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MĀDHAVA—A GREAT KERALA MATHEMATICIAN OF MEDIEVAL TIMES

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Mādhava (c. 1340-1425 AD) of Sangamagrāma (a village near the temple of Samgameśvara, modern Irinjalakkuda, lat 10^0 20´ N near Kochi/Cochin) is supposed to be one of the greatest mathematicians that medieval Kerala had produced. He was at the top of the lineage of the medieval Kerala mathematicians followed by Parameśvara, Nīlakaṇṭha, Jyeṣṭhadeva, Śankara, Nārāyaṇa, Acyuta Piṣāroṭi and others. A number of citations are available in the works of later scholars which show Mādhava's marvelous achievements in mathematics. Some of these are the values of π correct to 10 places of decimals, imposing corrections to infinite series after certain terms for quick and better results, derivation of Sine & Cosine power series for computing better Sine and Versed- sine tables which are unique by contemporary standard. His π, Sine and Cosine power series were rediscovered about 250 years later in Europe by scholars —Wilhelm Leibniz (1673), Newton (1675), De Lagney (1682), De Moivre (c.1720), Euler (1748) and others. The achievements of Mādhava in other areas are still to be explored and will be known only when new important manuscripts composed by him are brought to light through new editions and translations. Some of Mādhava's derivation and results are critically examined in this paper.

Key words: Concept of Sine and Cosine functions; Early Indian scholars' values of π; Lineage of Kerala school of mathematics; Mādhava; Mādhava's value of π in series, Mādhava's correction of π series for better result; Methods of derivation; Sine and Cosine power series; Sine and versed sine tables

INTRODUCTION

I personally feel that Mādhava was one of the greatest scholars of mathematics that Kerala had produced during the medieval period. The

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reference to Vakulavihāram (either Iriññārapalli or Irriññānavalli) as his house name in his *Venvāroha* (verse 13), and Sangamagrāma as his village name by Nīlakaṇṭha in his *Āryabhaṭīya-bhāṣya* (near famous temple of Samgameśvara, lat 10^0 20´ N), are used as pointer by many modern scholars including K.V. Sarma that he was from Kochi/Cochin in Central Kerala. His two works are now known, viz. *Sphuṭachandrāpti* (51 verses) and *Venvāroha* (75 verses), the former edited with commentary of Acyuta Piṣāraṭi and the latter edited and translated along with Introduction and notes by K.V. Sarma. Both these works deal with short cut method for finding true longitude of Moon by using anomalistic cycles. I am not going to discuss the merit of any of these works.

An effort is made to build a story on the basis of information what other great scholars, some are his pupils, others are commentators/ interpreters, had spoken about him and his contribution. You will also admit that reference by other scholars about Mādhava sets the rating of his scholarship which is indeed a matter of great appreciation. In fact, medieval Kerala mathematics had a long tradition which ran for almost 250 years if not more. The lineage starting with Mādhava runs as follows:

> Mādhava (c.1340-1425 AD) → pupil, Parameśvara (c.1400 AD) → son, Dāmodara (c.1440 AD) → pupils, Nīlakaṇṭha Somayāji (c.1500 AD) and Jyeṣṭhadeva (c.1530AD) → pupils of Nīlakaṇṭha ,Citrabhānu (c.1530 AD) and Śankara Vāriyar (c.1535 AD) → pupil of Jyeṣṭhadeva, Acyuta Piṣāraṭi (c.1550-1621 AD) → pupil of Acyuta, Nārāyaṇa Bhaṭṭātiri (c.1557AD), and so on.

Various epithets have been used with the name of Mādhava, some referred to him as *guru* (teacher), *paramaguru* (grand teacher), *golavid* (expert on spherical mathematics) etc. The later scholars have quoted important lines about the contribution of Mādhava, which are not found in his presently available works. To show the greatness of Mādhava, a few examples will be of great interest and may be cited in historical perspectives.

VALUE OF π GIVEN BY EARLIER SCHOLARS

Indian scholars have mostly defined π as the ratio of circumference of a circle to its diameter. Some excellent studies had already been made about various values of π used in Indian tradition, a summary of which will be of interest[1].

1. $\pi=\sqrt{10}$ (Umāsvāti, Varāhamihira, Brahmagupta, Śrīdhara and others)[2]; there are various theories known as to the origin of this value; there is some connection between $\sqrt{10}$ and 22/7 since they are connected with an old root-extraction formula known to the Jains, i.e. $\sqrt{(a^2 + r)} = a + r / (2a + 1)$, i.e. $\sqrt{(3^2 + 1)} = 3 + 1/7 = 22/7$;

   The Jain work, *Jivāvigama-sūtra* (*sūtra* 112) gives the circumference as 316 *yojanas* for a diameter of 100 *yojanas* fixing value of $\pi=3.16$.

