Indian Journal of History of Science, 47.4 (2012) 719–736

RELATION BETWEEN THE ARC AND THE RSINE IN *TANTRASAṄGRAHA* AND OTHER KERALA WORKS

M. S. Sriram[*], K. Ramasubramanian[†]
and R. Venketeswara Pai[‡]

(Received 9 April 2012; revised 27 July 2012)

In his *Tantrasaṅgraha*, Nīlakaṇṭha Somayājī has given a method for determining the arc (*cāpa*) corresponding to a given Rsine (*bhujā*), when both are small, using an iterative procedure. Nīlakaṇṭha also gives a method for finding the arc length for small Rsines, when the difference between the arc and the Rsine (*bhujā-cāpāntara*) is equal to an integral number of seconds of arc. These are described in greater detail in *Laghuvivṛti* and *Yuktidīpikā*—commentaries on *Tantrasaṅgraha*—and also in *Karaṇapaddhati* of Putumana Somayājī. In this paper, we discuss these methods of finding the (small) arc, given the Rsine.

Key words: *Bhujā* (Rsine), *Cāpa* (arc), *Bhujācāpāntara*, Iterative method.

[*]Department of Theoretical Physics, University of Madras, Guindy Campus, Chennai – 600 025; Email: *sriram.physics@gmail.com*

[†]Cell for Indian Science and Technology in Sanskrit, Department of Humanities and Social Sciences, IIT Bombay, Mumbai – 400 076; Email: *kramas@iitb.ac.in*

[‡]Amrita Vishwa Vidyapeetham, Amritapuri Clappana P O, Kollam – 690525; Email: *venpai79@gmail.com*

Introduction

The methods for finding the sine function for arbitrary angles form an important part of any Indian astronomy text, as most of the astronomical computations involve this function. Most of the texts provide the tables for sines at regular intervals, typically $3°45' = \frac{90°}{24}$. The sines for angles between two succesive multiples of $3°45'$ are to be found by interpolation.[1] After the celebrated Kerala astronomer Mādhava discovered the infinite series for sine and cosine functions, the Kerala texts on astronomy also use the sine series upto some terms, to obtain the sine of any angle.[2] For obtaining a reasonably accurate value of the sine of a small angle, or difference between the sines of two angles close to each other, the first two terms in the so called Maclaurin/Taylor series for the sine (that is, upto the cubic term) are adequate for most purposes. These are discussed in the second chapter of *Tantrasaṅgraha*,[3] where several approximate methods for computing the sines are presented.

One of them involves the inverse problem of finding the arc corresponding to a given Rsine, when both are small, using an iterative procedure. We discuss this method and compare it with the results obtained from the Maclaurin series method in the next two sections. In section 4, we discuss the determination of the Rsine and thereby the arc when the difference between them has a small specified value. We make a few concluding remarks in section 5.

Obtaining the arc (*cāpa*) from the Rsine (*bhujā*) using an iterative method

Verse 17 in chapter 2 (*Sphuṭaprakaraṇam*) of *Tantrasaṅgraha* prescribes a certain method for finding Rsine (*bhujā*) value of an arc (*cāpa*) which is small:[4]

शिष्टचापघनषष्ठभागतो विस्तरार्धकृतिभक्तवर्जितम्।
शिष्टचापमिह शिञ्जिनी भवेत् स्पष्टता भवति चाल्पतावशात्॥

śiṣṭacāpaghanaṣaṣṭhabhāgato vistarārdhakṛtibhaktavarjitam |
śiṣṭacāpamiha śiñjinī bhavet spaṣṭatā bhavati cālpatāvaśāt ||

Divide one-sixth of the cube of the remaining arc by the square of the *trijyā*. This quantity when subtracted from the remaining arc becomes the *śiñjinī* (the *dorjyā* corresponding to the remaining arc). The value is accurate because of the smallness [of the arc].

For the inverse problem of finding the arc corresponding to a given Rsine an iterative procedure is discussed in verse 37 in the same chapter:[5]

ज्याचापान्तरमानीय शिष्टचापघनादिना ।
युक्त्वा ज्यायां धनुः कार्यं पठितज्याभिरेव वा॥

jyācāpāntaramānīya śiṣṭacāpaghanādinā |
yuktvā jyāyāṃ dhanuḥ kāryaṃ paṭhitajyābhireva vā ||

The arc corresponding to a *jyā* may be obtained either by finding the difference between the *jyā* and the arc as given in the verse [beginning] *śiṣṭacāpaghana* etc., and adding that (difference) to the *jyā*, or from the table of *jyās* listed earlier.

