 DOI: 10.16943/ijhs/2016/v51i1/48378

What is Indian about Indian Mathematics?*

P P Divakaran**

(Received 12 April 2015; revised 12 December 2015)

Abstract

The principle and results of mathematics are universal and immutable, or so we believe: the principles are those which guide logical thought and the results emerge from their applications to abstractions encoding the world around us, primarily numbers (arithmetic) and space (geometry). There are, nevertheless, significant variations in the practice of mathematics in different cultures. Where India is concerned, several cultural traits, spanning its entire geography and history – northwest India in the earliest Vedic period to Kerala in the 16th century – can be identified. The present article is a preliminary, largely non-technical, inquiry into the roots of these unifying traits. The most pervasive influence seems to have been that of an oral and nominal mode of articulating and transmitting knowledge, in other words that of spoken language. Examples are given to illustrate how the resulting challenges to the doing of mathematics, generally considered to be an abstract and symbolic science, were accommodated or overcome, especially interesting being the impact of nominalism on decimal counting and on the notions of zero and infinity. Other topics discussed include the Indian approach to geometry and its differences from the Hellenic, the idea of proof in India and its evolution, and the flood of new ideas in 15th and 16th century Kerala which foreshadowed the modern mathematical mainstream. Some remarks are also offered on possible mutual influences across different cultures.

Key words and phrases: Cross-cultural transmission; Diagonal theorem; Excluded middle; Functions and calculus; Orality and nominalism: zero: infinity; Recursion and *saṃskāram*; *Śulbasūtra* and Euclid; Universality and cultural differentiators; *Upapatti* and *yukti*.

1. One Theorem, Three Cultures

To begin with, one has to ask what meaning beyond that of mere geographical association can be attached to the phrase "Indian mathematics" in my title. Before attempting to identify or even speak about whatever it is that is distinctively Indian in Indian mathematics, we must first be convinced that such a distinction is a meaningful one. This is an issue that has not had a great deal of focussed attention paid to it though historians of mathematics have often taken positions, generally subconsciously, on either side of the divide: 'mathematics in India' as opposed to 'the mathematics of India'. The former position accepts that the discipline of mathematics is a universally identifiable activity of the human mind, individually and collectively, the finest attestation of its capacity to reason things through minutely and precisely. It is the canonical view, for the very good reason that history largely vindicates it. Everyone may not be in agreement on a good characterisation of what this universality consists in and what its cognitive basis might be, but we all recognise the beast when we see it. To take a widely known example (and one which has special significance for India), the statement that $3^2 + 4^2 = 5^2$ is a true mathematical statement, recognised as such by even those who may not know why it has a deeper significance than say something like $3 + 4 = 7$. Related to it in no

* Based on a plenary lecture at a Seminar on "Methodological Aspects of Knowledge Production in Pre-Modern India" at the Centre for Contemporary Studies of the Indian Institute of Science, Bengaluru, 24-26 January 2015.

**Flat No. A11B, Rivera Retreat, The Vara, Ferry Road, Cochin - 683 013; Email: ppdppd@gmail.com

immediately obvious way is the equally clearly mathematical (and true) statement: In a rectangle, the area of the square whose side is its diagonal is the same as the sum of the areas of the squares whose sides are the two sides of the rectangle. The connection between the two is that, together, they imply that in a rectangle of sides 3 and 4 units, the length of the diagonal is a whole number, 5; or, equivalently, if a triangle has sides of 3, 4 and 5 units, the angle opposite 5 is a right angle. The geometric statement about areas is the diagonal theorem (those who are uncomfortable with this name can replace 'diagonal' by 'Pythagorean')[1] first formulated, in more or less the language I have employed, in the middle-late Vedic achitectural/geometric workbooks known as the *Śulbasūtra* (ca. 800 BC; Baudhāyana, Āpastamba). The numbers 3, 4 and 5 are the smallest examples of triples of whole numbers which have this geometric property and there are an infinite number of such triples. Diagonal triples go back a thousand years earlier and several thousand kilometers further west, occurring first on Mesopotamian clay tablets of the Old Babylonian period, around 1800 BC. The point for us is that this circle of ideas was seen as true and good mathematics as much in 19th century BC Mesopotamia as in 9th century BC India and 6th century BC (or perhaps a little later) Greece. There are any number of other instances in which pretty much the same mathematical developments, ranging from sharply defined individual results to whole new theoretical structures, have come up in disparate cultural backgrounds widely separated in space and time.

It would seem to be obvious then that if there is sense in invoking cultural qualifiers like 'Indian' or 'Mesopotamian' in relation to mathematics, it cannot refer to its substance but only to such intangibles as the value attached to mathematical activity, the manner in which mathematical discoveries were motivated and made, thought about, made use of and so on; in other words, to the cultural matrix in which this whole body of thought, mathematics, was situated. The way three different civilisations dealt with the diagonal theorem and diagonal triples ('the Pythagorean paradigm' in a convenient though historically misleading short phrase) already serves as the first sign-posts to what we might look for. The Babylonians wrote their mathematics down on durable material from which we learn that they knew some diagonal triples and also the geometric theorem for the special case of the square (which says that the square on the diagonal of the square has twice the area of the square on the side): there is a well-known tablet with the inscribed value of √2, the length of the diagonal, written in Babylonian sexagesimal figure of a square with its diagonals and an excellent approximation of the number notation; there are even figures of nested squares (on another tablet) which might have provided a visual demonstration of its truth. (The choice of the base for counting: 60 in Babylonia, 10 in India and none in Greece, is itself an interesting cultural differentiator). No statement of the geometric theorem (even for the square) has so far been brought to light. General geometric statements are not a strength of Babylonian mathematics but there is a remarkable tablet with a problem requring repeated use of triples to find a diagonal in three dimensions in an architectural context.

Pythagoras had a statement of the general geometric theorem and perhaps a demonstration of it, according to later, sometimes much later, Greek sources – their word is all we have for it, no textual material survived – but he probably did not have a list of triples. The pre-Euclid history of the Greek version of the theorem is murky and

[1] No theorem in Indian mathematics is named after an individual, no matter how revered. The diagonal theorem which remained nameless for a long time eventually came to be called *bhujā-koṭi-karṇ a-nyāya; karṇ a is* the diagonal of a rectangle or, equivalently, the hypotenuse of a right triangle.

what we know about its Greek enunci-ation and proof is essentially what is in Euclid (4th century BC). Much had changed in Greek intellectual life in the preceding two centuries and the Euclidean transformation of geometry into a rigorously axiomatic science was the most dazzling of the triumphs of the resulting deductive approach to the acquisition of knowledge. This is a very well known story as is the fact that when Europe rediscovered its Greek intellectual heritage in the middle of the second millennium AD, Euclidean geometry became the embodiment of the ideal to which all mathematical work aspired. Thus did the idea of a universe of 'pure' mathematics, sufficient unto itself, come into being and, in course of time, grow into the vigorous and apparently limitless enterprise that it is now. The theorem of Pythagoras remained and still remains an essential ingredient in many branches of mathematics – it is the foundation of all notions of distance in geometry – but for Euclid it was one among the many results that followed from his definitions and supposedly universally valid postulates through the exercise of equally unquestioned rules of logical reasoning. Euclid also brought the Pythagorean paradigm to a satisfactory conclusion: he had a general formula that produced all triples.

Indian geometry has been from the beginning less 'pure' than its Greek counterpart, making generous room within it for the mensurational (the 'metric' in geometry) aspects of the study of space: lengths of lines and, especially, areas of closed figures. The two facets of the diagonal paradigm together made for a perfect setting for the practice of this not-so-pure geometry or, perhaps, it happened the other way, the practice came first; in any case the Pythagorean paradigm set the direction for Indian geometry and remained its defining theme for as long as it lived. Already in the first propositions on the subject (Baudhāyana and Āpastamba), equal weight is given to the statement of the abstract theorem (along the lines cited above) and to a collection of (not too large) triples. In fact one of the later *Śulbasūtra*s (of Mānava, who probably lived at about the same time as Pythagoras) has a way of generating an infinite subclass of triples through a special case of Euclid's formula. We also know that the theorem and the triples were put to a practical and ritually important use, that of making right angles in the floor plans of altars *(vedi).*

The mechanisms of oral transmission of knowledge in Vedic India were very much more robust than they were in Pythagorean Greece (and have remained robust throughout history) and it is to this fact that we owe the survival of the *Śulbasūtra* texts. Their faithfulness to the originals is not in question, but their highly compressed *sūtra* format makes it almost certain that everything that their authors knew was not set down, though very likely to have been passed on down the generations. The old cliché, "absence of proof is not proof of absence", is particularly apt here: there are no proofs in these texts. Supplying proofs of the diagonal theorem, of varying degrees of ingenuity and elegance, has always been a staple of Indian mathematical history. The simpler ones tend to be diagrammatically prettier and there is no reason to doubt the opinion of the pioneering scholars of these texts that they served as visually convincing demonstrations of its truth. The compulsions of the axiomatic method would have made them unacceptable to Euclid; his own proof is visually unenlightening, as some of us may remember from our school days. And that illustrates another of the cultural differentiators of Indian and Greek geometry and, more generally, mathematics: what constitutes a proof, a verifiable demonstration, a *yukti* in the sense employed by Nīlakaṇṭha, the great 15th-16th century polymath from Kerala?

The focus in this article will naturally be on India, with Greek and later European work on related themes serving as the parallel mathematical universe for purposes of comparison. It must be borne firmly in mind that traditional mathematical scholarship in India came effectively to an end with the arrival of European colonial powers, say in the second half of the 16th century, at about the same time as the Greek-inspired mathematical rejuvenation of Europe took off. Formed as we are in the extraordinarily fruitful culture that is today's mathematical mainstream, it is not easy for us to keep our modernity aside while trying to penetrate the minds of the authors of the *Śulbasūtra,* or of an Āryabhaṭa or a Mādhava for that matter. Moreover, any enquiry into the unifying ideas that run through the enterprise of creating, validating and communicating mathematical knowledge will necessarily intrude into alien territory: logic and philosophy, linguistics, cultural history, etc., especially important in the Indian context being the primordial role of language and grammar in giving purpose and form to our mathematics, its Indian flavour as it were. The notes that follow are to be thought of as no more than a selective (and largely non-technical) account, illustrated by the simplest possible examples, of a very tentative first effort in that direction.

2. The geography of Indian mathematics

Mathematics in India has a long and continuous history. Going by textual evidence alone, the earliest example of mathematical creativity is the invention of a complete and systematic method of counting with 10 as the base (decimal enumeration) and the evidence, enormous quantities of it, comes from an unexpected source, the *Ṛgveda.* The *Ṛgveda* has close to three thousand number names distributed among all its ten Books, the supposedly early as well as the later Books I and X (Book VIII, one of the early 'Clan Books', is particularly rich in them). Postponing a closer look to the next section, here the author only note two points: i) the almost perfect regularity of the structure of the number names can only mean that the decimal system was well established already by the time the individual poems of the early Books were composed; and ii) the consensus among Vedic philologists is that the final redaction of the *Ṛgveda* in the form in which it is known today more or less – *via* Śākalya, Sāyaṇa, *et al.* – happened around 1200 - 1100 BC and that the earliest individual poems are to be dated a few centuries prior to it. The two points together mean that the perfecting of the decimal number system took place in tandem with the final stages of the evolution of Vedic grammar and the mathematically more interesting invention of the rules of metrically ordered verse. As for when and where: around 1400 BC give or take a century, wherever in northwest India the Vedic people were at that time.

