tu yaccheṣam
tat kṣepahṛtaṁ kṣepo guṇavargaviśodhite vyastam labdhiḥ
kaniṣṭhamūlaṁ tannijaguṇakāhataṁ viyuktaṁ ca
purvālpapadaparaprakṣiptyorghātena jāyate jyeṣṭham
prakṣepaśodhaneṣvapyekadvicaturṣvabhinnamūle staḥ
dvicatuḥ kṣepapadābhyāṁ rūpakṣepāya bhāvanā kāryā

Taking the lesser root (α), the greater root (β) and the interpolator (m) [of the equation $Dx^2 + m = y^2$] to be the dividend, the addend and the divisor [respectively of the linear indeterminate equation $mw - \alpha z = \beta$], the [indeterminate] multiplier (z) is to be determined by the method described earlier [i.e., by *kuṭṭaka*]. The prakṛti (D) being subtracted from the square of that (z), or the square of that multiplier (z) being subtracted from the prakṛti, the remainder ($|z^2 - D|$) divided by the [original] interpolator (m) is the [new] interpolator (m'); it will be reversed in sign in case of subtraction of the square of the multiplier [i.e., $m' = \frac{z^2-D}{m}$]. The quotient (w) is the [new] lesser root (α'); that (α') multiplied by the multiplier (z) and diminished by the product of the previous lesser roor (α) and the [new] interpolator (m') will be the new greater root β' [i.e., $\beta' = \alpha' z - \alpha m'$]. By repeatedly performing the above [operations], integral roots will be obtained corresponding to [an equation $Dx^2 + m = y^2$ with] an interpolator [of any of the forms] $\pm 1, \pm 2, \pm 4$. In order to derive integral roots for [the equation $Dx^2 + 1 = y^2$ with] unity as the additive from those roots [i.e., roots of the equation $Dx^2 + m = y^2$] corresponding to the interpolator ($m =$) $\pm 2, \pm 4$, the principle of composition is to be applied.

Acknowledgements. It is a great pleasure to recall the stimulating discussions on Pell's equation with Raja Sridharan over the years. In particular, it was Raja who has always been insisting on the necessity of a neat mathematical exposition on why the *cakravāla* works; the present exposition is a partial attempt in that direction.

The genesis of the paper can also be traced to a suggestion from R. Sridharan and M.D. Srinivas in 2009 to elaborate on Nārāyaṇa's treatment of the *varga-prakṛti.* I am grateful to M.D. Srinivas and M.S. Sriram for the text of the *Gaṇita Kaumudī* ([N]; [Sig2]).

Important inputs on an earlier draft, especially Remarks 6.7 and 7.3, have been provided by M.D. Srinivas. I also thank Somesh C. Bagchi, Neena Gupta, Vivek Hiremath, K. Ramasubramanian and B. Sury for carefully going through the drafts of the paper.

Notes

[1]This section is practically an abridged version of sections 1 and 6 of the author's article [Du2].

[2]The 14th century mathematician Nārāyaṇa Paṇḍita is not be confused with later writers named Nārāyaṇa who wrote mostly on astronomy and astrology. See [D2] for details.

[3]vataṁsa: a ring-shaped ornament, a garland, crest.

[4]*Kaumudī* means the moonlight or moonshine, which causes the *kumuda* (white water-lily or lotus) to blossom. The word is metaphorically used at the end of grammatical commentaries and other explanatory works in the sense of "elucidation", the implication being that the treatise designated *kaumudī* throws a gentle light on the subject it treats.

[5]Jayadeva's verses on the solution of the indeterminate equation $Dx^2 + 1 = y^2$ have been quoted in the text *Sundarī* of Udaydivākara composed in 1073 AD (cf. [Sh]). Nothing is known so far about the mathematician Jayadeva.

[6]By contrast, the smallest positive integer solution to $60x^2 + 1 = y^2$ is $(x, y) = (4, 31)$.

[7]Brahmagupta had remarked, way back in the 7th century, that a person who can find integer solutions of $92x^2 + 1 = y^2$ within a year is truly a mathematician. In the 17th century, Brouncker had met Brahmagupta's deadline! Silverman proposes [Si, p 217] that Pell's equation be named B^3 equation after Brahmagupta, Bhāskarācārya and Brouncker.

[8]The smallest positive integer solution for $96x^2 + 1 = y^2$ is $(x, y) = (5, 49)$; for $98x^2 + 1 = y^2$, it is $(10, 99)$. The example $D = 103$ of Nārāyaṇa too has this feature.

The fundamental solution for $102x^2 + 1 = y^2$ is $(10, 101)$; for $104x^2 + 1 = y^2$, it is $(5, 51)$. But the fundamental solution for $103x^2 + 1 = y^2$ is $(22419, 227528)$.

[9]We are not considering the Cattle Problem of Archimedes as the problem itself is not originally formulated as a problem of Pell's equation; besides no method for solving it is indicated.

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