2. $$\pi=62{,}832/20{,}000 = 3927/1250 = 3 + \cfrac{1}{7 + \cfrac{1}{16 + \cfrac{1}{11}}} = 3,\ 22/7,\ 355/113 \text{(convergents)}$$

   i.e. $\pi=3$ (1st order approx.), 22/7 (2^{nd} order approx.), 355/113 (3rd order approx.) (Āryabhaṭa I, Lalla, Bhaṭṭotpala, Bhāskara II and others)[3]. The value of π is termed as approximate (*āsanna*) by Āryabhaṭa I. The convergents are approximations, and a general method for calculating successive approximations by a technique (*kuṭṭaka*, pulveriser) was introduced by Āryabhaṭa I for finding the mean position of planets, unlike decimal system in modern times, where the rational fraction has large integers both in numerator and denominator. This method of using approximations was extremely popular among the later Indian scholars, and even the Chinese scholar It-sing, who came to India in the 7^{th} century AD, got fascinated and introduced it into China.

2.1. By taking circumference (C) as 21600 (=360x60) and by using the value of π =62,832/20,000, the radius (R) of the circle= $C/(2\pi)$ was found to be 3438 minutes, which has been occasionally used by Āryabhaṭa I and others.

3. π= 22/7= 3.142.. , correct to 2 places (Āryabhaṭa II, Bhāskara II)[4],

4. π= 355/113=3.1415929.., correct to 6^{th} places (Nīlakaṇṭha, Śankara Vāriyar, Nārāyaṇa)[5],

5. π=1,04,348/33,215= 3.14159265391.., correct to 9th places (*kecid*, an unknown person as noted by Śankara)[6] and

6. π= 28,27,43,33,88,233 / (9 × 10^{11}) =31415926536/10^{10} = 3.141592653592.., correct to 10 th places (Mādhava, Putumana Somayāji)[7].

The credit for the value is given to Mādhava by Nīlakaṇṭha and Śankara. The successive convergents are: 3, 22/7, 333/106, 355/113,67783/21576.

Mādhava's value, as quoted by by Nīlakaṇṭha and Śankara, is far better than others and considerably much closer approximation. Now the question arises: how Mādhava arrived at such a closer approximation, and whether he had any knowledge of incommensurability for determining the knowledge of π.

Mādhava's Value of π

Śankara[8], in four verses, said (*atrāha mādhava*) which suggests that Mādhava had actually suggested a method for finding the circumference of a circle by means of constructing a number of regular polygons, for the sum of the sides of the polygon would almost be equal to the length of the circumference of the circle. Step by step procedure was adopted to compute the side of a square- polygon for a circle, then half-side of the square-polygon (octagon), then half-side of the octagon (hexadecagon), then half side of the hexadecagon (32- gon) and so on indicating that the number of regular polygons had to be large for considerably accurate value. A few scholars had tried to tackle the problem by geometrical method, but apparently no solution has so far been concretized. However, I have tried here a trigonometrical reconstruction of the procedure, not attempted before, which will be of interest (Fig 1):

Let N = number of sides of a regular polygon inscribed in a circle of radius R, XY =one side of the polygon, T =middle point of XY, O= centre of the circle, 2θ =angle subtended by the side XY at centre O, then θ= 360^0 / XT= 360^0/ 2N = 180^0 / N, let P=perimeter of the polygon = 2N x XT = 2N x R Sin θ= 2N x R Sin (180^0 /N),

Then π = circumference / diameter = P/ 2R = N Sin (180^0 / N)

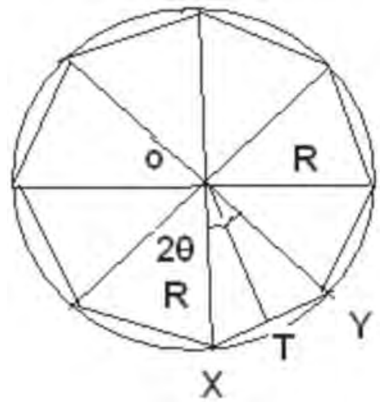

Fig. 1. Polygon of side N subscribing an angle 2θ at the centre o.

By putting the value of N = 4,5,6…, the value of π could be approximated. For instance,

N	$\pi = N \sin(180^0 / N)$
6	3
30	3.135853898
2000	3.141591362
4000	3.141592331
10,000	3.141592602
20,000	3.141592641
…….	……….

This shows the value of π converges very slowly, and to achieve Mādhava's value, N has to be very large.

Incommensurability of π and Mādhava's Correction

Mādhava had definitely a clear idea that π is an incommensurable quantity, and the Circumference (C) will never be completely divisible by Diameter (D), i.e. it will never terminate, otherwise why should he make attempts for better and better values of π ? This is also evident from his attempt to devise a value of π with the help of a slowly converging series, and tried to improve the result by suggesting corrections after a limited number of steps to attain the result much faster.