The earliest *Śulbasūtra*s were compiled further east, in the Kuru country around Delhi. After that there was a long break in significant mathematical activity until we come to the last centuries BC which saw a tremendous revival of interest in numbers and their potential infinitude, especially (but not only) among Jainas and Buddhists. To roughly the same period belongs Piṅgala, whose combinatorial investigation of prosody began with Vedic metres and ended up in the complete theoretical classification of all possible metres in a syllabary with two durations, *guru* and *laghu*, for each syllable, an achievement of the highest order. We do not know where he lived.

Around the 3rd century AD, the nature of mathematical activity began to change. Contacts with the Hellenic world, especially Alexandria in Egypt, brought into India the quantitative study of astronomical observations and the making of geometric models of planetary motion. This was a first; the idea of subjecting natural phenomena to mathematical analysis – the very foundation of modern science – whose results in turn could be used to predict events which had not yet come to pass was not part of Indian intellectual tradition and, later, it never was extended to other sciences than astronomy. In any case, astronomy became the driving force behind mathematical innovation to a very great extent, a privilged role it retained until the end. The Greek influence probably entered through Gandhara along with much else that was Greek but took firm root in the Malava region during the 4th and 5th centuries (the Siddhanta period, on account of several competing astronomical systems or *siddhānta*s). The last Siddhāntist was Varāhamihira (Ujjain, 6th century) but just before that Āryabhaṭa had already announced himself at Kusumapura in Magadha far to the east, where he composed his eponymous masterpiece bringing together Alexandrian astronomy and Indian geometry in a remarkable synthesis. The *Āryabhaṭīya* (499 AD) defined the direction that mathematical astronomy and mathematics itself were to take in India, much as Pāṇini's *Aṣṭādhyāyī* did for Sanskrit grammar and grammatical theory in general.

The Aryabhatan revolution inaugurated a period of extraordinarily rich and diverse mathematical achievements and it lasted until creative mathematics came to an end in a blaze of glory, in Kerala. In between, outstandingly good mathematicians popped up in virtually every region of cultural India. To name only the most influential, we have Bhāskara I and Brahmagupta (both 7th century) in western India and Bhāskara II (Bhāskarācārya) in the northern Deccan (12th century) sandwiching between them some fine mathematicians working in various parts of north India; Mahāvīra (9th century) and Nārāyaṇa (14th century) in the southern Deccan; the Chera royal astronomer Śañkaranārāyaṇa and his teacher Govindasvāmi (9th century) from almost the southern tip of Kerala; and finally the remarkable school established by Mādhava of which Nīlakaṇṭha and Jyeṣṭhadeva were prominent members, also in Kerala but further north in the Nila river basin (14th - 16th centuries). The centuries after Āryabhaṭa also saw the first documented transmission of mathematics from India to the Baghdad Caliphate, to China and to southeast Asia; the first surviving examples of positionally written decimal numbers, including a zero symbol in the form of a small circle, are in Sanskrit inscriptions from Cambodia (7th century).

One purpose of my running through the names of the chief protagonists in the story of Indian mathematics and their theatres of action, however sketchily, is to provide background for what follows: the names will help set the scene. The overriding impression, however, is not of how they differed one from another but how closely they were linked, over this vast span of time and geographical distance, by a sense of continuity in conceptual framework and technical apparatus. There is an identifiable DNA and it can be traced all the way back, in spite of the many mutations along the way such as those brought about by the great breakthroughs of an Āryabhaṭa or a Brahmagupta or a Mādhava. To cite only two out of many examples, while Āryabhaṭa's trigonometry and Brahmagupta's geometry of cyclic quadrilaterals are quite distinct in their goals and the means by which they were attained, their seeds are clearly discernible in the *Śulbasūtra.* The second example, even more dramatic, is Mādhava's invention of calculus for trigonometric functions: the trigonometry is Āryabhaṭa's but the crucial infinitsimal input relies on a much earlier idea, the recognition that numbers have no end.

There are excellent historical reasons for this uniformity of thought and method. Throughout histoy Indian mathematicians, like many other Indians – traders, pilgrims and proselytisers of new faiths, learned people, architects and artisans, soldiers and those seeking to evade them, etc. etc. – took to the road readily, often over very long distances, and the knowledge in their minds travelled with them. Plenty has been written about the gradual movement eastwards of the Vedic people, occupying most of the Ganga basin by the time of the Buddha, but Āryabhaṭa was not descended from that stock. He was from

Aśmaka *("aśmakīya"* and *"aśmaka-janapada-jāta"* according to Bhāskara I and Nīlakaṇṭha) some thousands of kilometers to the west; the plausible scenario is that he or his immediate ancestors were refugees from the invading Hunas, part of a great mid-5th century migration that also carried Mahayana Buddhism from its stronghold in Gandhara and thereabouts to less endangered areas of Gupta India, in particular eastern India. Bhāskara II's birth was near the modern town of Chalisgaon in northern Maharashtra but there is an inscription put up there by his grandson tracing the family back five generations to a paternal ancestor in the court of Raja Bhoja of Dhar – who, we may recall, fought Mahmud of Ghazni – in Malava. There is in fact a striking absence of 'published' mathematics (or astronomy) attributable to north India after the Afghan invasions and until the end of the 16th century. Mathematical learning became confined, by and large, to southern India.

But it was the same mathematics. Well before the time of the Afghan raids, Govindasvāmi, a devoted follower of Bhāskara I, and through him of Āryabhaṭa, was in Kerala, surely as part of the early wave of Brahmin settlers brought to the coast of Karnataka and Kerala by local kings and chieftains. The identified sources of these migrations are Ahicchatra on the Ganga and Valabhi in Gujarat where Bhāskara I probably taught. And the fact that Mādhava was a Tulu Brahmin, a recent arrival in the Nila valley from coastal Karnataka, itself a place of settlement of northern Brahmins, only serves to reinforce the northern connection further.

Textual and other internal evidence, as much as the mathematical content, fully vindicates the idea of a community of scholars scattered in various parts of India at various times but connected together in a permanent intellectual network, holding on to their traditional knowledge systems and the language, Sanskrit, in which the knowledge was expressed and preserved. Every student, no matter where he lived, began his studies with Sanskrit and its grammar before moving on to the specialised texts composed by his great predecessors, the *pūrvācārya*, no matter how far back in the past or how re-mote geographically. Despite its origins in Vedic ritual, the geometry of the *Śulbasūtra* was the common heritage of Hindu, Buddhist and Jaina mathe-maticians alike. Varāhamihira in Ujjain became thoroughly conversant with Āryabhaṭa's work in distant Kusumapura within about thirty years. A thou-sand years later, the hold of Aryabhatan ideas on the Nila mathematicians was so strong that some historians refer to them as the Āryabhaṭa school. One can cite any number of other instances: the past was part of the living present, at all times and everywhere. No Indian mathematician had to redis-cover the work of *pūrvācārya*s as Europe had to rediscover the achievements of the Hellenic civilisation before laying claim to its intellectual inheritance.

3. Orality and its legacy: the grammar of numbers

Going through an Indian mathematical text is a disorienting experience for a modern reader who has not been exposed to one earlier. It is all words, with no symbols and no equations; even the numbers are spelt out in words, as their literal names or linguistic substitutes for them.[2] This is as true of works composed in cryptic *sūtra*s as in more expansive verse or even more dilatory prose. We believe we know how the trend started: one cannot have a symbolic notation without writing and the scholarly consensus is that the Vedic culture was an oral, verbal culture which did not have writing or at least had no use for it when it

[2] There are also very few diagrams in the manuscripts, generally written down fairly late, but that is probably because of the difficulty of drawing reasonably precise figures on palm leaf with a metal stylus. There is good evidenc that figures formed part of doing and teaching geometry.

came to the sacred; and the *saṃhitā* texts, as much as the instructions for the making of *vedis* – and, hence, geometry itself – were sacred. That the very early mathematics was expressed non-symboloically is then not a surprise; the surprise is that the spurning of the symbolic and the written survived the arrival of writing – using, it may be added, a systematically organised, abstract and very symbolic syllabary. Perhaps the responsibility lay with the very conservatism that ensured mathematical continuity. The fact is that the reliance on the oral had enormous consequences, not only in the communication of mathematics but also in its exposition and in the way it was thought about.[3]

To begin at the beginning, the greatest influence of orality, mathematically, was on the genesis and evolution of a system for counting, that most fundamental of all mathematical skills. Numeracy, especially in its symbolic form, is now such a routine accomplishment that we barely pause to wonder at the sophistication of the principles that make the act of counting – and everything else that depends on it including much of mathematics – possible at all. At its root lies the idea of a base, 10 in India, but in principle any other – the 60 of the Babylonians for example – will do as well. Staying with the familiar decimal base, the role of 10 in apprehending precisely a large number is similar to that of say a unit of length, the meter, in measuring an otherwise unknown length. The basic arithmetical operation that applies to such measurements is division with remainder. Thus, take a number and divide it by 10, say 1947 = 194 x 10 + 7 and repeat the operation until the quotient becomes less than 10: 1947 = 194 x 10 + 7 = (19 x 10 + 4) x 10 + 7 and so on, resulting finally in $1 \times 10^3 + 9 \times 10^2 + 4 \times 10^1 + 7 \times 10^0$. The problem of the cognition of any number whatsoever is thereby reduced to that of numbers less than 10 that are the coefficients of powers of 10 which I will call, following Bhartṛhari (see below), atomic numbers. The key to the written representation of a number is in the realisation that these coefficients, specified in order, define the number uniquely; thus the sequence (1, 9, 4, 7), ordered to the right in decreasing powers of 10 and abbreviated to 1947, is shorthand for a polynomial when the variable is fixed at 10 (the coefficients are necessarily less than 10). The rules of arithmetical operations are all reflections of corresponding algebraic rules for polynomials though historically, of course, it happened the other way, from numbers and arithmetic to algebra.

The description above of the mathematics of enumeration is tailored to a written symbolic representation of numbers. The only extra ingredient it needs to be completely unambiguous is a symbol to mark an empty place, in other words a zero symbol as in 409 or 490. What can one do to carry out the same construction of numbers when one has no writing? The Indian answer is that one gives them spoken names. Begin with names for the atomic numbers, arbitrary but unanimously agreed, *eka, dvi, . . ., nava,* just like the arbitrary symbols 1, 2, . . ., 9. Then one assigns names to the places, the positions corresponding to the ordered powers of 10, also in principle arbitrary (although they may have – and some do – other meanings than numerical): *daśa, śata,* etc. Finally one combines these two sets of names to arrive at an ideal naming system for every number however large, inventing new names for higher and higher powers of 10 as the need is felt. But we are now dealing not with an abstract convention but a living language, Vedic Sanskrit, and the language has a grammar, and the grammar

[3] There is a partially preserved text, the Bakhshali Manuscript, that has a fair amount of symbolic notation and it is the only such. It is a singular document in other ways as well, not easy to fit into the standard chronology (4th or 5th century AD in my view). At least some mathematicians probably employed symbolic notation 'behind the scenes'. If they did, they kept it out of their formal writing, and it is difficult to know how widespread the custom was and whether there was anything like a uniform notational convention.

has rules for combining words so as to create other words with a specific, grammatically determined, meaning. Forming number names is a small part of the function of such general rules of nominal composition but they may not be violated. They must also have the universality and precision that mathematics demands: universal meaning that the rules should be categorical, not depending on the individual numbers being combined, and precise in the sense of the result being a uniquely identifiable number, conditions that are not indispensable in general linguistic usage.