In Figure 1, let PN represents the *jyā* whose corresponding arc length AP is to be determined. The radius of the circle R (*trijyā*) is given by $R = \frac{21600}{2\pi}$ since the circumference is taken to be 21600—the number of minutes corresponding to 360°. If $A\hat{O}P = \theta$, then the *jyā* corresponding to this angle is given by

$$jy\bar{a} = PN = l = R\sin\theta. \qquad (1)$$

When θ is small

$$\sin\theta \approx \theta - \frac{\theta^3}{3!}.$$

$$\text{Or,} \qquad R\sin\theta \approx R\theta - \frac{(R\theta)^3}{6R^2}. \qquad (2)$$

This is essentially the content of the first of the quoted verses '*śiṣṭacāpa-ghana...*', For the inverse problem of finding the arc from the Rsine, note that the difference (D) between the *cāpa* (arc) and its *jyā* (Rsine) is given by

$$D \approx R\theta - l = \frac{(R\theta)^3}{6R^2}. \tag{3}$$

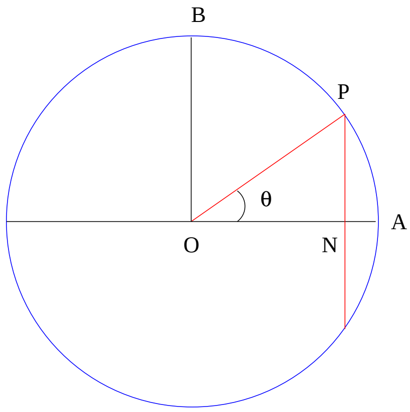

Fig. 1: Finding the arc length of a given *jyā* when it is small.

Now, given $R\sin\theta = l$, we have to find the arc, $R\theta$ from equation (3). However this has the cube of the very same arc which is to be determined. While commenting on this verse in his *Laghuvivṛti*,[6] Śaṅkara Vāriyar explains that an iterative procedure known as *aviśeṣakarma* would be employed to find the arc:

चापीचिकीर्षितां ज्यां कार्त्स्न्येन घनीकृत्य षड्भिर्विभज्य लब्धं पुनः त्रिज्याकृत्या च विभजेत्। तत्र लब्धं लिप्तादिकं ज्याचापान्तरम्। अहार्यत्वे पुनः षष्ट्या निहत्य त्रिज्याकृत्या विभजेत्। तत्र लब्धं विलिप्तादिकं ज्याचापन्तरमिति।

नन्वत्र शिष्टचापघनेत्यादिना इष्टचापतः तज्ज्याचापान्तरं क्रियते। न पुनः इष्टज्यातः तच्चापान्तरम्। सत्यम् ; अत एव अत्र अविशेषकर्म क्रियते। तद्यथा - उक्तवदानीतम् इष्टज्याचापान्तरम् इष्टज्यायां प्रक्षिप्य पुनरपि तद्घनतः पूर्ववदानीतं ज्याचापान्तरं मुहुराद्यज्यायामेव प्रक्षिपेत् यावदविशेषः।

अविशिष्टेन ज्याचापान्तरेण युक्ता इष्टज्या चापीकृता स्यादिति।

Find the cube of the given *jyā* and divide it by six. This may further be divided by the square of the *trijyā*. The result is the difference between the *jyā* and *capā* in minutes. If it is not divisible [if there is a fraction], then it has to be multiplied by 60 and then divided by the square of the *trijyā*. The result thus obtained will be the difference between the *jyā* and the *cāpa* in seconds.

Is it not true that, as per the procedure described in [the verse] *śiṣṭacāpaghana* . . . , we find the difference between the *jyā* and *capa* from the given (known) *cāpa* and not from the given *jyā*? Yes, it is true. It is only because of this, that an iterative procedure (*aviśeṣakarma*) is followed here where the difference between the *jyā* and *cāpa* is to be found from the given *jyā*. It is as follows: The difference between the *jyā* and *cāpa* obtained as described earlier must be applied to the given *jyā* and from the cube of that the [next approximation to the] difference between the *jyā* and *cāpa* must be determined. This again has to be applied to the given *jyā*, and the process has to be repeated till the result becomes *aviśiṣṭa* (not different from the earlier). This difference added to the given *jyā* will be the required *cāpa*.