Approximations for the value of π (=Circumference/Diameter=C/D) is found to have been started with the following basic series by almost all the Kerala mathematicians.

$$\pi = C/D = 4\,(1 - 1/3 + 1/5 - 1/7 + \dots) \qquad \dots(1)$$

It is an extremely slow converging series but it gives a reasonably correct value after computing a large number of steps, but how it was discovered nothing is clearly known. For example: C/D (after 19 steps)=3.194.., C/D (after 20 steps)=3.091.. and so on.

Śankara has recorded certain interesting approximation[9], which. Nīlakaṇṭha informs in his *Tantrasaṅgraha*[10] that the approximation is suggested by Mādhava. It says,

> “Multiply the diameter by four and divide by one; subtract from and add to the result alternately the successive quotients of four times the diameter divided severally by the odd numbers 3, 5,etc. Take the even number next to that odd number on division by which the operation is stopped; then as before multiply four times the diameter by the half of that and divide by its square plus unity. Add the quotient thus obtained to the series in case its last term is negative; or subtract if the last term be positive. The result will be very accurate if the division be continued to many terms”[11].

This in other words,

$$C/D = 4\left[1-1/3+1/5-1/7+\dots\pm 1/n \mp \frac{[(n+1)/2]}{[(n+1)^2+1]}\right] \qquad \dots(2)$$

He then continues,

> “ Now I shall write of certain other correction more accurate than this. In the last term the multiplier should be the square of half the even number together with one, and the divisor four times that, added by unity, and then multiplied by half the even number. After division by the odd numbers 3, 5 etc, the final operation must be made as just indicated” [12].

$$CD = 4\left[1-1/3+1/5-1/7+\dots\pm 1/n \mp \frac{[\{(n+1)/2\}^2+1]}{[(n+1)^2+4+1][(n+1)/2]}\right] \qquad \dots(3)$$

The no.(1) is a series which has a infinite number of terms. The series (2) and (3) have limited number of terms . What is interesting is that an end correction (*antyasamskāra*) is suggested in series (1) after a few steps, and by doing so, a tremendous achievements were obtained, not only in results but also by minimizing the number of operations or steps.

For example, number (1) which is an extremely slow converging series, Yano's calculation says that C/D = 3.194.. (after 19 steps), C/D =3.091…(after 20 steps), and millions of steps would be needed to arrive at Mādhava's value.

The series (2) gives much improved result, and the series (3) gives further improved result, i.e. C/D (after 19 steps) gives value correct to 9 places of decimals.

Methods of Derivation of π with Justification

How Mādhava arrived at such a end correction? A large number of hypothesis have been made by many scholars including Yano (1989), John (2002), and Ramasubramanian (2010). While Yano based his derivation on the basis of *Kriyākramakarī*, the later scholars depended upon *Yuktibhāṣā* of Jyeṣṭhadeva. The *Yuktibhāṣā* method, in short, suggests that the infinite series (1) could be written in the form:

$$C/D = 4\,(S_n \pm 1/a_n) = 4\,[S_{n-2} \pm (1/n - 1/a_n)] \qquad \ldots(4)$$

where S_n represents the sum terminating at 1/n (n=odd number by choice) and $1/a_n$ is the rational approximation to the remaining terms of the series.

On the same logic, $C/D = 4(S_{n-2} \pm 1/a_{n-2})$ …(5)

Items (4) and (5) being the same, it leads to $1/a_{n-2} = 1/n - 1/a_n$,

Or, $1/a_{n-2} + 1/a_n = 1/n$……. (6).

This is possible when $a_{n-2} = a_n = 2n$, but this is not admissible since a_{n-2} and a_n are always different, each of them may be closer to 2n and follow the same logic,

i.e. if $a_{n-2} = 2n$, then $a_n = 2n + 2$,

or if $a_n = 2n$, then $a_{n-2} = 2n-2$. This indicates that there is always a difference of 4 between a_{n-2} and a_n , and the 1^{st} order correction divisor agrees with the statement of *Yuktibhāṣā*[13], that it is ‘ double the even number above the last odd number divisor”, i.e. 2(n+1), since n is odd.

It is evident from the basic series (1) itself.

To ascertain the extent of inaccuracy(*sthaulya*) of the correction, *Yuktibhāṣā* further says, ‘the difference between the sum of the two successive corrections and the result of the division by odd number in between is to be known’,

i.e. the error factor $F(A)= (1/a_{n-2}+ 1/a_n - 1/n)$ ….(7)

are made, and the best estimate for (a_{n-2}, a_n)= [(2n-5, 2n-1); (2n-4,2n), (2n-3,2n+1), (2n-2, 2n +2), (2n-1,2n+3),..], are to be calculated. This is attempted by a number of scholars, present here, which show that the error magnitude is minimum for (2n-2, 2n+2). Obviously, $a_n = 2n +2= 2(n+1)$.

So the 1st order approximation, $F_1 (A)= 1/ a_n = 1/ [2 (n+1)]$, ….(8)

Most interesting part is that this intricate calculation was computed with the help of cowries.