There is no manual devoted to the science of numbers in the Vedic literature, perhaps because, unlike geometry, decimal counting and its evolution were organically part of the systematisation of Vedic grammar itself. In fact even the mathematical texts, over the centuries, pay little attention to the subject but for the numerous listings of the names of powers of 10. (A partial exception is the last great masterpiece of Indian mathematics (in Malayalam prose), *Yuktibhāṣā* of Jyeṣṭhadeva (ca. 1525)). The absence of a theoretical tract is more than made up for, fortunately, by the abundance of numbers in several Vedic texts, with the *Ṛgveda* at the head, of which many are compound numbers – those which are neither atomic nor powers of 10 – requiring the application of the rules of nominal composition in the formation and analysis of their names. For an appreciation of the the kind of analysis the decryption of a compound number name calls for, it helps to keep two facts in mind, one primarily mathematical and the other grammatical. Firstly, there are only two operations involved in making a polynomial, multiplication and addition (in logical order), which are to be encoded into the grammar of number names: thus $409 = 4 \times 100 + 9$. Grammatically, addition is conjunction and though there are several *samāsa*s by which it is implemented, they are easy to spot. Secondly, short of saying that every other other kind of composition represents multiplication, an independent grammatical analysis of multiplicative composition is more involved. The key here is that most numbers function as adjectives, so many cows for example; in 'four hundred cows', 'four hundred' is an adjective and 'four' is an adverb qualifying the adjective 'hundred', signifying the 4-fold repetition of the action of counting up to 100. The rules for the formation of such repetetive numerical adverbs in the *Ṛgveda* can be extracted and they are basically the same as codified by Pāṇini much later – allowance being made for some *chandasi* exceptions – as are the rules for their declensions *via* various affixes. With two or three irresoluble exceptions, every number name yields a unique number when Pāṇini is supplemented by an occasional appeal to the older authority of the word-for-word reading, the *padapāṭha*.[4]

The evidence of the *Ṛgveda* is doubly important in tracing the antiquity of decimal enumeration because there has always been a subconscious tendency, natural in view of the ubiquity of writing, to confuse the abstract principle with its symbolic manifestation. As far as the principle is concerned, the symbolic and the nominal are just equivalent representations of an abstract structure.

4. Orality and its legacy: beyond counting

The symbiosis between numbers and grammar touches only one – undeniably the most important – facet of how orality impacted mathematics (as it did many other fields of

[4] But the analysis is not always trivial. In the 19th century, before H. Kern's first published edition of the work came out in 1874, the *Āryabhaṭīya* was known to Western scholars only by name as *Āryāṣṭaśata* (for the 108 verses in its three substantive chapters), which is how Brahmagupta referred to it. H. T. Colebrooke, the first translator of Brahmagupta, and E, Burgess, ditto of the *Sūryasiddhānta,* both read *aṣṭśata* as 800, a mistake they would not have made had they paid attention to how repetitive adverbs *(āvṛtti-vācaka)* are to be formed.

activity). A receptive ear can hear its echoes resonating through the centuries, all the way down to the last significant mathematical writings of 16th century Kerala. Let us turn now to a brief and highly selective recapitulation of some of these aspects.

The most striking at first sight (one should perhaps say at first hearing), is mathematical language. With a few exceptions, texts were composed in (Sanskrit) verse, the earlier ones, including the *Āryabhaṭīya,* in an almost impenetrably dense *sūtra* format. Orality puts a premium on memory. In the absence of written texts, knowledge exists only in the memories of those who possess it and the resort to metrical verse, like the extreme concision, was an essential aid to memorisation. Versification in its turn brought up interesting and difficult mathematical questions and it took a long time for them to be precisely formulated and resolved, by Piñgala in his *Chandaḥsūtra* (3rd or 2nd century BC?). Deprived of writing, language is the articulation of sounds and phonetic accuracy is the guarantor of its integrity. The regularity of organisation of the Sanskrit syllabary, already clearly evident in the earliest Vedic texts, but given its final form at about the time of Pāṇini, is part of the response of the linguistsages to the challenge of preserving textual purity. Given that every syllable falls into one of two subsets depending on the duration of its articulation, the sequencing of syllables in a line of verse will determine a quasi-musical pattern (aided in Vedic recitation by the accents, which disappeared from later, classical, Sanskrit, probably with the arrival of writing) which is what a metre is. Piñgala characterised all such patterns for any syllabic length, thereby shutting the field down for good and, in the process, initiating the discipline of combinatorial mathematics. It is an astonishing piece of work; nothing like it existed in any other civilisation and we can see why: it needs a phonologically structured syllabary such as that of Sanskrit for a mathematical study of prosody, the science of *chandas,* to be at all feasible. The other remark is that Piñgala's formal methodology owes something to Pāṇini – that is not surprising; he was likely a near contemporary of Patañjali – particularly in the use of metamathematical/metalinguistic labels to designate subsets of objects sharing a common mathematical property or following a common linguistic rule. The curious fact is that Indian mathematicians seem to have paid him little attention until quite late and never to have made use of the notion of a set.

Piñgala comes into the picture in another role which has nothing directly to do with combinatorics but ties up with orality in a different way. He was the first to refer to the zero, *śūnya,* as a mathematically defined object, a number like any other. The idea of nothing, an emptiness, a vacancy, has been a subject of endless fascination to Indian learned men past and present, from grammarians to philosophers, going back to Pāṇini and his much-cited aphorism, *adarśanaṃlopaḥ, "*that which is not present is *lopa".* The story of the mathematical *śūnya* is more mundane. The *Ṛgveda* does not have it nor does any other text before the *Chandaḥsūtra.* And it occurs there in a context which makes its strictly numerical connotation absolutely clear: it is paired with *dvi* as a metamathematical marker, names of subsets occurring in a particular metre-classification problem.

The reason for the late appearance of zero as a number, a thousand years after the *Ṛgveda* and its profusion of numbers, is not difficult to see: an oral, nominal system of enumeration does not need it, unlike a written, symbolic system which, as we saw, cannot do without it; there are natural linguistic alternatives like 'there is no cow' to the unnatural 'there is zero cow'. Nor does arithmetic require its use. To multiply 409 and 51 for example, it is enough to say: (4 hundreds and 9 ones) times (5 tens and 1 one) is (4 times 5) thousands and (4 times 1) hundreds and (9 times

5) tens and (9 times 1) ones and go on from there, resulting in the answer (2 ten thousands and 8 hundreds and 5 tens and 9 ones) (= 20859 in symbols), with not a *śūnya* in sight.[5]

More puzzling is the absence of a symbolic zero after writing became widespread. The earliest known examples of writing in India, the edicts of Aśoka, have numbers in them but no zero. It is puzzling because Aśoka and Piñgala lived within maybe a century of each other. Even more mystifyingly, there is no zero in any of the many inscriptions from the following several centuries, on stone and coin, generally in Brāhmi characters, though they have lots of numbers. We have already referred to the fact that the first known, dated, symbolic zeros are from 7th century southeast Asia. Even if the zero in the shape of a dot *(śūnyabindu)* in the Bakhshali manuscript is earlier, perhaps as early as the 4th century, that still leaves a gap of about seven hundred years between Piñgala's mathematical zero (and the advent of writing) and its first written manifestation. What can account for this long delay in the passage from abstract idea to concrete symbol of a concept which, for many, is the very essence of the (decimal) place-value system?

Part of the answer is that it is no such thing; people could manage very well without it, and did, for a long time. The clue to the mystery of the missing zero really lies in that fact. The written Brāhmi number system went through a phase of experimentation before acquiring some sort of regularity, say by about the 2nd century AD (in the inscriptions of Nasik for example). It had symbols for the atomic numerals as well as individual single symbols for 10, 100 and 1000 (no zeros there), and these were combined to form multiples of powers of 10 by attaching atomic numerals as suffixes to the powers of 10: it is as though we were to choose to write 300 and 400 as 100^3 and 100^4 (remember that 100 itself was a single symbol). The additive part of the polynomial algorithm is then implemented by writing the terms to be added side by side, like $100^4 9$ for 409. We can recognise this strange system, symbolic but not positional, for what it is right away. It is none else than a faithful rendering into symbols of the names of atomic and power-of-10 numbers, and also of the grammatical rules that bind them together, from the pre-existing verbal nomenclature; suffixing and juxtaposition are, symbolically, multiplicative and additive composition respectively. And, naturally, there is no 0 because it is not needed. Brāhmi numbers are visual symbols not for the numbers themselves but for their grammatically determined names; they really are a testimony to the absolute primacy of the spoken language.

We have no idea when written numbers finally freed themselves from the tyranny of orality and became fully positional. Aside from the undated Bakhshali manuscript, the first dated written positional numbers are from the end of the 6th century. They do not have a zero but that is probably a matter of chance and, in any case, less important in the grand scheme of the evolution of numeration than that they are truly positional. It is this scheme that then travelled to Persia and the Abbasid kingdom and eventually to Europe.

The pervasive influence of language and the resistance to symbolic notation had other consequences such as the premium put on the precision of technical terminology. The fact that Sanskrit is a language that values syntactical rigour above all meant that technical terms, very long when necessary, could be manufactured with a high degree of specificity, mitigating to a limited extent the disadvantages of a narrative style of doing mathematics. But, without a matching degree of linguistic (and mathematical) sensitivity

[5]This is trivial but it had to be said. What it reflects is the fact that no 0 is required in the usual way of writing polynomials nor in algebraic operations with them; a term with coefficient 0 is just absent, *lopa* in the Pāṇinian sense. The 0 symbol is the price to be paid for abbreviating the full polynomial as it is normally written to the sequence of its coefficients as in a written number.

on the part of the auditor or reader, it could also lead to misunderstanding and error. Examples of how the original exactitude of nomenclature was lost in sloppy readings are easy to find but let us restrict to one, historically significant, which has only recently been cleared up. Of the major achievements on which Brahmagupta's greatness rests, one is the creation of an elaborate theory of cyclic quadrilaterals, four-sided figures whose four corners lie on a circle. It is a remarkable body of work which took its inspiration from the *Śulbasūtra* and the diagonal theorem (the continuity mentioned earlier) and carried it to a spectacular conclusion. But the mathematics is not the issue here. Brahmagupta invented the term *tricaturbhuja* for the object of his interest and it caused much confusion among later scholars including some famous traditional commentators (presumably not descended from him through a line of oral transmission) – what could a 3-sided 4-sided figure be? – who chose to interpret it as a conjunctive compound: a trilateral and a quadrilateral, a reading not supported by either mathematics or grammar. The confusion disappears once it is realised that Brahmagupta's theory of the cyclic quadrilateral is built on an imaginative and powerful exploitation of one key property: while it has four sides, only three can be assigned independent lengths, the fourth then being fixed; a cyclic quadrilateral really is a triquadrilateral in this sense. Grammar vindicates this reading.[6]

5 Geometry

When it comes to geometry, we are all Euclideans. Euclid defines geometry for us as the decimal system defines numbers, we know no other kind. It bears recalling why. First comes admiration for the magnificent edifice he erected, proposition built upon proposition, easy or difficult, and then the realisation of the deeper truth, that all of it, even theorems as far removed from being self-evident as can be – think of the nine-point circle – follows relentlessly from a handful of apparently self-evident postulates, with an equally self-evident set of rules of reasoning as guiding principles. The *Elements* was the first, and for a very long time the only, mathematical work in which every result emerged out of the inexorable logic of deductive reasoning, without room for ambiguity or doubt, the perfect example of a deductive system of knowledge.