Karaṇapaddhati of Putumana Somayājī,[7] composed around 1730 AD also discusses this method and the iterative procedure, in verse 19, chapter 6 of the text:

स्वल्पचापघनषष्ठभागतो विस्तरार्धकृतिभक्तवर्जितम्।
शिष्टचापमिह शिञ्जनी भवेत् तद्युतोऽल्पकगुणोऽसकृद् धनुः॥

svalpacāpaghanaṣaṣṭhabhāgato vistarārdhakṛtibhaktavarjitam |
śiṣṭacāpamiha śiñjanī bhavet tadyuto'lpakaguṇo'sakṛd dhanuḥ ||

The cube of a small arc should be divided by six and obtained result is [further] divided by the square of the radius. The

> remainder obtained [by subtracting this from the arc] gives the Rsine (*śiñjanī*) [of the arc]. The Rsine of the small arc added to that (the cube of the arc divided by the square of the radius multiplied by six) would be the arc when the process is repeatedly done.

The iterative procedure referred to in *Laghuvivṛti* and *Karaṇa-paddhati* can be described as follows. It is known that for a small arc

$$\begin{aligned} R\sin\theta &= R\theta - \frac{(R\theta)^3}{3!R^2} \\ \text{or,} \qquad R\theta - R\sin\theta &= \frac{(R\theta)^3}{6R^2} \quad (= D \text{ say}). \end{aligned} \tag{4}$$

In the first step of iteration, the text prescribes to replace the arc $R\theta$ by $R\sin\theta$ in the RHS, so that the arc-Rsine difference D is taken to be

$$D \approx D_1 = \frac{(R\sin\theta)^3}{6R^2}, \tag{5}$$

and the arc $R\theta_1$, in the first iteration is given by:

$$\begin{aligned} R\theta_1 &= R\sin\theta + D_1 \\ &= R\sin\theta + \frac{(R\sin\theta)^3}{6R^2}. \end{aligned} \tag{6}$$

This value of $R\theta_1$, is used to find the difference between the arc and $R\sin\theta$ in the next stage of iteration:

$$D \approx D_2 = R\theta_1 - R\sin\theta = \frac{(R\theta_1)^3}{6R^2}, \tag{7}$$

and the arc $R\theta_2$ in the second iteration is given by:

$$\begin{aligned} R\theta_2 &= R\sin\theta + D_2 \\ &= R\sin\theta + \frac{(R\theta_1)^3}{6R^2}. \end{aligned} \tag{8}$$

Proceeding along the same line to obtain the higher iterates,

$$R\theta_i = R\sin\theta + \frac{(R\theta_{i-1})^3}{6R^2}. \tag{9}$$

Dispensing with the common factor R in these expressions, we obtain the following expressions for the angle in the first, second and third steps of the iterative procedure:

$$\begin{aligned} \theta_1 &= \sin\theta + \frac{(\sin\theta)^3}{6} \\ \theta_2 &= \sin\theta + \frac{(\theta_1)^3}{6} \\ &= \sin\theta + \frac{(\sin\theta)^3}{6} + \frac{(\sin\theta)^5}{12} + \frac{(\sin\theta)^7}{72} + \frac{(\sin\theta)^9}{1296} \end{aligned} \tag{10}$$

and

$$\begin{aligned} \theta_3 &= \sin\theta + \frac{(\theta_2)^3}{6} \\ &= \sin\theta + \frac{(\sin\theta)^3}{6} + \frac{(\sin\theta)^5}{12} + \frac{(\sin\theta)^7}{18} + \mathrm{O}(\sin\theta)^9, \end{aligned} \tag{11}$$

where we don't write the higher order terms explicitly. In fact, it can be seen that the coefficients of $\sin\theta$, $\sin^3\theta$, $\sin^5\theta$ and $\sin^7\theta$, are not altered from those appearing in the expression for θ_3 when we compute expression for θ in higher orders of iteration, so that we have

$$\theta = \sin\theta + \frac{(\sin\theta)^3}{6} + \frac{(\sin\theta)^5}{12} + \frac{(\sin\theta)^7}{18} + \ldots \tag{12}$$

This is the series for the angle θ in terms of $\sin\theta$, obtained using the method of iteration which is implied in the text *Tantrasaṅgraha*, and which is explicitly stated in its commentaries *Laghuvivṛti* and *Yuktidīpikā*, as well as in *Karaṇapaddhati*. It may also be mentioned here that the iterative method described here is an algebraic method for finding a numerically better root of an equation even in modern times.