As to 2nd order approximation, all the approximations arising out of the error factor(7) has been assessed and found that $A=4=2^2$ is the best fit,:

$$F_2(A) = \frac{1}{(2n+2) + \dfrac{A}{(2n+2)}} . = \frac{(2n+2)}{[(2n+2)^2 + 4]} = \frac{(n+1)/2}{(n+1)^2 + 1} \quad …..(9)$$

This justifies Mādhava’s derivation of the series (2).

Likewise, the 3rd order derivation is fixed when $B=16=4^2$ is found to be the best fit from the correction factor (7):

$$F(A) = \frac{1}{(2n+2)+\dfrac{4}{(2n+2)+\dfrac{B}{(2n+2)}}} = \frac{1}{(2n+2) + \dfrac{4(2n+2)}{[(2n+2)^2 + 16]}}$$

$$= \frac{[(2n+2)^2 + 16]}{[(2n+2) + 16(2n+2) + 4(2n+2)]} = \frac{[\{(n+1)/2\}^2+1]}{[(n+1)^2+4+1][(n+1)/2]} \quad \ldots(10).$$

This verifies the derivation (3).The two correction factors are in fact applications of continued fraction which are discussed as minimization algorithm. The correction factors (8), (9) and (10) give the values of π as 3.141594652591010 (1^{st} order approx, correct to 5 decimal places), 3.141592652790990 (2^{nd} order approx.,correct to 8 places), 3.141592653590510 (3^{rd} order approx.,correct to 10 places) respectively. The 3^{rd} order approximation of Mādhava was adopted by Paramesvara.The 5^{th} order was adopted by Raja Śankara Varma in his *Sadratnamālā*[14] which fixes

π = 3.14159265358979324130 (correct to 17 places).

Mādhava's rectification of *n* th term of a slowly converging infinite series gives more and more accurate value of π , discovery of sine and cosine power series , which were rediscovered about 250 years later by European scholars. Lot of improvements were made by Kerala scholars in fixing the value of radius or diameter w.r.t. a circle and their refinement, sine table, sine and cosine series and other related functions.

Sine Table

Indian trigometrical tradition conceived three trigonometrical functions starting from Āryabhaṭa I onwards. These are: *jyā*, *kojyā* (or *koṭijyā*), and *utkramajyā*. For a point P in a quarter circle of centre O and radius R, if an *iṣṭacāpa* i.e. the arc (s) is divided into n equal arc bits, and AP makes an angle θ/n ; if s=rθ, then in modern notation (see Fig. 2),

Jyā AP = *bhuja-piṇḍajyā* (b)= PM= r sin (θ/n)= r sin (s/r n),

Ko-jyā AP =*koṭi-piṇḍajyā* (k)=OM= r cos (θ/n)= r cos (s/r n) and

Utkrama-jyā AP=*śara-piṇḍajyā* (u)=AM= OA-OM= r- r cos (θ/n)

= r versin (θ/n) = r vers (s/r n).

Obviously b_j =r sin(js/n), k_j = r cos (js/n),

and u_j = r vers (js/n)= r[1 –cos(js/n)].

As sine-values (bj) increase, versine-values (uj) also increase; but, as versine-values (uj) decrease, the cosine-values (kj) decrease in the same proportion . Āryabhaṭa I in his *Āryabhaṭīya* (ii.12) obtained the Rsine values by dividing quarter circle (90^o) into 24 equal divisions each of 3° 45´ (or 225´) i.e. at 225´, 450´, 675´…5400´, and constructed his sine table by

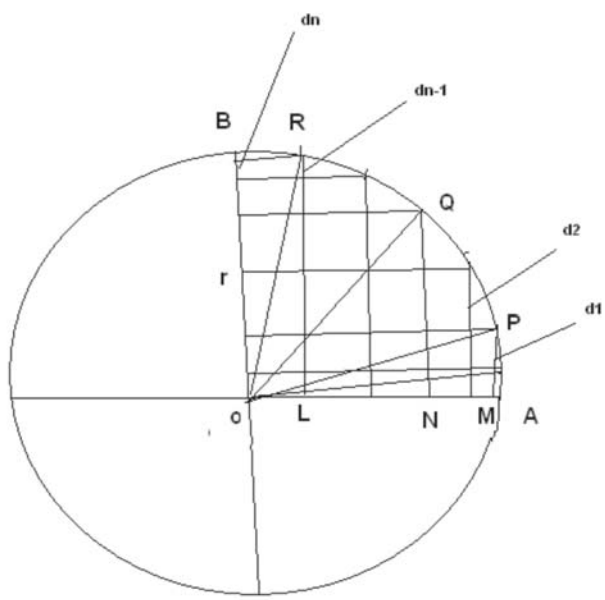

Fig 2. r sine differences, first order

taking into account b_1, b_2,…..b_{24} as r sines and d_1 ,d_2 ,…..d_{24} as 24 corresponding r sine differences, when $d_1 = b_1$, $d_2 = b_2 - b_1$,…$d_{n+1} = b_{n+1} - b_n$, and r (radius)= $d_1 + d_2 + \ldots + d_n$. Āryabhaṭa I gave a rule to calculate d_1, d_2 ,….d_n which runs thus:

(1) $d_{n+1} = b_1 - 1/\, b_1[\Sigma\, b_n]$ where n=1,2,…n. (ĀBh of Āryabhaṭa I);

(1.1) $d_{n+1} = d_n - 1/\, b_1\, [\Sigma\, d_n]= d_n - b_n/b_1$ (Parameśvara's comm. on ĀBh);

(1.2) $d_{n+1} = d_n - b_n/b_1\, (d_1\text{-}d_2)$ (Nīlakaṇṭha's comm. on ĀBh);

Siddhāntic astronomers[15], adopted various values of r as

r = 3438´ (Āryabhaṭa I, Sūrya, Lalla, Sumati, Āryabhaṭa II, Bhāskara II);

= 120´ (Varāhamihira, Bhāskara II, Brahmadeva);

= 150´ (Brahmagupta, Lalla);

= 3437´ 44´´ 19´´´ (Govindasvāmī, Udaydivākara);

= 3437´ 44´´ (Vaṭeśvara, Parameśvara); and so on.

Varous methods were used by these astronomers to calculate the r sine values. Bhāskara II, however, had applied the 2nd order differences for the purpose. He knew that 1st order r sine differences slowly decrease for lower to higher arc but the 2nd order differences increase slowly[15] .

However, Mādhava used a more refined value

$$r = \frac{21600 \times 10^{10}}{2 \times 31{,}41{,}59{,}26{,}536} = 3437' \, 44'' 48'''.$$

The corresponding 24 sine values at an interval of 3^o 45´in the 1st quadrant, as given by Mādhava, are as follows[16]:

(1) 224´ 50´´ 22´´´	(2) 448´ 42´´ 58´´´	(3) 670´ 40´´ 16´´´
(4) 889 45 15	(5) 1105 01 39	(6) 1315 34 07
(7) 1520 28 35	(8) 1718 52 24	(9) 1909 54 35
(10) 2092 46 03	(11) 2266 39 50	(12) 2430 51 15
(13) 2584 38 06	(14) 2727 20 52	(15) 2858 22 55
(16) 2977 10 34	(17) 3083 13 17	(18) 3176 03 50
(19) 3255 18 22	(20) 3320 36 30	(21) 3371 41 29
(22) 3408 20 11	(23) 3430 23 11	(24) 3437 44 48 (=R)

These values are correct to 8 to 9 places of decimals. How Mādhava arrived at such accurate values?

Mādhava's sine table (*jyā* table or value of half sine chord) was given aslo for a quarter circle drawn at 24 equal intervals of the arc at 225´, 450´, 675´…5400´

Sine and Cosine Series

Yuktibhāṣā has quoted the relevant passages containing sine and cosine series, as available in the *Tantra-saṅgraha* of Nīlakaṇṭha. It has left distinct hints that the results contained in the lines were of Mādhava. It says,

> “Multiply the arc by the square of itself (multiplication being repeated any number of times) and divide the result by the product of the square of even numbers increased by that number and Square of the radius (the multiplication being repeated same number of times). The arc and the results obtained from above are placed one below the other and are subtracted systematically one from its above. These together give the *Jivā* (*r* sin *θ*) collected here as found in the expression beginning with ***vidvān*** etc. Multiply the unit (i.e. radius) by the square of the arc (multiplication being repeated any number of times) and divide the result by the product of square of even number decreased by that number and square of the radius (multiplication being repeated same number of times). Place the results one below the other and subtract one from its above. These together give the ***śara*** (*r*—*r* cos *θ*) collected here as found in the expression beginning with ***stena***”.[17]

If t_n and t'_n be the *n*-th expression for *jivā* and *śara*, then for a small are s and radius r, the rule says,

$J\bar{\imath}va = (s-t_1) + (t_2 - t_3) + (t_4 - t_5) + \ldots\ldots..,$ where

$$t_n = \frac{s^{2n}.s}{(2^2+2)(4^2+4).....\left[(2n)^2+2n\right].r^{2n}} \left(n=1,\ 2,\ 3.......\right).$$

The successive terms t_1, t_2, t_3……. are,

$$t_1 = \frac{s^3}{3!r^2}, t_2 = \frac{s^5}{5!r^4}, t_3 = -\frac{s^7}{7!r^6}, t_4 = \frac{s^9}{9!r^8},.......$$

$$J\bar{\imath}va = s - \frac{s^3}{3!r^2} + \frac{s^5}{5!r^4} - \frac{s^7}{7!r^6} + \frac{s^9}{9!r^8} - \frac{s^{11}}{11!r^{10}} + ... \quad .(1)$$