For anyone brought up in this world of mathematical certitude (in both senses), the first encounter with the geometry practised in India, from the *Śulbasūtra* onwards, can be disconcerting. There are no lists of postulates, in fact there are no postulates. Their role is taken over by (unspecified) notions on which everyone (presumably) agreed – a common store of knowledge – such as: two lines each perpendicular to a given line will not intersect. Definitions are implicit in the constructions, not spelled out; nowhere does one find a circle, drawn with a pair of cord-compasses, defined as the locus of points at a constant distance from a fixed point. Educated common sense takes the place of Euclid's "common notions". Altogether, rather than the strictly segregated categories of unquestionable assumptions Euclid needed to get going, we find a more fluid foundation of intelligent good sense, unquestioned at a given time but not unquestionable.

Indian geometry is much more visual than that of Euclid, in the sense that diagrams convey a good idea of the geometric truth behind them (as some of the figures below illustrate), rather than just serve as props in the logical argumentation. The reason probably is that it has generally stayed in touch with the real world: it began in the architecture, streetscapes and decorative plastic arts of the Indus Valley – an

[6] There is a verse in the *Āryabhaṭīya* which refers to trilaterals and quadrilaterals together. The term employed is the conjunctively correct *tribhujāccaturbhujā*.

area whose study is still in its infancy – and it retained its architectural bias in Vedic times before becoming the mathematical vehicle of astronomy with Āryabhaṭa. The basic building block is the circle, easy to draw with the cord-compass, two sticks and a length of cord stretched taut between them. Circles lead naturally to pairs of mutually perpendicular lines in several different ways. The one that found favour in India, already in the Indus artefacts and right through till the end, arises as follows: when two circles intersect, the line conneting the two points of intersection, the common chord, is perpendicular to the line joining the centres; if the circles are of equal radii, the two lines are, in addition, divided equally by their point of intersection. There is a method for constructing a square with the cord-compass at the beginning of the earliest *Śulbasūtra,* that of Baudhāyana, which uses nothing but this property (which I will call the Indian orthogonality property). Rather than reproduce Baudhāyana's instructions for drawing it (see Fig. 1), I will let the reader's eye wander over the various circles and the lines joining their intersections, take in its visual appeal and then let the geometric truth emerge. (For the unconvinced, it is an instructive exercise to prove, starting from Euclid's axioms, that *ABCD* is in fact a square).

The construction is (or should be) something of a pivotal point in any enquiry into the Indianness of Indian geometry. Historically, sequences of intersecting circles generating

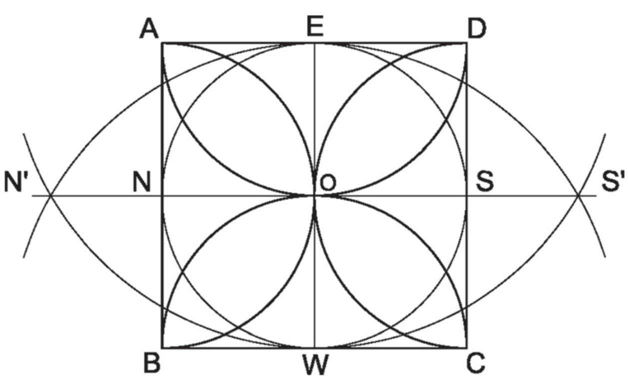

Fig. 1. Square from circles (no diagonal theorem).

pleasing patterns, of which the square in Fig. 1 represents the 'unit cell', occur in quite a few Indus Valley artefacts. The similarity is so uncanny that it is not possible to doubt that Indian orthogonality represents a direct line of continuity between Indus and Vedic cultures, one of the very very few such tangibles that we have. It is also quite unGreek. In Euclid's austerely deductive geometry, the circle comes late – its first appearance is in Book III – well after what for him are the elementary building blocks, lines, and the constructs that follow immediately, their intersections, angles between them, figures composed of them, especially triangles (including the Pythagorean property of right triangles). The most fundamental relationship between a pair of lines is perpendicularity in India, not parallelism as for Euclid, so much so that the term *tribhujā* always meant a right triangle rather than a general one. One can cite many such oppositions. Taken together, they are a good reason to be cautious when questions of mutual influences come up as they have, repeatedly, over more than a century.

Mathematically, the prime significance of the 'square from circles' construction is that the diagonal theorem and diagonal triples play no part in it. Indeed, there is no sign of the Pythagorean paradigm in any of the (admittedly few) places we might expect to find it among Harappan remains, including in the perpendicular street grids; Indian orthogonality is enough. In the *Śulbasūtra*, on the other hand, it is omnipresent. Where did it come from – indigenously discovered or through contact with the intervening Mesopotamian civilisation? We will return to this question in section 9 below but, for the present, only note that we have good reasons to believe that the Vedic sages knew the diagonal theorem to be a theorem; it was not guesswork, they had proofs.

Before that, let us turn briefly to another, simpler, construction from the *Śulbasūtra* which also has a link with the Indus civilisation as well as with an equally far off future development, namely Āryabhaṭa's trigonometry. It is a method (the only one documented in India) for determining the *prācī*, the true east independent of location and season, by means of the shadow of a vertical rod: fix the rod at the centre of a circle and mark the two points at which the tip of its shadow touches the circle (*W* and *E* in Fig. 2) during the course of a day; then *WE* points to the east.

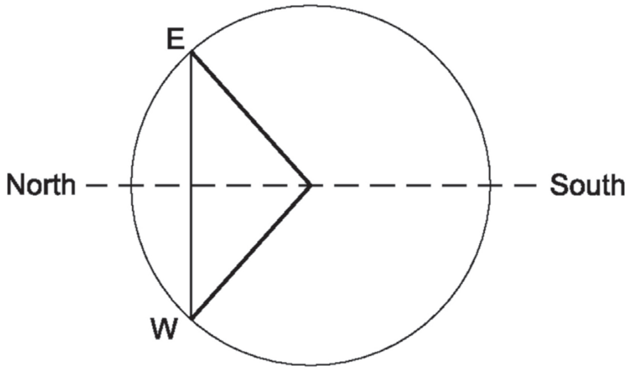

Fig. 2. Determination of the *prācī*.

The plausible continuity of geometric themes of which we have already seen an example argues for this simple method as having originated in the cardinal orientation of streets and buildings in the Indus Valley cities. The connection with the trigonometry of the future is equally unmissable: it is impossible to look at the figure without seeing at once that the north-south line – also needed by the ritualists – cuts the chord *EW* perpendicularly into two equal halves (Indian orthogonality, effectively), each of which is the half-chord (*jyārdha*) or the sine. All of trigonometry is encapsulated in this diagram and in the diagonal theorem applied to the right triangle of which the half-chord is one side. The half-chord as a concept does not occur in Greek geometry. We might say that trigonometry as a discipline has its distant origin in the needs of ritual architects; it is very Indian in the way it is rooted in practical needs, is easily visualisable and brings together two of the signature elements of Indian geometry, the circle and the diagonal theorem. The direction east retained its special role as the axis of reference even after geometry lost its sacral moorings, in the choice of celestial coordinates in astronomy as well as in pure geometry where it became the equivalent of the modern positive *y*-axis. More generally, from a cultural perspective, geometry seems to have undergone two contextual transformations: from secular (Indus Valley, as far as we can tell; so very little is known about its ritual practices) to sacred (*Śulbasūtra*; perhaps an example of the bestowing of magical powers on newly discovered knowledge) and back to secular. That each step in the evolution took such a long time, we might also say, is just another signature of Indianness.

The *Śulbasūtra* of course have no proofs. As far as the diagonal theorem is concerned, a collection of them have been supplied over the centuries by various commentators. The simplest are direct and visual, based on cutting up the relevant areas appropriately and refitting the pieces differently; in Euclid's language, the notion involved is congruence of the pieces. Their credibility as representing *Śulbasūtra* thinking derives from several factors taken together, each of which may not be definitive. Firstly, the statement of the geometric theorem in the *Śulbasūtra* is strictly geometric, without reference to numbers or computation. Number-free, cut-and-fit geometry of areas is characteristic of many constructions in the *Śulbasūtra*, some of which are the same as the suggested proofs, especially in the case of the square. Furthermore, long after geometry had transcended visual reasoning, we still find cut-and-fit proofs, only for the diagonal theorem, as late as in the writings of Nīlakaṇṭha and Jyeṣṭhadeva. The robustness of the transmission chain is a persuasive guarantee that a commentator of say the 11th century who ascribes his particular reading of a proposition to Baudhāyana or Āpastamba is likely to be repeating a line of argument that goes back a long way, maybe even all the way to them.

For the square, the traditional proofs are so elementary that Nīlakaṇṭha considered the theorem to be self-evident. Fig. 3 shows two variants. (The lengths of the side and the diagonal are denoted by a and d respectively, merely for the purpose of identification).

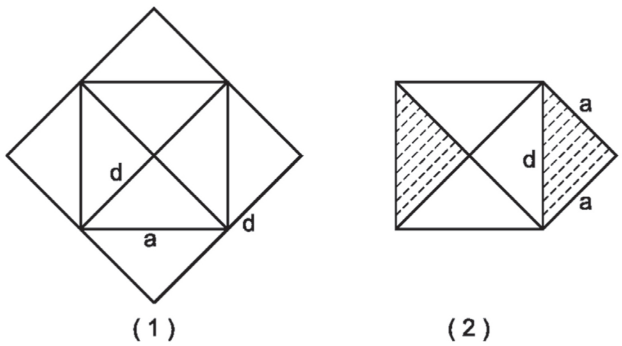

Fig. 3. The diagonal theorem for the square

In the first variant for example, the square on the diagonal, suitably drawn (the outer square), is cut into eight congruent (isosceles) right triangles, four of which fit together to form the square on a side (the inner square), and that is all of the proof. For the rectangle, the simplest proofs are based on the same area-matching idea (there is quite a collection of them) but a degree of geometric imagination is also demanded. It seems likely that the theorem was first formulated for the square and then generalised; Baudhāyana's presentation is in that order. The strategy of first dealing with the simplest special case and then stretching it as far as it will go is very typical, though it is not always as straightforward as it is here; Brahmagupta's theory of cyclic quadrilaterals is a good illustration of how powerful and technically subtle it could become.