Comparison of Maclaurin series method and the Iterative method

The Maclaurin series expansion for θ in powers of $\sin\theta$:[8]

$$\theta|_{\text{Maclaurin}} = \sin\theta + \frac{1}{6}(\sin\theta)^3 + \frac{3}{40}(\sin\theta)^5 + \frac{5}{112}(\sin\theta)^7 + \ldots \tag{13}$$

There does not seem to be a Maclaurin expansion for $sin^{-1}\theta$ in the Kerala school, like the series for $tan^{-1}x$ attributed to Mādhava, and elaborated in several texts. Though the method described in the Kerala texts for finding the arc from the Rsine is valid for small θ only, since the term upto the cubic power alone appears in the expansion for $\sin\theta$, it would be interesting to use it on the whole range of $0 - 90°$ for θ. In Table 1, we compare the computed values of θ from the Maclaurin series to different orders, as also the first, second and the third iterates for θ from the iterative procedure described above.

$\sin\theta$	upto $(\sin\theta)^3$ term in Maclaurin series, as well as the first iterate and the error	upto $(\sin\theta)^5$ term in Maclaurin series and the error	second iterate in TS and the error	upto $(\sin\theta)^7$ term in Maclaurin series and the error	3rd iterate in TS, and the error	exact value of θ in radians; value in deg. in paranthesis
0.1	0.10016667 (0.00075%)	0.100167417 (0.000004%)	0.1001675014 (0.008%)	0.1001674211 (0.0000001%)	0.100167508 (0.00008%)	0.100167421 (5.7392°)
0.3	0.3045 (0.0632%)	0.30468225 (0.0034%)	0.3047055527 (0.0042%)	0.3046920134 (0.002%)	0.304715089 (0.009%)	0.30469264 (17.4576°)
0.5	0.5208333 (0.5282%)	0.523177083 (0.0805%)	0.523547514 (0.0098%)	0.5235258557 (0.014%)	0.52391757 (0.06%)	0.523598776 (30°)
0.7	0.75716667	0.769771917	0.7723474468	0.7734484479 (0.25%)	0.776786858 (0.18%)	0.775397497 (44.4270°)
0.9	1.0215	1.06578675	1.077649448	1.08713929 (2.9%)	1.108584139 (1%)	1.119769515 (64.1581°)
0.95	1.09289583	1.150929404	1.167563011	1.182105176 (5.68%)	1.215270975 (3%)	1.253235838 (71.8051°)
1	1.16666667	1.241666667	1.264660494	1.286307524 (18.12%)	1.337109201 (14.8%)	1.570796327 (90°)

Table 1: Comparison between the Mādhava's iterative method and the Maclaurin series expansion.

It is seen that the iterative method works remarkably well even for large angles. For $\sin\theta = 0.5$ which corresponds to an exact value of θ (in radian) $= 0.8235987756$ (30°) the answer from the iterative method (upto third order) has an error of 0.18%, and even for a high value of $\sin\theta = 0.9$, that is $\theta = 1.119767515$ radians (64.1581°), the error is only 1%. In general, for low values of θ (even upto 30°) the Maclaurin series method is more accurate than the iterative method, as seen from the first three rows of the table. However, for the higher values of the angle, the iterative method is more accurate than the Maclaurin series method.

The arc and the Rsine for a fixed difference between them

Apart from the iterative procedure described above, *Tantrasaṅgraha* also gives an ingenious way to obtain the Rsine (*bhujā*) and thereby the arc when the difference between them has a small specified value, in verses 38, 39 of chapter 2 of the text:

त्रिखरूपाष्टभूनागरुद्रैः त्रिज्याकृतिः समा।
एकादिघ्नया दशाप्ता या घनमूलं ततोऽपि यत्॥
तन्मित्ज्यासु योज्याः स्युः एकद्व्याद्या विलिप्तिकाः।
चरदोःफलजीवादेः एवमल्पधनुर्नयेत्॥

trikharūpāṣṭabhūnāgarudraiḥ trijyākṛtiḥ samā|
ekādighnayā daśāptā yā ghanamūlaṃ tato'pi yat||
tanmitjyāsu yojyāḥ syuḥ ekadvyādyā viliptikāḥ|
caradoḥphalajīvādeḥ evamalpadhanurnayet||

The square of *trijyā* is 11818103 (in minutes). Multiply this by 1, 2 etc., divide by 10 and find the cube roots of these results. If the *jyā* (whose arc is to be found) has a measure equal to these (the above cube roots), then 1, 2 etc., seconds have to be added to them. Thus the arc of the R sine of small angles involved in the *caradoḥphala* may be obtained.

