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Book Review

Taming the Unknown— A History of Algebra from Antiquity to the Early Twentieth Century

Victor J Katz and Karen Hunger Parshall, Princeton University Press, 41 William Street Princeton, New Jersey 08540 and Princeton University Press 6 Oxford Street, Woodstock, Oxfordshire, OX20 1TW; Pages 485

Charlie Chaplin in his book "My autobiography" (Penguin books, 1978) writes (Page 320): "When I realize how distorted even recent events have become, history as such only arouses my skepticism. Whereas a poetic interpretation achieves a general effect of the period. After all, there are more valid facts and details in works of art than there are in history books."

About the history of mathematics, R L Wilder in his book "Evolution of Mathematical Concepts" (Open University Press, John Wiley edition 1968) writes (page xix): "The late historian, George Sarton, considered the chief reason for studying the history of mathematics to be its humanistic value. Descriptive history, devoted to narration of significant mathematical inventions and anecdotes concerning their originators, certainly meets this criterion. But deeper understanding and broader perspective can be achieved by a study, at the same time, of the interaction between mathematics and its cultural environment."

Later in the same book Wilder adds (page 21): "Such considerations only give a hint about the complexity of the problem regarding the relations between a group of people and their culture. Evidently the culture is "something" they have inherited from their forebears; from the latter they get their languages, religions, social customs, skills, tools, and if sufficiently 'civilized', their mathematics. But this 'something' that has been passed on to them makes up their whole way of life; they not only have to live their lives in dependence on the culture they have inherited, but the only way they can make 'progress' is to work from within the framework of that culture. Moreover, the changes or improvements that they can make are limited by the state of the culture as they inherit it."

The reader having got so far in the review might wonder (with some justification) whether this review is about the contents of the book of Katz and Parshall or Raymond Wilder! We quote Wilder (and Chaplin) to emphasize that a writer of a topic such as the history of Algebra, which spans such a wide canvas needs to have knowledge of the mathematics done by several cultures and more importantly be sensitive to the conditions of the cultures at the time in which the mathematics was done. Very few people have such broad scholarship and sensitivity and it is nice that those who do like Katz and Parshall have written this book and shared their knowledge with others. Topics as basic as the History of Algebra or the History of numbers are not just about mathematics but also involve the history of humanity as a whole. To venture to write about these is quite a challenge (not to mention the hazard of being accused by nasty reviewers of inadmissible omissions!)

To give an example of the authors sensitivity, after quoting the following moving passage of al-Bīrūnī an Islamic mathematician of the 11th century (page 136): "The number of sciences is great, and it may be still greater if the public mind is directed towards them at such times as they are in the ascendancy and in general favor with all, when people not only honor science itself, but also its representatives. To do this is, in the first instance, the duty of those who rule over them, of kings and princes. For they alone could free the minds of scholars from the daily anxieties from the necessities of life, and stimulate their energies to earn more fame and favor, the yearning for which is the pith and marrow of human nature. The present times, however, are not of this kind. They are the very opposite, and therefore it is quite impossible that a new science or any new kind of research should arise in our days. What we have of sciences is nothing but the scanty remains of bygone better times."

Katz and Parshall add: "Still, it is useful to remember, when dealing with the subject of Islamic mathematics, that the mathematical enterprise in Islam lasted longer than both the era of classical Greek mathematics and the age of "modern mathematics".

When we study mathematics, as children, we learn about numbers, and how to add and multiply numbers and later about polynomials, and the corresponding operations with polynomials, and much later in the undergraduate and graduate years learn modern abstract algebra. It is to be noted that the development of numbers took forty thousand years, and that the algebra we study in the early years took a much longer time to develop than modern abstract mathematics and therefore has a fascinating history.