It is easy to make these visual proofs conform to Euclidean axiomatism; a good definition of the square and the notion of congruent right triangles will do the trick. One can wonder if Euclid might have devised an equally trivial proof if he had had the idea of first looking at the square (which is not the same as the isosceles right triangle) and gone on from there. It is not impossible that the theorem reached him in its general form, the very easy special case having got mislaid somewhere, some time.

The final big advance in geometry was Mādhava's marriage of two themes from the past, trigonometry and the idea of infinity, to create the new discipline that, much later and in an alien culture, came to be named calculus. We will catch a glimpse of what is Indian in it in a later section but, like Brahmagupta's work, proper justice cannot be done to it in a general review.

6 What is a proof?

Very few of the texts setting out original results for the first time bother to give even an indication of how they were obtained; the *Śulbasūtra* are only the first illustration of this tendency. Commentaries are a little better; they explain difficult points, add their own interpretations and so on but, with rare exceptions, will disappoint those looking for sharply enunciated and logically laid out demonstrations. Even in the great masterpieces, terminology is often not properly defined, conditions that apply not explicitly stated (Āryabhaṭa, Brahmagupta, ...). This gave rise to a widely shared belief that the masters did not have communicable proofs for the results they discovered and in the theories they created or, worse, did not have universally agreed criteria for what a proof should be. Modern scholarship has begun to undo this misunderstanding, mainly by recognising that the real purpose of the canonical texts was to leave a permanent record of knowledge newly gathered, not to justify it; that was done in the parallel pedagogic activity of face-to-face in-struction. In the time-honoured Indian tradition of discussion and debate (orality again), the ideal way of learning was to have the master or someone of his intellectaul progeny explain and clarify and respond to questions, like Bṛhaspati with Indra (see the next section).

But even if we ignored this primary mode of knowledge transmission, it would be naive to imagine that mathematical knowledge – the best of it far from being "self-evident to the intelligent" – came out of informed guesswork or computational astuteness alone (supplemented perhaps by divine revelation?).[7] It is a telling fact that in India's vast mathematical inheritance, there are very very few results (and none that is genuinly deep and original) which are plainly wrong, two errors (on volumes of solids) by Āryabhaṭa (of all people) being the most notorious. For mere conjectures, this is an impossibly high rate of success as we know in the context of modern mathematics. Equally naive is to suppose that there was no idea of what a proof should aim to be. Mathematicians were not given to bringing logic or philosophy into their writings – they barely managed to get all their mathematics in – but it would be absurd to think that they were ignorant of or untouched by the spirited disputes and fine hair-splittings of the various schools of philosophy/logic, a great deal of it about the fundamentally epistemic question of how we know something to be true. Indeed, the one mathematician who did write about foundational issues, Nīlakaṇṭha, shows himself to be very well informed on such matters. It is time to lay to rest the misconception that the best of gems from the ocean of true and false knowledge were brought up by any agency other than the exercise of individual intelligence, *svamati*.

Āryabhaṭa has given us exactly one short verse about where good mathematics comes from; fortunately, his most perceptive interpreter, Nīlakaṇṭha, has much more to say about how we acquire knowledge and about how we are to know it to be true. There is a background. Nīlakaṇṭha was an exceptionally good mathematical astronomer and his guru's guru, Parameśvara, an equally good observational astronomer who revised the basic parameters governing planetary motions, inherited ultimately from Āryabhaṭa, on the ground that they no longer accounted for his observations. In three books written at the end of a long life championing his grand-guru's work (in an already full and fulfilling life), Nīlakaṇṭha expresses himself freely on the enterprise of acquiring and validating knowledge. The primary instrumentality is that of our senses, even in mathematics (it must be remembered that he was partial to almost tactile, geometric, reasoning in arithmetical and algebraic problems, by 2- and 3-dimensional cut-and-fit methods). The 'raw data' are to be subjected to analysis and (tentative) inferences drawn by means of our mental faculties, to be then exposed to criticism by the knowledgeable and taught to young students. If the conclusions are at odds with the revelations of sacred books *(śruti)* or the word of mortals however influential *(smṛti),* the *śruti-smṛti* twins are to be rejected. (It must also be remembered that, as far as a rational world-view is concerned, the thousand years after Āryabhaṭa was a period of regressive Purāṇic ascendancy. A 9th century astronomical text by one of his followers makes fun of *paurāṇ ika-śruti* on the subject of eclipses; this battle is an old one).

Nīlakaṇṭha is critical of pure theory as, according to him, theories are endless and inconclusive. There is no room in his inductive world view for any sort of axiomatism as indeed there was not in Indian philosophical systems from the time of the Buddha and the *śrāmaṇa* movements; no first causes and first principles, no unquestionable "postulates" and "common notions". And no strictly deductive proofs: if all knowledge is contingent, how can it be otherwise for metaknowledge, the knowledge that something (anything) is or is not true? The principles of plane

[7] The last but one stanza of the *Āryabhaṭīya* says: "From the ocean of true and false knowledge, by the grace of *brahman,* the best of gems that is true knowledge has been brought up by me, by the boat of *my own intelligence"* (my emphasis; the Sanskrit word is *svamati).* It must be added that, grammatically, this *brahman is* to be taken as the immanent spirit of the cosmos (neuter in gender), not the male god of creation, Brahmā with a capital B.

geometry and the rules of arithmetic are trustworthy not because they derive from the axioms of Euclid and of Peano (far into the future) but because they work.

Nīlakaṇṭha practised what he preached. Especially in the *bhāṣya* of the *Gaṇita* chapter of the *Āryabhaṭīya,* his proofs of Āryabhaṭa's propositions are models of clarity and logical completeness. As though in acknowledgement of these newly advertised virtues, the word *yukti* for proof ('reasoned justification' in the context) began to displace the older term *upapatti* in writings from Kerala. The significance of this shift of emphasis in the organisation of proofs can be gauged from the title, *Yuktibhāṣā,* of the masterpiece written by his disciple Jyeṣṭhadeva, ostensibly as a mathematical guide to his astronomy but much more than that in reality: a textbook covering the high points of the mathematics of the day, complete with detailed proofs of all propositions.

The question of what constitutes a proof outside a strictly axiomatic-deductive framework has been a subject of intense debate over the last century or so. Without speculating about Nīlakaṇṭha's credentials as an early intuitionist (it has been suggested), it seems safe to say that the Indian position in its maturity would have been that it is a futile quest, endless and inconclusive. As a matter of *vyavahāra,* practical considerations, most mathematicians would have been happy to leave the verdict on whether they had proved something or not ultimately to the judgement of their peers. Bhāskara II says at one place that a piece of mathematics whose *upapatti* does not meet with the approval of the learned assembly is like rice without butter; the simile is interesting in that he implies only that it is unpalatable.

Nīlakaṇṭha does not refer to the one epistemic given that apparently had universal acceptance among Indian mathematicians, which is the rejection of the principle of the excluded middle. The stance had major consequences, chief among them the repudiation of the powerful proof device of *reductio ad absurdum*, reduction to absurdity, proof by contradiction. In the axiomatic philosophy, to exclude the middle is to assert that logic is bivalent: a proposition is either true or not true (or, synonymously, false; it is good to be pedantic because the issue is not free from linguistic ambiguity). To prove that a proposition is true, one starts by supposing that it is false and shows that the axiom system (and the rules of logic) within which the proposition is framed necessarily leads to a contradiction which, by bivalence, means that the proposition is true: a proposition and its negation cannot both be false. Among the earliest and simplest examples of its use is Euclid's (or perhaps someone else's) proof that the square root of 2 is not a rational number, i.e., that it cannot be expressesd as a fraction. The proof consists in showing that assuming that it is a fraction (the negation of the proposition to be proved) leads to a contradiction with a property of fractions which says that they can always be simplified by removing common factors from the numerator and denominator. Now $\sqrt{2}$ is a 'number' the *Śulbasūtra* authors were familiar with geometrically as the diagonal of the unit square (which is also how the Greeks knew it) and Baudhāyana gives a value for it as a fraction, making it clear that it is not exact. The step to the realisation that no fraction can be the exact value of $\sqrt{2}$ was not taken until the very end when Nīlakaṇṭha declares (very uncharacteristically, with no *yukti*) that it cannot be determined.

The notion of irrationality appears straightforward when formulated negatively (not a fraction) but a satisfactory positive definition (what is it then?) is subtle and was not even attempted till the 19th century. Correspondingly, all reasonably elementary proofs of irrationality are by contradiction; it is in fact not easy to see that even the sophisticated, more structurally formulated, proofs are fully free from the taint of *reductio* at all levels of their total architecture. In any case, one can easily understand that a culture which refused to exclude the middle would have had serious reservations about such proofs. Indian scepticism about the philosophical utility of logical bivalence goes far back, at least to the Buddha's reluctance to categorise what is and what is not, well before Nāgārjuna formalised tetravalence: that {yes, no, yes-and-no, neither-yes-nor-no} exhaust all logical possibilities.

Bivalence requires (as indeed does tetravalence) an unambiguous logical meaning to be attached to linguistic negation: what does 'not' mean? Formally, to be able to usefully identify a set S′ not having a particular property, one must have a universal set U which circumscribes all the possibilities one is willing to consider and of which the set S which has the property is a sub-set; S′ is the complement of S in U. In the irrationality problem U is taken (tacitly in the early days, more consciously nowadays) to be the set of all real numbers, numbers which are the lengths of all (geometric) straight lines (never mind how that is defined). To see why the universe cannot be dispensed with, we only have to note that all the arithmetical steps of Euclid's proof remain unchanged if we ask whether $\sqrt{-2}$ can be expressed as a fraction (once we extend numbers to include negatives). It cannot be, but it is not a real number either as we know. Indeed, one Indian mathematician, very late, claimed that square roots of negative numbers cannot exist since the square of every number is positive. (The faint reflection of bivalence that is seen here had the sanction of some schools of logic but only in non-existence proofs). The Buddha's discourses did not, of course, specify a universe of propositions, encompassing as they did questions of life and death, of cosmogony, metaphysics, ethics, etc. Within such a wide and ill-defined horizon, it is not obvious how to define negation in a formally acceptable way.

The high point of Nīlakaṇṭha's engagement with irrationals is a remark-able passage in his *bhāṣya* of the *Āryabhaṭīya* claiming that the value of π (defined as the ratio of the circumference of a circle to its diameter) can only be given approximately because there is no single unit of length that will measure both the circumference and the diameter without a remainder in at least one of them, in other words that π cannot be a fraction. How did he know? Once again, he does not say. My wishful hypothesis is that he had some kind of a proof by contradiction – technically it would have been within his reach since the main tool in many elementary proofs, the theory of continued fractions, was part of the equipment of the Nila school – which he kept to himself because it would not have passed the test of philosophical approval.