The procedure is explained in the commentary *Laghuvivṛti* as follows:

अथवा एकद्व्यादिविलिप्तारूपं यज्ज्याचापान्तरं तद्विधायिनीः बह्वीः जीवाः पठित्वा तत्तुल्यासु अभीष्टज्यासु एकद्व्यादि विलिप्तारूपं ज्याचापान्तरं इष्टज्यायां प्रक्षिप्य तच्चापं कर्तव्यमिति। तत् कथमिति चेत् -
तत्र त्रिज्यायाः कृतिः त्रिखरूपाष्टभूनागरुद्रैस्तुल्यसङ्ख्या

प्रसिद्धा। तत्र त्रिज्याकृतेः एकद्व्यादिनिहतायाः दशभिर्विभज्य लब्धात् फलात् घनमूलमानयेत्। तत्तत्तुल्यासु इष्टज्यासु क्रमादेकद्व्यादिविलिप्तिकाः ज्याचापान्तरत्वेन ग्राह्या इति। तथानीतज्याचापान्तरम् इष्टज्यायां प्रक्षिप्य चापीकरणं कार्यमिति। तद्यथा -

Or if the difference between the *jyā* and the arc length is equal to 1″, 2″, 3″ etc. then construct the table listing the *jyās* corresponding to these differences. If the *jyā* whose *cāpa* is to be determined happens to be (very close to) one of the values listed in the table, then add this difference between *jyā* and *cāpa* (1″, 2″, 3″ etc.) to the *jyā* to get the required *cāpa*. If you ask, how this should be implemented –

It is well known that the square of the *trijyā* = 11818103. Multiply this by 1, 2, 3 etc., divide by 10, and take the cube roots of the resulting quantities [in minutes etc]. If the *jyā* whose *cāpa* is desired to be found happens to be one of the values [listed in the table], then it is to be understood that the corresponding difference between the *jyā* and *cāpa* is going to be only 1″, 2″, 3″, etc. The difference between the *jyā* and *cāpa*, obtained thus, may thus be added to the given *jyā* to get the desired *cāpa*. This may be done as follows.[9]

We explain the procedure in modern notation in the following. The difference between the *cāpa* and its *jyā* is given by

$$D = R\theta - l \approx \frac{(R\theta)^3}{6R^2}. \tag{14}$$

In the above equation all the quantities are expressed in minutes. When the difference $D = 1''$, which is one-sixtieth of a minute, we obtain

$$\frac{(R\theta)^3}{6R^2}. = \frac{1}{60}. \tag{15}$$

This implies that when $D = 1''$ the corresponding arc is given by

$$R\theta_1 = \left(\frac{1.R^2}{10}\right)^{\frac{1}{3}}. \tag{16}$$

Similarly when $D = 2''$, the corresponding arc is given by

$$R\theta_2 = \left(\frac{2.R^2}{10}\right)^{\frac{1}{3}}, \tag{17}$$

and so on. In general, when $D = i''$, the corresponding arc is given by

$$R\theta_i = \left(\frac{i.R^2}{10}\right)^{\frac{1}{3}}. \tag{18}$$

Here $R\theta_i$'s correspond to the arcs when the difference between the *cāpa* and the *jyā*, $R\theta_i - R\sin\theta_i$ (D) is i''. *Laghuvivṛti* gives the explicit values of the *jyās*, $l_i = R\sin\theta_i = \theta_i - i''$, in the *Kaṭapayādi* notation as *lavaṇaṃ nindyaṃ*, etc. In Table 2, we list these *jyā* values $R\sin\theta_i$ and the arcs $R\theta_i$, for $i = 1, 2, \ldots 24$. In the second column we give the values of the *jyās* in the *Kaṭapayādi* notation, whereas the third column gives the indicated numeral values. The fourth column gives the arc which is the sum of columns 1 and 2. The fifth column gives the value of the arc length as computed by using (18), which inolves the computation of the cube root of $\left(\frac{iR^2}{10}\right)$ for different values of i ($i = 1 \ldots 24$). In doing so, we have also used the exact value of the *trijyā* (in minutes), that is, $R = \frac{21600}{2\pi}$. Given the fact that some approximation in the *trijyā* value and the extraction of the cube root is involved in the computation of arc length, it is remarkable that the value of the arc as per the text differs at most by $2''$ from the exactly computed value of the arc length.