When s = r *θ*, the eqn (1) reduces to

$$J\bar{\imath}va = \sin\ \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \frac{\theta^9}{9!} - .... \quad .....(2)$$

Again, as per rule, *Śara* = $(t'_1 - t'_2) + (t'_3 - t'_4) + \ldots\ldots$

where,

$$t'_n = \frac{s^{2n} .r}{(2^2-2)(4^2-4)\ldots[(2n)^2-2n] . r^{2n}} \quad (n=1,2,3\ldots)$$

So, *Śara* = $r - r\cos\theta = (t'_1 - t'_2) + (t'_3 - t'_4) + \ldots\ldots$

Or, $r\cos\theta = r - t'_1 + t'_2 - t'_3 + t'_4 - t'_5 + \ldots$

The successive terms t_1', t_2', t_3', …. are :

$$t_1' = \frac{s^2}{2!\,r},\ t_2' = \frac{s^4}{4!\,r^3}, \ldots.\ t_6' = \frac{s^{12}}{12!\,r^{11}} \ldots\ldots.$$

$$r\cos\theta = r - \frac{s^2}{2!r} + \frac{s^4}{4!r^3} - \frac{s^6}{6!r^5} + \frac{s^8}{8!r^7} - \frac{s^{10}}{10!r^9} + \frac{s^{12}}{12!r^{11}} - \ldots\ldots\ldots\ldots (2)$$

When $s = r\theta$, the eqn (2) reduces to

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} - \ldots\ldots\ldots\ \ldots (3)$$

Fortunately the passages beginning with *vidvān* and *stena* referred to in the above verses have been preserved in both *Āryabhaṭīyabhāṣya*[18] of Nīlakaṇṭha and *Karaṇapaddhati*[19]. In the former it has been clearly stated that the values of the first five terms t_5, t_4, t_3, t_2, t_1 of the eqn (1) and of t'_6, t'_5, t'_4, t'_3, t'_2 and t'_1 of eqn (2) were given by Mādhava (*evāhamādhavaḥ*) when s = 5400´ and r = 3437´ 44´´43´´´. The values are : *vidvā* (=44"′ = t_5), *tuna bala* (= 33" 6"′ = t_4), *kavīśanicaya* (= 16′5" 41"′=t_3), *sarvārthaśilasthiro* (= 273′57" 47"′= t_2), *nirvirdhānga – narendrarung* (=2220′39" 40"′= t_1) and *stena* (= 6"′ = t_6′′), *stripiśuna* (= 5" 12"′ = t_5′) *sugandhinaganud* (= 3′ 9" 37"′ = t_4′), *bhadrangabhavyāsana* (= 71′ 43"24 = t_3′), *mīnānganarasimha* (= 872′3"5"′ = t_2′), *unadhanakrtbhūreva* (= 4241′9"0" = t_1).

These values when substituted in eqn (1) containing terms from t_1 to t_5, *jīvā* comes out to be 3437′44"48"′, the 24th sine value given in the table of Mādhava (here s = 5400′). (This means that Mādhava gave a general rule for sine and cosine series which were verified in later scholars dividing the arc into six divisions and adding the *bhujāntaras* or *bhuja* differences).

When s is replaced gradually by 225′, 450′, 675′ …… Mādhava's sine table is obtained. In a similar way, the values with *stena*, when substitutes in eqn. (2), the versed or cosine table is obtained. This is evident, as adduced by the authors of *Tantrasaṃgraha* and *Karaṇapaddhati*, that the eqns. (1) and (2) were used for the computation of the sine and versed sine or cosine tables.

How Mādhava arrived at the equations (1) and (2) is not yet clearly known. The *Tantrasaṃgraha*[20] of Nīlakaṇṭha and *Karaṇapaddhati*[21] have both given indications how step by step corrections for small arc could be executed.

The *Yuktibhāsā*, however, has given the complete rational of the eqns (1) and (2). Its author Jyeṣṭhadeva [22](c. 1500-1600), in an effort to find an expression for the difference between any arc and its sine chord, divided the circumference of the quarter of a circle into n equal divisions and considered the first and second sine differences. He then found the sum of the first n sine differences and cosine differences by considering all sine chords to be equal to corresponding arc and the small unit of the circumference to be equal to one small unit. A few scholars including including Ramasubramnian, Srinivas and Sriram have tried to compute in detail the 1st and 2nd differences of Rsines, Rvers sines and Rcosines, which though interesting is avoided for any further discussion.

Only one point to stress is that the sine values are not actually equal to its arc length, so further correction was applied *ad-infinitum* to each of the terms of the values obtained for *jīvā* and *śara*, which ultimately gives rise to the eqns (1) and (2). It would be quite likely to presume that the rational was first initiated by Mādhava before Jyeṣṭhadeva could justify for its rationalistion.