A final comment on the subject of method. Just as conspicuous as the absence of *reductio* is the ubiquity of a style of reasoning that can be called, in very general terms, recursive. A recursive process is, loosely, one which results from the iteration of an elementary process in which the output of the nth step in the iteration is fed back as the input of the (n + 1)th step.[8] In this generality, the idea is of wide applicability. Frits Staal in particular has written extensively about the presence of recursive patterns in the syntactic structure of rituals, in the chants which accompany them, and in grammatical constructions of various sorts: *padapāṭha* and *prātiśākhya* in their role as aids to memory (orality again) and, most effectively, in the possibility of the repeated application of nominal composition, potentially without end. That alone makes it an essential component of the Indianness of Indian mathematics. Its mathematical avatars can, naturally, be highly technical and we will have to be content with a selective and qualitative overview.

[8] Logicians and computer theorists work with more precise definitions. The subject is of much current interest partly because recursive algorithms are economical and effective in computation: maximal precision from minimal lines of code.

The prototype recursive structure is the one that defines that primordial object, the set of all decimal (more generally, place-value or based) numbers. The elementary process is division with remainder. As we saw in section 3, the decimal entry in the place of ones of a number N is obtained as the remainder n_1 when N is divided by 10: $N = 10N_1+n_1$; n_2 in the place of tens as the remainder when the quotient N_1 is divided by 10: $N_1 = 10N_2 + n_2$; and so on. Having seen also that decimal counting is as old as the earliest literary forms of Vedic Sanskrit, we may well imagine that it might have been the unacknowledged model for the recursive elements in nonmathematical, in particular linguistic, structures. And, as we shall see in the next section, acknowledgement did come eventually, from Bhartṛhari.

Within the domain of mathematics itself, the range of applications of the recursive idea is amazingly diverse: definitions and constructions, algorithms, expansions of functions as infinite series, etc. – as is to be expected in India, the dividing lines are not always very sharp – not to mention its ultimate metamathematical consummation as a method of proof, mathematical induction. Depending on the problem, the elementary process which will be iterated as well as the initial input are matters of imagination and choice. Apart from decimal counting itself, its earliest use is as an algorithm for the square root of a number which is not a square (a 'number that cannot be determined') to arbitrary accuracy; there is an explicit recursive formula for it in the Bakhshali manuscript and it may well have been used in deriving the *Śulbasūtra* approximation for $\sqrt{2}$. Recursive logic runs through Āryabhaṭa's *kuṭṭaka* technique for the solution of the linear indeterminate (Diophantine) equation as well as Jayadeva's solution (*cakravāla*) of Brahmagupta's quadratic analogue of it. But it is in the work of the Nila school that the method finds its full power and glory under the very appropriate name of *saṃskāram*: every iterative step is a refining of the approximate output of the previous step for better accuracy and, carried *ad infinitum*, leads to the exact answer 'in the limit'. The general technique of *saṃskāram* was as indispensable in the implementation of Mādhava's programme as the infinitesimal philosophy was to its conception. The details are intricate and, regrettably, we have to leave it at that. As an inadequate substitute, here is a quick look at how it works for the square root.

The starting point in all *saṃskāram* computations is an educated, consciously approximate, first guess at the answer. For the square root of a number N, the first guess is the square root of the square number that is closest to N; for example, if N is 109, the first guess for its square root is 10. The exact answer is $10 + x$ for some unknown x; all we know is that $(10 + x)^2 = 100 + 20x + x^2 = 109$ and that x is less than 1 since 11^2 is greater than 109. The quadratic equation for x can be approximately solved without taking square roots by neglecting x^2 in comparison with $100 + 20x$. Thus 10 is the first input and the elementary process is the easy solution of a linear equation which, in the first step, is $100 + 20x = 109$, giving $x = 9/20$. The input in the next iteration is $x = 10 + 9/20$ and so on; the process is repeated as many times as needed for a given requirement of precision (it is actually a very efficient algorithm). An alternative viewpoint is to think of the square root as a function of N and carry out the iteration indefinitely, arriving thereby at an infinite series expansion of $\sqrt{N}$ for any N.

7. Naming and knowing: the problem of infinity

While language and grammar provided the nourishing ambience in which mathematics flourished, we must also note that the flow of ideas was not all one way. The reductive power of the decimal paradigm in making the unknowable infinitude of numbers accessible through an apprehension of a finite set of atomic numerals had an appeal that went well beyond the purely mathematical, in particular to those who wrestled with the unboundedness of language itself. The concern about how to comprehend the infinite potentiality of linguistic expressions through the finite means at our command was already present in very early writings on grammar; there is Patañjali's fable about Bṛhaspati teaching grammar to Indra by going over grammatical expressions one by one and getting nowhere after a thousand years of the gods, leading him to conclude that they can be understood only by means of some (finite number of) rules both general and particular. General and particular rules are precisely what go into the formation of numbers. Several centuries later, in an act of reciprocal generosity, decimal numbers – inevitably, I would like to believe – became the model for Bhartṛhari (5th-6th century) for how potentially unlimited structures in a language are to be constructed from its elementary building blocks. The perfectly named *Vākyapadīya* ("Of Sentences and Words") says: "Just as the grasping of the first numbers is the means for the grasping of other (or different) numbers, so it is with the hearing of words". (Note that words are heard). The gloss, written by himself according to some scholars, leaves no doubt about what he meant:

> As the numbers beginning with one, serving different purposes, are the means of understanding numbers like hundred, thousand, and so on, and are thought of as constituent parts (*avayava*) of hundred and so on, so the apprehension of a sentence is based on the precise meaning of words such as Devadatta, the understanding of which is innate (or inherent).[9]

Later on, he speaks of the atoms (*aṇu* and *paramāṇu*) of sound or speech (*śabda*) gathering together, by their own capacity, like clouds: all linguistic objects are formed by the agglomeration of atoms of *śabda*, just as all numbers are formed by the rule-bound coming together of atomic numbers.

Bhartṛhari was many things, linguist and philosopher of language, epistemologist and cognitive theorist, but a mathematician he was not. Yet, the distinction he makes between the cognition of the atomic numbers, *ādya-saṃkhyā*, which is an innate capacity, and the compound numbers formed by their clustering together is accurate. The comprehension of these two classes of numbers depends on quite distinct mental processes. To know a compound number, we need to apply the abstract arithmetical rules underpinning the decimal system (the equivalent of the 'general rules' of Patañjali) no matter how effortless it may appear in practice – that is what the choice of a base such as 10 does. There cannot be any rules for numbers less than the base and that is why their cognition is an innate faculty – and their names or symbols entirely arbitrary.

Concerns about how we 'know' a number are very much older than Patañjali, going as far back as the genesis of the decimal system itself. There is a line in a hymn to Agni from the early Book IV of the *Ṛgveda* which equates the act of counting to that of seeing and attributes the capacity to do so to Agni, god of fire and light. Louis Renou who first made the connection (later taken up by Frits Staal) notes that *khyā* from which *saṃkhyā*is derived is the verb root for seeing or looking. Support for this identification comes from Vedic words, still current, formed by attaching prefixes other than *saṃ* to *khyā* all of which have meanings derivable from illumination and/or vision (examples: *ākhyā, prakhyā, vikhyā,* etc.). And there are numerous passages in the *Ṛgveda* that speak of Agni's faculty of "comprehending all things in this world minutely and correctly", arising out of his power to illumine. Numeracy was a divine gift as language itself was; I like to

[9] For mathematicians, the change in the value of the same numeral as a function of its position was of course not news. Nevertheless, the sense of novelty never seems to have worn off. *Yuktibhāṣā* has an enlightening passage in its first chapter titled *Saṃkhyāsvarūpam* in which the value of a numeral in the place of ones is called its *prakṛti* and in higher places its *vikṛti.*

imagine that the abundant presence of numbers in the visionary poetry that the *Ṛgveda* otherwise is is a celebration of this supreme achievement, that of bringing to the world of mortals what had belonged to the gods.[10]

It is not a matter of chance that the concept of the infinite has been a background presence in the paragraphs above. No linguist, certainly not after Pāṇini, could have overlooked the infinite generative power of grammar, the inexhaustibility of grammatical expressions and the impossibility of enu-merating them all, that Patañjali's fable highlights, just as no one interested in numbers and how they are built up could have ignored their interminability. The prehistory of numerical infinity as the unattainable limit of counting numbers – not just as a proxy for some vague metaphysical notion – is, correspondingly, much older than that of the complementary idea of the zero. The awareness, or at least the suspicion, that numbers had no end was already very much in the air in the earliest Vedic times. The highest power of 10 in the *Ṛgveda* is 10^4 with the name *ayuta,* but almost immediately afterwords, the *Yajurveda,* in both its recensions, has long lists of powers of 10 by name, pushing the highest power up to 10^{12} and further, in one list in the *Taittirīyasaṃhitā,* to 10^{19}. An even more interesting set of nine lists has numbers increasing in constant steps rather than exponentially, generally beginning with small numerals and ending around 100. There are two reasons that make these lists interesting and it is rewarding to look at them closely:

1. They start for instance like *ekasmai svāhā, tribhyaḥ svāhā,* . . . (depending on the particular list) and after reaching 100 *(śatāya svāhā),* invariably end with a sort of coda: *sarvasmai svāhā* (and so does one of the power lists: after *loka* = 10^{19}, it is *sarva).* It is impossible not to conclude that we are seeing here the first explorations of the limits of counting: no matter how one arranges the count, there is no end to numbers; all one can do is to stop at some point and let *sarva* be the bridge that spans the totality of all numbers, rather like the modern notational convenience of $\cdot\cdots$. The fascination with ever longer lists of numbers kicked off by the *Yajurveda* continued for a millennium and culminated in the well-known episode from the *Lalitavistāra* of Prince Siddhārtha's pre-marital examination in mathematics in which he astounds the world with his knowledge and understanding of impossibly (and unnamably) large numbers.
2. All numbers are cited by name. That meant inventing new names for the higher powers of 10 (compoud number names being then taken care of by grammar). They were taken from the existing vocabulary, *uṣas, samudra* and so on, without a direct numerical connotation, a practice which quickly got out of hand and led to the breakdown of a canonical association of name and number. It would seem that the pressure to find more and more names got to the point where no one bothered to create a system that could accommodate the logic of the multiplicative process by which higher powers were – in the mind – constructed. The 5th century Buddhist philosopher Vasubandhu speaks in one of his works of 60 named powers of 10 but actually gives only 52 of them, explaining that the other 8 have been *vismṛtam,* lost to memory. It may appear to be no more than an amusing episode to us but the problem of matching numbers and names in a unique fashion was a serious one; it is in fact possible to argue that the compulsive need to have an unambiguous name for every conceivable number was what came in the way

[10] Parallels can be found in other places and at other times. Prometheus took from the gods not only fire but also the skill of numbers. In India, at a later time when the Mahāyāna Buddhists were preoccupied with the endlessness of numbers and the infinite multiplicity of the cosmos in space and time, the infinitely benevolent Avalokiteśvara, Bodhisatva of the infinitely radiant Amitābha, was attributed Agni's gift of acute vision.

of an explicitly expressed recognition of their inexhaustibility.