In *Laghuvivṛti*, it is stated that this method (of obtaining the arc from the looking table) is accurate only if the arc and the Rsine are small:

यद्यपि सुसूक्ष्मचापीकरणोपायः पूर्वमेव प्रदर्शितः तथापि अल्पीयस्याः जीवायाः चापीकरणमेवं कर्तव्यम्, इति इहापि प्रदर्शितम्। अत उक्तम् - चरदोःफलजीवादेः एवमल्पधनुर्नयेत् - इति।

Though the procedure for obtaining more accurate values of the arc length has already been stated, for smaller *jyās* the

The diff. D=*cāpa*–*jyā* in seconds	The given value of *jyā* in *Kaṭapayādi*	min	sec	The textual value of *cāpa* min	sec	The computed value of *cāpa* min	sec
1	*lavaṇaṃ nindyaṃ*	105	43	105	44	105	43.56
2	*kapilā gopī*	133	11	133	13	133	12.42
3	*cararāśayaḥ*	152	26	152	29	152	29.04
4	*tavārthitayā*	167	46	167	50	167	49.80
5	*laghunoddiṣṭaḥ*	180	43	180	48	180	47.34
6	*rājñaḥ pra·ḷ·ayaḥ*	192	02	192	08	192	07.02
7	*dhāmnāṃ trinetraḥ*	202	08	202	15	202	14.82
8	*narakapuram*	211	20	211	28	211	27.12
9	*savadhūṭīndraḥ*	219	47	219	56	219	55.14
10	*jalasūradrī*	227	38	227	48	227	46.80
11	*himavān guruḥ*	234	58	235	09	235	07.98
12	*triśaṅkuvaraḥ*	241	52	242	04	242	03.18
13	*varado vajrī*	248	24	248	37	248	35.88
14	*tilabhūrmeruḥ*	254	36	254	50	254	48.90
15	*kālena tatra*	260	31	260	46	260	44.58
16	*nṛpaticaraḥ*	266	10	266	26	266	24.78
17	*tilakaṃ sāndraṃ*	271	36	271	53	271	51.12
18	*dhāvati sarit*	276	48	277	06	277	04.86
19	*na me kuñharaḥ*	281	50	282	09	282	07.20
20	*nivṛttajaraḥ*	286	40	286	60	286	59.10
21	*śreṣṭhaka·ḷ·atra*	291	22	291	43	291	41.46
22	*mamāśā dhātrī*	295	55	296	17	296	14.94
23	*dhūpo'gnīnāmbu*	300	18	300	41	300	40.26
24	*tilavanagaḥ*	304	36	304	60	304	58.02

Table 2: Look-up table from which the values of arc lengths of small *jyās* can be directly read, when the difference between the *jyā* and the *cāpa* is equal to an integral number of seconds.

arc lengths may be obtained by this method (from the look-up tables). That is why it is stated: The small arc length of the *cara-doḥphala* etc. should be obtained by this method.

This procedure is summarised in *Yukti-dīpikā* in the following manner:[10]

उक्तं चापघने षड्घ्नत्रिज्यावर्गे कलासमम् ॥
दशांशे तत्कृतेः चापज्यान्तरं विकलासमम् ।
एकादिघ्नात्ततस्त्रिज्यावर्गतो दशभिर्हृतात् ॥
घनमूलं तु यल्लब्धं तत्तुल्ये धनुषि स्थिते ।
एकद्व्याद्या विलिप्ताः स्युः चापज्याविवरोद्भवाः ॥
तदूनं चापमर्धज्या तद्युता ज्या च तद्धनुः ।
कार्योऽविशेषश्चापाप्तौ चापाल्पत्वे दृढं च तत् ॥

It has been stated implicitly (in verse 17 of the text) that the difference between the *jyā* and *cāpa* will be equal to $1'$ (one *kalā*), when the cube of the arc length is equal to six multiplied by square of the *trijyā*. The same will be equal to $1''$ (one *vikalā*) when the cube of the arc length is equal to one-tenth of the square of *trijyā*.

Now, the square of the *trijyā* divided by 10 is multiplied by 1,2,3, etc. Then the cube roots of the results are taken [and stored separately]. These correspond to the arc lengths, when the difference between the *jyā* and *cāpa* is equal to $1''$, $2''$, $3''$, etc., respectively. When differences are subtracted from the arc length we get the *jyā* and when they are added to the *jyā* we get the arc length. *Aviśeṣakarma* must be done in order to get accurate results for the *cāpa* from the *jyā* whose values are small.

Here it is stated that when the arc $R\theta_i$ has the value $\left(\frac{i.R^2}{10}\right)^{\frac{1}{3}}$, the *jyā* is denoted by subtracting—– from this: $R\sin\theta_i = R\theta_i - i$. However when the *jyā*, $R\sin\theta_i$ has the value $\left(\frac{i.R^2}{10}\right)^{\frac{1}{3}}$, the arc $R\theta_i$ is not exactly equal to $R \sin\theta_i + i$. In this case, the arc has to be obtained using the iterative procedure, as described earlier.