In western mathematics Newton (1642-1727) is often given credit for the expansion of sine and cosine series No. (2). The result was established later algebraically on a solid foundation by De Moivre (1707-38) and Euler (1748)[23]. It is clear from the discussion that the Indian scholar Mādhava (1350-1410) used and possibly established the series (1), (2) and (3) of course in finite form before Newton, De Moivre and Euler, and laid the foundation of his sine table.

Concluding remarks

Mādhava is credited by his pupils/commentators for many important discoveries. The verses in later works are referred to as, 'that what is said by Mādhava', 'what is said here by Mādhava' (*atha mādhavena ukta, atrāha mādhava*), etc. These verses are not found in the available texts of Mādhava. It is quite suggestive that Mādhava might have written more works. Efforts might be made to explore and compile all relevant passages quoted by later scholars as well as to hunt for his other works if they are not yet lost. The mathematical computation by cowries is another important process that is emphasized by *Yuktibhāṣā* in finding the best fit for the approximation in infinite series. Warren has also reported that he met a brāhmin on the south Indian sea-beach who could calculate the eclipses in much faster than European could do at his time. My only plea is that some attempt should be made to compile some information on the lost technique of the cowries used in mathematical operation. There is no doubt that the method is different what was used in abacus by the Chinese.

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Notes

1. Datta, 1926; Gupta, 1975; Datta, Singh and Shukla, 1980; Yano,1989.
2. *Tattvārthādhigama-sūtra* with *Bhāṣya* of Umāsvāti (c,150 AD),iii.11; *Jambūdvīpa-samāsa*, iv.9; *Vṛhatkṣetra-samāsa* of Jinabhadra Gani, i.7; Brāhmasphuṭasiddhānta, xii.40;Triśatikā, rule 45;Gaṇitasārasamgraha,vii.60; Mahāsiddhānta, xv.88; See also.Datta, 1929,p. 131; see also, 1926, pp.25-43. The result was known earlier to Chinese scholar Tsu Chhung Chih (430-501 AD).
3. *ĀBh*,ii.10; *ŚiDVṛ* ,i.1-2, ii.3; *Bṛhat Sam* comm., p.53; *vyāse bhanandāgni* (3927), *hate vibhakte khabāṇasūryai* (1250), *paridhiḥ susūkṣmaḥ* (*Līlāvatī*, rule 199).
4. *Mahāsiddhānta* of Āryabhaṭa II, xv.92f; Śripati recognized both 22/7 and Āryabhaṭa I's value of 3927/1250 (Līlāvatī, p.34); Bhāskara also used both the values and says that '*dvāvimśati*(22)*ghne vihṛte'tha śailaih* (7) *sthūlo'thavā syād vyāvahārayogyah* (*Līlāvatī*, rule 199; Kriyākramakarī, p.399).The value was known to the Greek scholar Archimedes (c.250 BC).

5. *trīśa* (311)*ghnacakraliptā vyāso'rtheṣvagni* (553) *bhirhṛtah* (*Tantrasamgraha*,TSS 188,i.7); *vyāse śareṣvagni* (553)*hate vibhakte rāmendurūpai* (311) *paridhih susū kṣmah—kecid punaratraivāsannataram paridhimuddiśya pāthāntaram vyadhuh*, says Śankara(*Kriyākramakarī*, p.377). Nārāyaṇa says, "Divide the given perimeter into 710 parts, with 113 of them as the radius describe a circle and thus construct the circular temple" (*Tantra-samuccaya*, ii.65, ed T. Ganapati Sastri, TSS, 1919).

6. Śankara says, *kecid punah sūkṣmatamam paridhi pathitavantah—vṛttavyāse hate nā gavedavahṇyabdhikhendubhih*(104348)/ *tithyaśvivibudhe*(33,215)*bhakte susūkṣmah paridhirbhavet* //(*Kriyākramakarī*, p.377).

7. Quoted by Nīlakaṇṭha in his comm on ĀBh.ii.10, TSS 101, Trivandrum,1930; Śankara also says that Mādhava has given a more closer approximate value of circumference (*mādhavācāryai punarata api āsannatamām paridhi samkhyān uktavān*) when the diameter is known as:

   bibudhanetragajāhihutāśana triguṇavedabhavāraṇabāhava /

   navanikharvamite vṛttivistare paridhimānamidam jagadurbudhāḥ //

   '*Bibudha* (gods, 33), *netra* (eyes, 2), *gaja* (elephant, 8), *ahi* (serpents, 8), *hutāśana* (fires, 3), *tri* (three, 3), *guṇa* (qualities, 3), *veda* (*samhitās*, 4), *bha* (*naksatras*, 27), *vāraṇa* (elephant, 8), *bāhava* (arms, 2) [2,827,433,388,233], may be accepted by intelligent people of the world as the measure of the circumference when the diameter of a circle is nine *nikharva* (9×10^{11})'.(quoted in *Kriyākramakarī* comm. of Śankara). Among all the values adduced by Sankara, Mādhava's value is the best which is correct to 11 places of decimals.like fixing the value of radius of a circle with great accuracy (correct to 11 places).