The reluctance to venture beyond the nominal was bound to give way sooner or later. Prince Siddhārtha in the *Lalitavistāra* adopts a clever recursive technique to break free from the tyranny of names: just describe the construction, by a process of iterated powers of an already large number, for example the number of grains of sand in as many riverbeds as there are grains of sand in one riverbed, and so on. The number of universes of the Mahāyāna cosmogony occurs in the discourse as does the word uncountable, *asaṃkhyeya.* Very soon afterwards (the Siddhārtha fable is probably from the 3rd century AD), real (not legendary) mathematicians had good mathematical reasons to get to grips with infinity in a more abstract setting. Both Āryabhaṭa and Brahmagupta worked on a certain class of equations called indeterminate and it turned out that they had an unlimited number of different solutions, an observation that was – finally! – expressed through the unambiguous word *ananta.* One would have thought that someone, a master like Bhāskara II, would have come out and said that the property of being *ananta* was a property primarily of the counting numbers, but that did not happen.

Ironically, the so-called classical period also produced that most articulate champion of the power of the spoken word, Bhartṛhari, who might well have been a contemporary of Āryabhaṭa. At several places in the *Vākyapadīya,* especially in those parts where he lays out the general contours of his programme, the doctrine is invoked according to which an abstract 'thing' is cognisable only by virtue of its being articulable. Examples: "There is no cognition in this world which does not involve the word *(śabda)*. *All* knowledge *(jñāna)* can be said to be intertwined with the word". Or: *"All* knowledge of what must be done in this world *(itikartavyata)* is tied to the word". The synonymy that connects seeing and knowing with naming has its roots, of course, in the Vedic oral tradition; indeed, according to Monier-Williams, *khyā* itself has the sense of 'to be named' in the early Vedic corpus.

The Word never lost its power, even among mathematicians: to name was to know, to cause to exist. *A* millennium after Bhartṛhari and two millennia and a half after the *Yajurveda,* we find the idea (ontological nominalism(?) if we must have a name for it) reasserted by Jyeṣṭhadeva in the *Yuktibhāṣā.* Following the obligatory list of the names of powers of 10 (up to 10^{17}), Jyeṣṭhadeva brings together name, existence and the concept of infinity as a number all in one marvellously revelatory line: " Thus, if we endow numbers with [repeated] multiplication [by 10] and [the consequent] positional variation *(sthānabhedam),* there is no end to the *names* of numbers; hence we cannot *know* the *numbers themselves* and their order" (my emphases, of course).

The paradox is that the fear of the unknowable did not stop Mādhava and those who followed him from employing sequences of numbers tending to infinity as a key input in the creation of spectacularly new mathematics, that of calculus on the circle. Jyeṣṭhadeva explains how it is done, with care and a sharp mathematical sensibility. The relatively unexciting role of infinity in the new mathematics is that the final results themselves are in the form of infinite series whose successive terms are described recursively but quite explicitly; they were, in that sense, 'known'. The far deeper one was that of producing geometrical quantities, for instance an arc, as small as we please, by dividing a finite arc by a number as large as we please. It does not bother Jyeṣṭhadeva that the divisors could not be given names and therefore could not be known; it was enough that a sequence of unboundedly increasing numbers did the job. Nevertheless, there must have been a conflict, as evidenced by the trouble he takes to explain how the nominal imperative was subordinated to the demands of the mathematics as it evolved. In the description of the actual process of division, the divisor is first kept fixed at *parārddham* (10^{17}) which is the largest *named* power of 10 in his list. It is then explained that the method becomes increasingly (infinitely) more accurate as 10^{17} is replaced by increasingly (infinitely) higher powers of 10 and that *parārddham* is just a notional number, used in the description for definiteness of terminology. In other words, Jyeṣṭhadeva resolved the conflict by first paying his dues to nominalism and then letting the mathematics dictate what actually had to be done. It is perhaps a measure of his unease that the relevant passages are among the few where the writing falls short of its usual clarity.

From a mathematical point of view, this technique of 'division by infinity' is superior to the early European attempts at defining infinitesimals *ab initio*. Consequently, the Nila calculus is sounder, conceptually and technically, than that of Newton and Leibniz (though far more narrowly circumscribed in scope). The distinction may appear to be one of perception but it can be traced, ultimately, to the two cultures' understanding of the infinite. Central to calculus is the notion of a limit which in turn needs the notion of an infinite sequence – of numbers, geometrical points, whatever – for its formulation. For European mathematics, infinity as a mathematically well-defined concept was not easy to grasp, lacking as it did an intuitively natural model for it such as the one provided by decimal numbers. There are little asides in Newton's private notebooks expressing his prideful delight at his discovery of "the doctrine recently established for decimal numbers" – about three millennia old by then in India – which he proposed to use as a guide in the definition of infinite series.

8. New directions

The final brilliant act in the story of Indian mathematics – as the Nila school turned out to be – marked the advent of many other breakout ideas which moved it away from its traditional moorings and towards what we can recognise now as modern mainstream mathematics. It is not possible to do justice to this new orientation in a survey like the present one. From today's perspective, its most salient feature is that mathematics began to be driven more and more by its own internal logic and not only by possible applications, astronomical or otherwise. The development of calculus was not really called for by any astronomical problem; methods for dealing with them to the necessary numerical precision already existed. And, within that general framework, the technical apparatus deployed owed little to the past, pointing instead to what was still to come in Europe. To give one example, Mādhava derived the sine and cosine series in a sequence of steps which can be summarised in the modern vocabulary of calculus as follows: i) set up the differential equations satisfied by the functions, ii) convert them into integral equations, and iii) solve them by iteration *(saṃskāram)*. Merely to list these steps is to establish their credentials as essential components of modern analysis as it evolved in the 18th and 19th centuries.

Foundational advances outside the domain of calculus narrowly defined include the introduction of an abstract algebraic approach in a problem of no great practical value (estimating truncation errors in series) and the consequent recognition of the need to answer (and, indeed, ask) questions like: what is a polynomial? Jyeṣṭhadeva has an amazingly modern characterisation of the natural numbers as being defined primarily by their property of succession, as close to axiomatic mathematics as India ever came. This was in turn prompted by the need to justify entirely new techniques of proof – mathematical induction – where traditional styles of demonstration failed. The idea of proof itself underwent a subtle change primarily, as we saw in section 6, through Nīlakaṇṭha's reflections on the question of how we know mathematical propositions (and astronomical doctrine, but that does not concern us) to be true. Their impact is evident in the new rigour that is a hallmark of *Yuktibhāṣā:* proofs are unambiguous and complete, with a noticeably formal, structured character to them.

What is interesting for us in these signals of a modernising mathematical mindset is that, together, they constitute a reassertion of the universal character of all good mathematics; Indian mathematics became less Indian so to say. The old order began to give way, gradually, under the impact of Mādhava's originality; it is not easy for a modern reader who has 'seen it all' to grasp at first sight what an extraordinary transformation of the ethos of mathematics *Yuktibhāṣā* represents when compared to earlier writings. First and foremost is the notion of a function as defined over its entire natural domain, the circle – calculus is fundamentally about functions with some continuity properties – not just as its values at a few selected points as was the case earlier. Then there are the other manifestations of the new autonomy: the assured handling of infinite series and infinity itself in the abstract, free of the constraints of nominalism; the abandoning of the search for geometric justifications of arithmetical and algebraic identities; the first formulations of abstract algebraic concepts and methods; the first whiff of axiomatisation in the treatment of numbers; the stress on formally presented proofs; and so on, each of them a first intimation of a radical move away from long-held habits of thought. And all of them anticipated, in an embryonic form, the evolutionary stages mathematics passed through in Europe following the Cartesian revolution;[11] perhaps it is a truism that every great advance in mathematics is a step towards its universal core. It is another story that creative mathematics was brought to an end in Kerala – and in India – before these shoots could take root and flourish, at about the time that Europe was embarking on its great voyage of scientific and mathematical discovery. The tragedy is that the opening up in the 16th century of extensive maritime channels of communication between India and Europe, instead of inaugurating an era of mutually stimulating exchanges, had the effect of bringing the curtain down on mathematics in India; what was a new beginning in one culture turned out to be an end in the other.

9. Cross-cultural currents?

The achievements of Mādhava and the other Nila mathematicians and the new directions they pointed to were subsumed in due course in the explosive growth of mathematics in Europe that began in the 17th century. And that leads to a question the author have avoided so far, that of the influences of different mathematical cultures on one another or, in the spirit of my title: what is non-Indian in Indian mathematics and, conversely, what Indian traits can be identified in other mathematical cultures? Much has been written, and over a very long time now, about mathematical give and take among different cultures, without producing a proportionate degree of enlightenment. The subject is a complex one involving chronological priority, possible mechanisms of contact and, most decisively, assessments of commonalities in mathematical themes and their working out. What makes a rigorous and objective identification of mathematical cross-currents across cultural borders difficult, more so than in the case of, say, linguistic borrowings, is the very universality and immutability of the quarry: given propitious circumstances, prepared and well-motivated minds are as likely to unearth a mathematical "best

[11] The one exception to this blanket statement about the Cartesian trigger is of course the axiomatic geometry of Euclid. But it was only after Descartes that Euclid became the ideal to aspire to in other areas of mathematics (not to speak of other sciences and, even, all human knowledge). An axiom system for numbers had to wait until Peano in the late 19th century.

of gems" at a given time and place as at any other; we might even take that as an operational definition of its universal identity and appeal.

In actuality, however, comparative history of mathematics has not always been a shining example of an objective and rigorous discipline. Within the Indian context, it was perhaps natural that colonial historians were not particularly interested, especially from mid-19th century onwards, in whatever data could be reliably extracted or in what to do with them; the driving urge was to derive everything from Classical Greece.[12] (It must also be said that some of the earlier colonial historians – Colebrooke, Playfair, Whish and some others – were much more open-minded and did a commendable job with the sparse material they had in hand). In the period following the publication of the *Śulbasūtra* (by Thibaut, 1875), some Indianists dated Baudhāyana and Āpastamba to the late centuries BC for no more discernible reason than to preserve Euclid's primacy at least chronologically, regardless of the philological evidence that both of them predated Pāṇini. The Pythagorean theorem was a particularly sensitive issue – and it must be remembered that this was before the decipherment of the Babylonian mathematical tablets and the flutter they caused among historians of Classical mathematics. The emphasis then shifted: is it possible that the theorem of the diagonal came to India from Babylonia?

The honest answer is that we still do not know with certainty. Absolutely nothing is known about any kind of contact between 18th century BC Mesopotamia and the Vedic people who, presumably, were making their first settlements in northwestern India at that time. We do know about extensive trade exchanges between Mesopotamia and the Indus civilisation but that was at its height 7 or 8 centuries earlier and had come to an end. To confuse (or, maybe, clarify) matters further, Indus artefacts have nothing suggesting an acquaintance with the diagonal theorem. In brief, and to come back where we started, the known history of the Pythagorean paradigm is very well accounted for if we are ready to accept it as one of the great universal truths – an *element* in the sense Proclus gave to Euclid's *Elements* – which more than one culture discovered for itself. The same temporal mismatch applies to the other great universal idea from antiquity, place-value counting, and the confusingly different ways it was implemented: Indus Valley probably counted in base 8, Babylonia, later, in base 60 and Vedic India, still later (and orally), in base 10.