Mindful of this Katz and Parshall explain in Chapter 1, why they chose to write a book on this topic (page 3): "We also believe that this is a story worth telling, since it is a history very much worth knowing. Using the history of algebra, teachers of the subject, either at the school or at the college level, can increase students' overall understanding of the material. The 'logical' development so prevalent in our textbooks is often sterile because it explains neither why people were interested in a particular algebraic topic in the first place nor why our students should be interested in that topic today. History, on the other hand, often demonstrates the reasons for both. With an understanding of the historical development of algebra, moreover, teachers can better impart to their students an appreciation that algebra is not arbitrary, that it is not created 'full-blown' by fiat. Rather, it develops at the hands of people who need to solve vital problems, problems the solutions of which merit understanding. Algebra has been and is being created in many areas of the world, with the same solution often appearing in disparate times and places."

We give a brief account of some of the topics in the book. There are a large number of topics, and many mathematicians who are mentioned, but for lack of space we focus only on some topics related to the solutions of equations.

In the second chapter the mathematics of ancient Egypt and Mesopotamia is discussed with emphasis on the solution of quadratic equations via the geometric technique of completing the square.

In Chapter 3, the Geometrical Algebra of Euclid and Apollonius is discussed, and also a solution of a certain cubic due to Archimedes.

Chapter 4 is concerned with the work of Diophantus noting importantly (page 68) that Diophantus avoids dealing with irrational and negative quantities. It is also noted that Diophantus deals with the solution of a quartic equation that reduces immediately to the solution of a quadratic.

The next chapter is about the Chinese contribution to algebra and includes the Chinese contribution to the solution of simultaneous equations via a technique similar to Gaussian elimination and also contains an account of an elimination process for polynomial equations of several variables discovered by Medieval Chinese mathematicians which is reminiscent of the modern procedure of finding the resultant of polynomials. The classical Chinese Remainder theorem is also discussed.

The next chapter discusses Algebraic thought in Medieval India and contains the Indian contributions to linear and quadratic indeterminate equations and also the solution of Bhāskara II of a special quartic equation which does not reduce immediately to a quadratic equation. It would be interesting to find out more about the tradition in which this example arose.

The next chapter is about Algebra in Medieval Islam and includes work done by al-Khwārizm on quadratic equations and work done by Islamic mathematicians towards the solution of cubic equations.

The next chapter is concerned with Fibonacci and the types of equations that he considered. He too considered cubics, quartics, sextics reducible to quadratics.

The authors note that Fibonacci gives an approximate solution to a cubic not reducible to a quadratic and does not hint at how he arrived at this approximate solution. They also consider in detail the work done by Medieval Italian mathematicians like Gerardi and Dardi on cubic and higher degree equations.

Before discussing the famous solution of the cubic of Del Ferro-Tartaglia-Cardan, in Chapter 9, Katz and Parshall remark Chapter 8 (Page 192): “There was thus a centuries-long Western tradition, in Italy at least, of working with equations of degree higher than two before the celebrated sixteenth-century work of NiccoloTartaglia, Girolamo Cardano, and others.”

Chapter 10 is concerned with the Analytic geometry of Fermat and Descartes and Euler’s attempted proof of the Fundamental Theorem of Algebra.

The next four chapters describe the history of modern algebra starting with Lagrange who interpreted the solutions of the cubic by considering the symmetries of its roots, which led after the efforts of many mathematicians, like Abel, Ruffini and Gauss, to the work of Galois, who defined the Galois group of a polynomial, which was later to be interpreted as a subgroup of the permutations of its roots. Also,the work of Dedekind, who motivated by problems in Number theory defined ideals, the work of Cayley, who introduced matrices and of Grassmann, who introduced the notions of Vectors and Vector Spaces (over the real numbers) are discussed in these chapters. These basic notions which have their corresponding historical roots in the solutions of quadratic and cubic equations in one variable, the solutions in integers of indeterminate quadratic equations, and the solutions of linear simultaneous equations (problems considered by various cultures in the past), finally led to the general definition of groups, rings, ideals, vector spaces, and modules, and to Modern Abstract Algebra thanks to the pioneering work of Emmy Noether, Emil Artin and Van der Waerden and others.