8. *Kriyākramakarī* comm. on *Līlāvatī*, ed by K.V.Sarma, VVRI, Hoshiarpur, 1975, p.377

9. *Kriyākramakarī* comm. on *Līlāvatī*, ed K.V.Sarma,p.379.

10. *Tantrasaṃgraha*. ed. K.V. Sarma, comm. in verse, p.101, vss.271-274,

11. *vyāse vāridhinihate rūpahṛte vyāsasāgarābhihate /*

    triśarādiviṣamasamkhyābhaktamṛṇam svam pṛthak kramāt kuryāt //

    yatsamkhyayā'tra haraṇe kṛte nivṛttā hṛtistu jāmitayā /

    tasyā urdhagatāyā samasamkhyā taddalam guṇo'nte syāt //

    Nīlakaṇṭha indicates in his *Tantrasaṃgraha* comm.(vss. 271-74), vide K.V.Sarma's edition, p.101 that the rule is of Mādhava;. Quoted also in *Kriyākramakarī* comm. on the Līlāvatī, p.379; this popular verse also appears in the *Tantrasaṃgraha* Mss (ii.1-3) of Tripunittura Sanskrit College Library and Adyar Library; tr by Datta, Singh and Shukla, 1980, p. 156.

12 *tadvargo rūpayuto hāro vyāsābdhighātatah prāgvat /*

tābhyāmāptam svamṛṇe kṛte dhane kṣepa eva karaṇīya //

labdhah paridhih sūkṣmo bahukṛtvo haraṇato'tisūkṣmaḥ syāt //

Nīlakaṇṭha indicates in his *Tantrasaṃgraha* comm. vss. 271-74,(vide K.V. Sarma's edition, p.101) that the rule is of Mādhava;. Quoted also in *Kriyākramakarī* comm. on the Līlāvatī, p.379 ; tr by Datta, Singh and Shukla, 1980, pp.156; the *Tantrasamgraha* Mss (ii.4-5 1/2) of Trippunitura College Library and the Adyar Library has noted the same correction in different wordings, which runs thus:

asmāt sūkṣmataro'nyo vilikhyate kaścanāpi samskāraḥ /

ante samasamkhyā-dala-vargassaiko guṇassa eva punaḥ //

yugaguṇito rūpayutassamasamkhyādalahato bhavedhāraḥ /

13. Yuktibhāṣā, 2008, p.74

14. Sadratnamālā, vide John (2002), pp.181-82

15. Bag, 1979, pp.256-57.

16. *śreṣtham nāma variṣṭhānām*(22 05 4220 i.e.0224´50´´22´´´), *himādrirvedabhāvanaḥ* (85 24 8440 i.e. 0448´42´´58´´´), *tapanobhānusūktajño* (61 04 0760 i.e. 670´40´´16´´´),…. *devo viśva sthalībhṛguh* (84 44 7343 i.e.3437´44´´48´´´)// *tatparā dikalāntāstu mahājyā mādhavoditā* i.e. rhe values are for the correspondinding arc of 225´, 450´,….5400´, as said by Mādhava.

[*Tantrasaṃgraha,* ed S.K.Pillai,TSS 188,p.19, Trivandrum, 1958; Rai, R.N., "Sine Values of *Vaṭeśvarasiddhānta*", *IJHS*,7.1(1972)11; Bag, A.K.(1976),p.54].

17. *nihatya cāpa vargena cāpam tattatphalāni ca|*

haret samūlayugvargaistrijyāvargahataih kramāt||

cāpam phalāni cādhodhonyasyoparyupari tyajet|

jivāptyai, sangraho 'syaiva vidvān-ityādinakrtaḥ||

nihatya cāpavargena rūpam tattatphalāni' ca|

hared vimulayugvargaistrijyāvargahataih kramāt||

kintu vyāsadalenaiva dvighnenādyam vibhājyatām|

phalānyadhodhah kramaśo nyasyoparyupari typajet ||

śaraptyai, sangraho 'syaiva stenastrîtyādinā krtaḥ |

Vide *Yuktibhāsā,* Pt. I, edited with notes by Ramavarma (Maru) and Tampuran and A.R. Akhilesvara Iyre, Trichure, 1948, p. 190; Bag. 11.1 (1976) 54-57.

18. *Āryabhatīyabhāsya* of Nīlakaṇṭha, TSS 101, p. 113; Ganita-*Yuktibhāṣā,*1948 p. 145.

19. *Karaṇapaddhati*, vi. 14-15.

20. *Tantrasaṃgraha* of Nīlakaṇṭha, 1958 ed, ii. 12 $\frac{1}{2}$

21. *Karaṇapaddhati*, vi. 19

22. Gaṇita Yuktibhāṣā, 2008, pp.221-26.

23. *Chamber's Encyclopaedia,* 10, p. 227, 1935, Cajori, F. *A History of Mathematics,* 2nd revised and enlarged edition, p. 206

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