More generally, the readiness to accept temporal antecedence (sometimes accompanied by superficial similarities) as the determinant of transmission is still a running motif in much modern writing. Many examples can be given but one that has found fairly wide acceptance will suffice here. The background is the wholesale importation into India of the Ptolemaic model of planetary motion – as well as the supremely important idea that such motions could be described mathematically – during the so-called Siddhānta period (the two or three centuries preceding Āryabhaṭa), a true and well-documented example of transmission. But Āryabhaṭa had no use for the mathematics that (presumably) came with it; he invented what he needed, mainly trigonometry. Trigonometry was a bringing together of two central elements of *Śulbasūtra* geometry, the geometry of the circle and the diagonal theorem, resulting in the key concept of the half-chord as the natural linear object associated to an arc. It is this object, the sine of an angle, that helps turn much Greek geometry of the circle into easy exercises in trigonometry, an activity popular with some modern historians.

[12] The urge was of course not confined to mathematics. To take an example not too far removed from it, it was seriously argued that the notion of syllogism in Indian logic was dirctly descended from Aristotle, a "prepostorous notion" in the words of one preeminent modern logician.

Āryabhaṭa's famous table of half-chords takes no more than a day's work, if that, once the ideas are in place (though it must be said that some of the auxiliary ideas are themselves very deep). Now, Ptolemy in Alexandria had made an admirably fine table of full chords, not half-chords, in the 2nd century AD but, in the absence of the simplifications that trigonometry afforded, it was laborious work. It shares nothing mathematically, in spirit or in technique, with Āryabhaṭa's work, except that they both dealt with arcs of circles (and were useful to astronomers). Nevertheless, it is an old habit, still vigorously alive in certain circles, to say explicitly or through implication that Āryabhaṭa's table – and other facets of his trigonometry – is to be traced to a direct influence of Ptolemy, or to a predecessor of his whose surviving writings are, regrettably or fortunately, far too fragmentary for a meaningful comparison to be feasible.

The *Almagest* of Ptolemy and the *Āryabhaṭīya* are among the most thoroughly analysed of the astronomical texts of antiquity. The fact that that did not prevent conjecture from being put forward as historical fact – at constant risk of reasoned repudiation – is as good a measure as any of how mere age or received opinion (the Indian equivalents would be *śruti* and *smṛti*) can trump the evidence of our eyes and ears and our minds. (Nīlakaṇṭha should be alive now). In the middle of the last century, Bartel van der Waerden, a historian of Greek science and its relationship with Mesopotamia and a very distinguished mathematician, went so far as to declare in effect that no great mathematical advance is made more than once independently and that all subsequent 'rediscoveries' are derived causally from that original epiphany. Insights revealed to the chosen, once and never again, do not sound very much like the unvarying outcome of a universally shared faculty of the human mind. But it is a temptingly attractive position for a historian to take as it absolves him or her from the responsibility of searching for evidence of anything beyond chronological priority. If van der Waerden had not died in the 1990s, when the availability of the relevant Indian material in European languages was still poor, he might have come to realise that his dictum can cut both ways; few even among the knowledgeable knew, even as recently as that, what Mādhava had achieved two and a half centuries before calculus was invented in Europe, which is about the same time span as separated Āryabhaṭa from Ptolemy (and at a time of enormously easier communications). The surprise is that the dictum, not always consciously acknowledged or consistently applied, still has a following among today's historians, vastly better informed, or at least with the means to be.

It is not impossible that new data may come to light one day that will require a reevaluation of the earliest phase of Indian mathematical history and its linkages with the outside world. The broad picture we have today, especially from the time of Āryabhaṭa onwards, is of a tradition that is comfortably self-sufficient, perhaps overly conservative and even insular. How else can one account for the fact that during the first centuries AD, a time when Indo-Hellenic cultural contacts were so intimate and over such a broad front – from astronomy to architecture and sculpture – Euclid, and Greek geometry generally, was so assiduously ignored or rejected? Whatever be the explanation, Indian science as a whole was the poorer for it. It never produced an Archimedes (who too seems to have been unheard of in India), someone with a vision wide enough to include natural science outside astronomy as a part of mathematical thought; there was in fact no physics in India at any time if we exclude, as we should, purely metaphysical speculations (on atomism for example).

The dynamics of the episodes of a reverse flow, from India, is much better understood, simply because the evidence in support is of much better quality. Here again, possible very early cross-currents, for instance from the Indus civilisation to Mesopotamia, remain uncharted. In contrast, the much later spread of decimal enumeration, first to Persia and the Abbasid Caliphate and thence farther west, reaching Europe in the 13th century, is very well documented. The same wave carried Āryabhaṭan astronomy and the mathematics that went with it, trigonometry in particular, and played a huge part in the second coming of Mesopotamian science in the following centuries. Something similar happened with China as well at about the same time, the agents of transmission being Buddhist men of learning. That the new arrival did not take to Chinese soil as well as it did to the sands of Arabia, and declined in step with the decline of Buddhism in that country (in fact was abolished by royal decree at one point), is perhaps another example of the role of cultural predispositions in the welcome given to alien knowledge. All of these instances illustrate perfectly a conjunction of the conditions necessary for a watertight case for transmission: priority, established modes of communication, credible documentation and, most critical of all, a discernible impact on indigenous mathematics, mutations of the mathematical DNA so to say.

It is legitimate to wonder now what the great surge in European mathematics of the 17th century – the birth of modern mathematics, quite simply – might have owed to the rich but dying Indian tradition. There is no doubt at all that the 'new doctrine of numbers', decimal enumeration and its offshoots, had an absolutely decisive impact, acknowledged by the mathematicians themselves like Newton and Laplace if not always by historians, most visibly in the maturing of algebra as an autonomous discipline (as was also the case in India). Brahmagupta's work was known in Islamic Spain but the direct inspiration for the founder of modern number theory, Fermat, was Diophantus of Alexandria (3rd-4th century AD), who already had been translated into Latin by then. Trigonometry, like decimal numbers, reached Europe through Arabs, but later, and its adoption into 'pure' mathematics (as opposed to, say, cartography) was slower, perhaps because of the hold of the geometric legacy of Euclid. Mainstream geometry was almost exclusively Euclidean, supplemented by the conics of Apollonius. Above all, Descartes' synthesis of geometry and algebra through abstract graphical reprsentations of functions, the most transformative event of an eventful century, had no Indian antecedents at all; no one on the banks of the Nila drew the graph of the sine function.

It is against this background that we have to judge whether word of Mādhava's exploits might have reached Europe and, if it did, influenced the genesis of infinitesimal calculus there. A positive answer would, of course, pass the priority test. Channels of communication also existed, especially with Kerala. But in the mountains of paper the Portuguese generated, no mathematical document with a link to Kerala has so far been found (to my knowledge; it must be added that two of their likely repositories were destroyed: in Portugal as a result of the Lisbon earthquake of 1755 and in Kerala in the burning down of the great Jesuit library of Kochi by the Dutch at almost exactly the time the young Newton's first ideas on calculus were germinating). The more serious difficulty concerns internal evidence; first indications – a proper study remains a task for the future – are that the cultural markers on the two avatars of calculus are as distinctively individual as can be, given the fundamental thematic unity. To mention only the most prominent, there is no trace in Europe at all of the Nila method of approaching the infinitesimal limit by dividing by infinity. Its place is taken by the subtle and original notion of the derivative (Newton's fluxion) or the differential (Leibniz), without invoking directly the idea of infinity at any point; presumably they were more comfortable with quantities which are zero but not quite zero than those which are infinite but not quite infinite. Secondly, Europe was preoccupied with many strictly local problems which needed only the differential half of calculus for their solution, such as the construction of tangents to curves and the determination of their (local) high and low points (maxima and minima of functions); the problems treated by Mādhava were global, of which the differential half was just a preliminary to the integral half (rectification, quadrature).

There are good historical reasons for these variations in taste and choice. The generality of the structural approach that Europe discovered in Euclid's geometry and carried forward through its algebraisation contributed significantly to the vision Newton and Leibniz had of calculus. They recognised from the start that the concepts they were working with were applicable to very general functions and curves and said so explicitly; and it is probably not an accident that Fermat, before them, had made seminal contributions to the founding of both analytic (Cartesian) geometry and calculus. Add to this the role physical science, in particular mechanics (which also was in the process of being created as an exact science), played in the work of both Fermat and Newton and we have an idea of the ambition that the new discipline was born with. The Nila mathematicians came nowhere near that kind of breadth and scope, not because of any inherent limitation in their methods as we know now – division by infinity is as versatile as the European infinitesimal limit – but because they were content with solving the handful of particular problems they were interested in; it was a cultural thing. Alto-gether, it is fair to say that if there is a convincing case for any significant influence on European calculus of what happened in distant Kerala, that case is still to be made. The circle of ideas surrounding the discipline of calculus is one of the great universals of mathematics and it should surprise no one that more than one gifted mind found its way to it.

A final thought. Just as our mathematical ancestors were creatures of their milieu, so are those of us who study their achievements, as best we can, of ours. In these days of scholarship by assertion, that is all the more reason for us to heed Nīlakaṇṭha's advice about separating true knowledge from false: trust our senses for the data and our thinking minds for what we can deduce from them; ignore all else, whether revealed in the sacred texts (*śruti*) or handed down by mere mortals (*smṛti*), when they go against facts and logic.

Bibliography

Sarasvati Amma, T.A. *Geometry in Ancientand Medieval India*. Motilal Banarsidass, Delhi, 1979.

Bavare, Bhagyashree and Divakaran, P.P. Genesisand Early Evolution of Decimal Enumeration: Evidence from Number Names in *Ṛgveda*. *IJHS* 48.4 (2013):535.

Divakaran, P.P. The First Textbook of Calculus: *Yuktibhāṣā*. *J. Ind. Phil*. 35 (2007): 417.

Divakaran, P.P. Notes on *Yuktibhāṣā*: Recursive Methods in Indian Mathematics in C.S. Seshadri (Ed) *Studies in the History of Indian Mathematics*, Hindustan Book Agency, NewDelhi, 2010.

Datta, Bibhutibhushan and Singh, Avadhesh Narayan, *History of Hindu Mathematics* Vol. I. Bharatiya Kala Prakashan, Delhi, 1935, reprinted 2004.

Joseph, George Gheverghese, *A Pasage to Infinity: Medieval Indian Mathematics from Kerala and its Impact.* Sage, New Delhi, 2009.

Narasimha, Roddam. Epistemology and Language in Indian Astronomy and Mathematics, *J. Ind. Phil.* (2007): 521.

Plofker, Kim. *Mathematics in India*. Princeton University Press, Princeton, 2009.

Sen, S.N.and Bag, A.K. *The Śulbasūtras of Baudhāyana, Āpastamba, Kātyāyana and Mānava.* Indian National Science Academy, New Delhi, 1983.

Staal, Frits. *Ritual and Mantras: Rules without Meaning.* Motilal Banarsidass, Delhi, 1990, Indian edition 1996.

Discovering the Vedas: Origins, Mantras, Rituals, Insights. Penguin, New Delhi, 2008.

Tampuran, Ramavarma (Maru) and Akhilesvarayyar, A.R. *Yuktibhāṣā Part I-General Mathematics*. Mangalodayam, Trichur, 1948.

vander Waerden, B.L. *Geometry and Algebrain Ancient Civilizations*. Springer, Berlin, 1983.