In *Karaṇapaddhati* (verse 20, chapter 6) also, the same procedure for finding the arc/Rsine when their difference is an integral of seconds is stated:

एकद्वित्र्यादिसंख्याघ्नत्रिज्यावर्गनयांशतः ।
घनमूलं हि चापज्या स्वसंख्योनविलिप्तिकम् ॥२०॥

> Multiplying the square of the radius by the number 1, 2, 3 etc. and dividing [that] by 10 (*naya*). [let] the cube root of the resulting quantity be obtained. Those integral numbers in seconds when subtracted from this would be the Rsines corresponding to the arc.

The corresponding values of the Rsines are given in the *Kaṭapayādi* notation as *gūḍhā menakā*, etc. in the commentary of *Karaṇapaddhati*. These are listed in Table 3, only with the corresponding values of the arcs which differ from the Rsines by i seconds and the actual computed values of the arc, as in the earlier Table. In fact most of the entries are the same as in the earlier Table, except for the five entries, as indicated. It is surprising that in these cases the value are less acurate than the arcs in *Laghuvivṛti*, though *Karaṇapaddhati* is a later text.

Discussion

Indian mathematics and astronomy abound in approximations and simplifications for various expressions, that appear in the calculations. For instance, we find an interesting expression for the sine function in *Mahā-bhāskarīya* of Bhāskara I,[11] which in modern notation reads as:

$$\sin\theta = \frac{16\theta(\pi-\theta)}{5\pi^2 - 4\theta(\pi-\theta)}. \tag{19}$$

The diff. ($jy\bar{a}-c\bar{a}pa$) in seconds	The given values of *jyā* in Kaṭapayādi notation	in numerals		The textual val. of *cāpa*		The computed val. of *cāpa*	
		min	sec	min	sec	min	sec
1	*gūḍhā menakā*	105	43	105	44	105	43.56
2	*pūjyo gāṅgeyaḥ*	133	11	133	13	133	12.42
3	*candraḥ śrīmayaḥ*	152	26	152	29	152	29.04
4	*stambhaḥ sthitikṛt*	167	46	167	50	167	49.80
5	*gūḍho'hni dīpaḥ*	180	43	180	48	180	47.34
6	*prājño rādheyaḥ*	192	02	192	08	192	07.02
7	*dhanī trinetraḥ*	202	09*	202	15	202	14.82
8	*ugraḥ kukkuraḥ*	211	20	211	28	211	27.12
9	*satvadhīḥ puraḥ*	219	47	219	56	219	55.14
10	*svargaṃ surāṣṭraṃ*	227	34*	227	48	227	46.80
11	*himavān gauraḥ*	234	58	235	09	235	07.98
12	*rāmo'yaṃ vīraḥ*	241	52	242	04	242	03.18
13	*vārijaṃ bhadraṃ*	248	24	248	37	248	35.88
14	*tāṇḍavaṃ miśraṃ*	254	36	254	50	254	48.90
15	*kalau nācāraḥ*	260	31	260	46	260	44.58
16	*ājyāptiḥ kṣīrāt*	266	10	266	26	266	24.78
17	*caṇḍaḥ kesariḥ*	271	36	271	53	271	51.12
18	*dhāvati sarit*	276	49*	277	06	277	04.86
19	*umeṣṭo haraḥ*	281	50*	282	09	282	07.20
20	*adbhuto hāraḥ*	286	40	286	60	286	59.10
21	*krūrā yoddhāraḥ*	291	22	291	43	291	41.46
22	*śiśurmadhuraḥ*	295	55	296	17	296	14.94
23	*dhairyaṃ jñānāṅgaṃ*	300	19*	300	41	300	40.26
24	*tilaugho nīlaḥ*	304	36	304	60	304	58.02

Table 3: The values of small arcs and the corresponding *jyās* that can be read-off from a table when the difference between the two is equal to certain integral number of seconds, as given in the commentary of *Karaṇapaddhati*. The given values of *jyās*, which differ from those in Table 2 are indicated with an asterick mark.

This expression yields results that are correct to two decimal places for all the values of θ (in radians) in the range (0 to $\frac{\pi}{2}$). Various methods for computing the sines for small angles are described in *Tantrasaṅgraha* and its commentaries, *Laghuvivṛti* and *Yuktidīpikā*.[12] In Indian works on astronomy and mathematics, iterative methods are used in several situations. One of the earliest instances is that for finding the *mandakarṇa*[13] which involves the radius of the epicycle, which itself is proportional to the *mandakarṇa*, in the seventh century text *Mahābhāskarīya* of Bhāskara I.[14]

In this paper we have discussed an iterative procedure for finding the arc given the sine, as described in these works as well as *Karaṇapaddhati*. Though this procedure is stated to be valid only for small angles, it is reasonably accurate ($\sim 1\%$) even for any angle as large as $\sim 60°$. We have also discussed the related procedure for finding thr arc/Rsine when the difference between them is an integral number of seconds, as presented in these works.