The work of many other mathematicians and their contributions to Abstract Algebra has also been written about in detail in these final chapters (for example the Wedderburn theory of Non-commutative Rings, whose roots are more modern going back to the work of Hamilton and Grassmann). We however do not go into the details.

The book contains a wealth of examples that are well worth a careful study, (for example, a quadratic equation solved in an unusual manner by the Mesopotamians on Page 28). It includes some very nice photographs and pictures of mathematicians, I would have been happy to see a few more. It also quotes original sources giving the reader an idea of how difficult it must have been in earlier eras to understand mathematics in the absence of notation we take for granted.

For example when talking about Abū Kāmil (an Egyptian mathematician of the ninth century) the authors remark (Page 150): “It is important to remember, however, Abū Kāmil’s algebra, like all Islamic algebra texts of his era, was written without symbols. Thus the algebraic manipulation “made almost obvious by modern symbolism”is carried out completely verbally.”

They add a little later: “It cannot be denied that Abū Kāmil was doing algebra using words, but it also seems clear that his readers could easily have gotten lost trying to follow his involved verbal expressions.”

Many parts of Chapters 1 to 10 of the book are accessible to college students with some knowledge of mathematics. The later chapters require knowledge of abstract algebra at the undergraduate and graduate level.

We realise that whilst giving any account of so rich a topic as the evolution of Algebra writers have to make choices, and any account is bound to be incomplete. Mindful of this, we mention, the work of some Ancient Indian Mathematicians, whose inclusion would blend well with the material discussed in the book. We mention these topics also because of our personal interest in the History of Indian Mathematics and the History of modern Abstract Algebra.

We begin with an important historical figure, in the development of Indian mathematics (not mentioned in the book), namely Nārāyaṇa Paṇḍita.

As is mentioned in the book of Katz and Parshall, the development of the sine cosine and arctangent series by the Kerala School required a formula for the asymptotic sum of powers of natural numbers. That the time was ripe for the Kerala School to emerge is evidenced by the work of Nārāyaṇa Paṇḍita in the 14th century who wrote down a combinatorial identity, later written down also by Fermat from which this asymptotic sum can be easily derived. This point does not seem to be mentioned when the history of the Kerala School is discussed because possibly the mathematicians of the Kerala School do not seem to have referred to Nārāyaṇa Paṇḍita directly.

The work of Nārāyaṇa Paṇḍita represents a culmination of a tradition in Indian Combinatorics going back to Piṅgla (around 300 B.C). His work represents a turning point in the history of Indian mathematics, in that it gets abstract and developed for its own sake.

The evolution of algebra through the solution of equations is related, to the assaults that the notion of a number has had to take from the infinite, going from rational numbers to numbers that could be expressed as square roots and cube roots of “known” quantities, to algebraic numbers that could not be expressed by radicals, and finally to transcendental and ideal numbers. Already when one tries to solve general linear equations in one variable, one is led to negative numbers and the mysterious number “zero”, both of which were freely used by Indian mathematicians from the time of Brahmagupta. Also, Indian mathematicians knew that numbers in base arithmetic behaved like “polynomials”, that is they were aware of the “algebraic” nature of the decimal place value system, which made it possible for them to give simple algorithms for square and cube rootsof numbers for example found in Āryabhaṭīya.

These points are relevant to the discussion in Chapter 6 on Indian mathematics.

Secondly in the Chapter 11 of the book, where the history of the Fundamental theorem of Algebra is considered it would be interesting to trace the evolution of Euler’s attempted proof of the theorem (which was completed later by Laplace and Gauss) leading eventually to a proof of the Fundamental theorem of algebra given by

Emil Artin who used Galois theory and Sylow's theorem.