The iterative procedure mentioned in these works is for the specific purpose of finding the arc from the Rsine, and involves a cubic equation of the form:

$$y = x - \frac{x^3}{6}, \tag{20}$$

where, y (Rsine) is a known quantity and x (arc) is the unknown quantity which has to be solved. However, this can be used for solving any cubic equation of this form, and hence it can be considered as an early attempt for finding the root of a cubic equation.

Acknowledgements: The authors would like to thank the anonymous referee for making some valuable suggestions for improving the paper.

Notes

[1]See for instance, Shukla K S and Sarma K V, 1976; Kapileswara Sastry, 1995.

[2]Sarma K V, et.al., 2008

[3]For details, see Ramasubramanian and Sriram, 2011.

[4]Ibid., p. 73.

[5]Ibid., p. 90.

[6]Pillai S K, 1958.

[7]Sambasiva Sastri, 1937; Koru P K, 1953.

[8]The so-called Maclaurin series for the expansion of $f(x)$ in powers of x appearing in a treatise of Maclaurin in 1742, had been noted as a particular case of "Taylor series" by Brook Taylor in 1715 itself. The Taylor series itself had already appeared in a work of James Gregory in 1668 and also in a work of Jean Bernouli before Taylor. See Gupta R. C., 1997 for more details.

[9]The values of the *jyās* given in the succeeding verses *lavaṇaṃ* . . . , are listed in third column of Table 2.

[10]Sarma, 1977.

[11]Shukla K S, 1960.

[12]For a discussion on different types of series for sine and cosine function in Kerala Mathematics see Plofker Kim, 2005.

[13]This term refers to the hypotenuse associated with the correction due to the eccentricity of a planet's orbit, termed as the 'equation of centre' in modern astronomy.

[14]For more details see the article by Deepak P. Kaundinya, et.al., appearing in the same issue of the journal.

Bibliography

Gupta R. C., 1997
'False Mathematical Eponyms and Other Miscredits in Mathematics', *Gaṇita Bhārāī*, Vol. 19, Nos.1–4, pp. 11–34.

Kapilesvara Sastry, 1995
(ed.) *Sūryasiddhānta* with *Tattvāmṛta* (a lucid commentary with annotation and notes in Sanskrit), Chaukhambha Sanskrit Bhavan, Varanasi.

Koru P. K., 1953
(ed.) *Karaṇapaddhati* of *Putumana Somayājī*, with Explanatory Notes in Malayalam, Cherp.

Pillai S. K., 1958
(ed.) *Tantrasaṅgraha* of Nīlakaṇṭha Somayājī, with the comm. *Laghuvivṛti*, Trivandrum.

Plofker Kim, 2008
in *Contributions to the History of Indian Mathematics*, eds. G. Emch, M. D. Srinivas and R. Sridharan, Hindustan Book Agency, New Delhi.

Ramasubramanian K. and Sriram M. S., 2011
Tantrasaṅgraha of Nīlakaṇṭha Somayājī, with Eng. tr. and detailed Explanatory Notes, Hindustan Book Agency, New Delhi.

Sambasiva Sastri K., 1937
(ed.) *Karaṇapaddhati* of *Putumana Somayājī*, Trivandrum.

Sarma K. V., 1977
Tantrasaṅgraha of Nīlakaṇṭha Somayājī, ed. with *Yuktidīpikā* (for Chapters I–IV) and *Laghuvivṛti* (for Chapters V–VIII) of Saṅkara Vāriyar, Hoshiarpur.

Sarma K. V., et.al., 2008
Gaṇita-yukti-bhāṣā of Jyeṣṭhadeva, ed. and tr. by K. V. Sarma with Explanatory Notes by K. Ramasubramanian, M. D. Srinivas and M. S. Sriram, Hindustan Book Agency, New Delhi; (Repr. Springer 2009).

Shukla K. S., 1960
(ed.) *Mahābhāskarīya* of Bhāskara I, with tr. with Explanatory Notes, Lucknow.

Shukla K. S., and Sarma K. V., 1976
(ed.) *Āryabhaṭīya* of Āryabhaṭa, ed. with tr. and Explanatory Notes, INSA, New Delhi.