The evolution of the notion of a number from a mathematical quality, that is the length of a line segment to an abstract quantity that various operations like addition, multiplication, taking square roots and cube roots ….. can be performed has taken many centuries, as has the evolution of algebra from a tool for solving equations to an abstract subject dealing with objects that have nothing to do with numbers. This is very nicely summed up by Katz and Parshall (page 150): "These examples show that Abū Kāmil was willing to use the algebraic algorithms that had been systematized by the time of al-Khwrizm with more general numbers than rational ones. In particular, he dealt easily with square roots and square roots of expressions with square roots."

They add a little later: "Moreover, although his solutions would be sides of squares in his geometric derivation, he essentially disregarded the geometry and thought of all his solutions as "numbers", just as the words for the unknowns would suggest. It did not matter whether a magnitude was technically a square or a fourth power or a root or a root of a root. For Abū Kāmil the solution of a quadratic equation was not a line segment, as it would be in the interpretation of the appropriate propositions of the *Elements.* It was a "number", even though Abū Kāmil could not perhaps give a proper definition of that term. He therefore had no compunction about combining the various quantities that appeared in the solution, using general rules."

The authors end the book in a similar manner, with the following lines describing the classic book of Van der Waerden on algebra, and the evolution of Modern Algebra: "Van der Waerden's has proved to be a very potent conceptualization. It marked a critical shift from a notion of algebra as a means for finding the roots, whether real or complex, of an algebraic equation that, as we have seen, had held sway at least since the ninth century. The publication in 1930-31 of his compelling presentation in *Moderne Algebra* thus marked a turning point in the history of algebra. In defining a common language that would come to be shared by algebraists internationally, it marked, at least symbolically, the "moment" when algebra became *modern."*

One is reminded of the story of the evolution of the concept of number summarised by Wilder (page 35 of the book cited above): "Whether counting started in a single, prehistoric culture and spread thereafter by diffusion or developed independently in various cultures (as seems most likely), is perhaps not too important for our purposes, interesting as it may be to speculate thereon. The scarcity of knowledge about modern man's forebears has not greatly impeded the study of biological evolution; and since it seems impossible to find out *when* man developed counting, relative to his biological origin and geographical spread, we may as well get on with what we know from the archaeological and historical records. Even the use of the word 'started' seems inadmissible indeed inasmuch as counting could hardly have 'started' in either the individual or the historical sense. Even if it evolved in a simple primitive centre, it did *evolve* and, as in the case of many cultural elements, the event would allow *dating* only by a convention."

The authors Katz and Parshall are well aware that the dating of the birth of modern Algebra is similarly a convention in the sense of Wilder, hence they use the word symbolically and put the word moment in inverted commas and italicise modern.

The following quote of Hasse (Algebraic Number theory edited by Cassels and Frohlich, Academic Press 1967, pages 276- 277) describes similarly how Homological Algebra invented to describe ideas in topology was used in Number Theory: "If I have understood rightly, it was here my task to delineate for the mathematicians of the post-war generation a vivid and lively picture of the great and beautiful edifice of class field theory erected by the pre-war generations. For the sharply profiled lines and individual features of this magnificent edifice seem to me to have somewhat lost somewhat of their original splendour and plasticity by the penetration of class field theory with cohomological concepts and methods, which set in so powerfully after the war."

It is hoped that inspired by the contents of the last four chapters of this book are expanded by mathematicians and historians of Mathematics to write books focusing on the specialised topics discussed in this book like Group theory, Ring theory, Linear Algebra, Wedderburn theory,… showing how the theorems discussed in Undergraduate and Graduate Algebra (like for example the orthogonal diagonalisation of real quadratic forms discussed in the book) were proved by the mathematicians of yesteryear thus revealing the original splendour of the subject. This would be of interest to the community of mathematicians.

We have only given a glimpse of this in this review of the tangled tale narrated well by Katz and Parshall recommending heartily that it be read.

Raja Sridharan
School of Mathematics,
Tata Institute of Fundamental Research,
Homi Bhabha Road, Mumbai 400005;
Email: sraja@math.tifr.res.in