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Ideas of Infinitesimals of Bhāskarācārya in *Līlāvatī* and *Siddhāntaśiromaṇi*

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Abstract

The core concept of Calculus is motivated by the dynamic examples of astronomy dealing with instantaneous velocities of planets. The first attempt at formalisation of these ideas was made during the periods of Bhāskarācārya and Mādhava and, Isaac Newton and G F Leibniz developing the entire Calculus, and later Cauchy laying the foundation for modern Calculus based on the rigorous treatment of the concept of limit.

In this paper Bhāskarācārya's algorithm to deal with expressions involving multiples of zero (treated as $infinitesimals$) and zero-divisors (zero as divisors), is considered with a brief reference to the similarity it bears with the ideas of Newton and Leibniz. Bhāskarācārya says that this mathematics is of great use in Astronomy.[1] Therefore, the infinitesimal concepts suggested in his *Līlāvatī* and implied in his geometric treatment of instantaneous sine-difference equivalent of the differential equation $d\sin\theta = \cos\theta\ d\theta$, (in Leibniz$'s$ notation) are considered in some detail as given in Gaṇeśadaivajña's own commentary (*Buddhiviāsinī*) and his commentary *(Vāsanābhāṣya)* of *Siddhāntaśiromaṇi*.

Keywords: Calculus, Infinitesimal, Bhāskarācārya, *Vāsanābhāṣya, Khaguṇa, Khahara,* Fluxion, Limit.

1 Introduction

Though the concept of infinitesimals is a development of mathematics of relatively recent times (17th century) it has its roots in the ancient world of mathematics in the form of intuitive ideas prevalent in Greek, Arabic, Chinese and Indian civilizations among others.

However, these concepts were of primitive nature based on geometric ideas such as:

1. The tangent as a secant in its limiting position leading to the idea of slope as derivative.

2. The circumference as the limit of the perimeter of inscribed regular polygon of large number of sides leading to the idea of π as a limit of a sequence.

3. The area of a circle as the sum of indefinitely large number of infinitely small sectors leading to the idea of integration as a limit of the sum.

These are static examples. The core concept of Calculus is motivated by the dynamic examples of astronomy dealing with instantaneous velocities of planets. The earliest development of this was during the periods of Bhāskarācārya (12th century ce) and Mādhava of Kerala School of Mathematics (14th

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[1]The statement found in *Vāsanābhāṣya* of Bhāskara while commenting on verse 47 of *Līlāvatī* goes as follows: अस्य गणितस्य ग्रहगणिते महानुपयोगः।

century ce) and, Sir Isaac Newton and G F Leibniz (17th century ce), the founders of Calculus, and much later, D'Alembert and Cauchy, who laid the theoretical foundation for modern Calculus without involving infinitesimals. The earlier conceptual difficulties of Calculus were due to the controvertial indeterminate form $\frac{0}{0}$, which arises, for example, from the $intuitive$ concept of infinitesimals as explained below.

2 Newton's notions of infinitesimals

Newton calls the variables x and y $fluents$ and considers them as moving points on a curve, $\dot{x}$ and $\dot{y}$ being rates of motion called *fluxions* (instantaneous velocities). He calls the products $\dot{x}o$ and $\dot{y}o$ the $moments\ \ of\ \ fluxions$ (which are instantaneous displacements or increments) where o is infinitesimally $small$.[2] The limit of the ratio $\dfrac{\dot{y}o}{\dot{x}o}$ of $ultimate\ \ quantities$ is what he calls as $ultimate\ ratio$ (the so called $\dfrac{dy}{dx}$). Since the ratio of *fluxions* is same as that of *moments of fluxions,* in later years there was confusion between these two terms, and the term $fluxion$ itself was indiscriminately used to denote the increments Hayes, *A treatise on fluxions*. For convenience he denotes $\dot{x}o$ simply by o and $\dot{y}o$ by $f(x+o)-f(x)$, where $y = f(x)$. The ultimate ratio $\dfrac{\dot{y}o}{\dot{x}o}$ at $o = 0$ assumes the indeterminate form $\frac{0}{0}$.

Different views of infinitesimals

This section compares algorithms of Newton, Leibniz, Euler and Cauchy.

Newton's Fluxions: Variables x and y are called $fluents$, the time-rates of changes $\dot{x}$ and $\dot{y}$, the *fluxions*, and $\dot{x}o$ and $\dot{y}o$ are the moments of *fluxions*,[3] o being an infinitesimal. $\dfrac{\dot{y}o}{\dot{x}o}$ is called the $ultimate$ ratio, thereby implying that o $ultimately$ equals zero,[4] and it is denoted by $\dfrac{\dot{y}}{\dot{x}}$, called the derivative. $\dot{y}o = \left[\dfrac{\dot{y}}{\dot{x}}\right]\ \dot{x}o$ is the moment of *fluxion* $\dot{y}$.

Leibniz's Differentials: The infinitesimals of the variables x and y are called the differentials dx and dy, where $dy = f\ '(x)dx$. One may treat them as the $ultimate$ $(indivisible)$ things.[5] $\dfrac{dy}{dx}$ is the quotient of differentials or differential coefficient as in $dy = \left[\dfrac{dy}{dx}\right]\ dx$. This is a useful tool where dx $(\neq 0)$ is finite though sufficiently small.

Euler's Differentials: The differential dx can diminish indefinitely till it equals[6] 0.

$$dy = f(x+dx) - f(x) = \left[\frac{dy}{dx}\right]\ dx,$$

where $\left[\dfrac{dy}{dx}\right]$ can be expressed in the form

$$\frac{f\ '(x,0)\ dx}{dx} = \frac{f\ '(x,0)\ 0}{0},$$

virtually defined as $f'(x)$, whereas $dy = f\ '(x)\ dx \neq 0$, for practical calculations in science.

Cauchy's Limits: A variable h can diminish *indefinitely close enough* to 0 called its $limit$. In symbols: $h < \epsilon\ (> 0)$ or lim h = 0, or $h \to 0$,

[2]Zero: The Biography of A Dangerous IdeaSeife, *Zero: The Biography of A Dangerous Idea*. o is the lowercase *omicron*, the first letter o of the Greek word *ouden* meaning nothing. Greeks denoted zero by the letter o. Here the word *small* for all practical purposes means, a small-difference, small enough for the accuracy warranted by the technology of those times. The word *`small'* in italics, in what follows, is used in this sense.

[3]Newton's algorithm is given in ``Mathematical Principles of Natural Philosophy'' and Heys's ``Treatise on theory of fuxions''Hayes, *A treatise on fluxions*.

[4]Quantities and the ratio of the quantities which in any finite time converge continually to *equality* and before the end of the time approach nearer the one to the other than by any given difference, become *ultimately equal* at $o = 0$ Smith, *History of Mathematics*.

[5]The Calculus Gallery, Masterpieces from Newton to LebsegueDunham, *The Calculus Gallery, Masterpieces from Newton to Lebsegue*, p. 24.

[6]Calculus GalleryDunham, *The Calculus Gallery, Masterpieces from Newton to Lebsegue*, p. 53.

$h \neq 0$. $\dfrac{f(x+h)-f(x)}{h}$ can be expressed in the form

$$\frac{f\,'(x+h\theta)h}{h} = f\,'(x+h\theta),$$

where $0 < \theta < 1,\ h \neq\ 0.$

$$\lim_{h \to zero} \frac{f(x+h)-f(x)}{h} = \lim_{h \to zero} f'(x+h\theta) = f'(x).$$

In short we can as well define

$$\frac{f(x+h)-f(x)}{h} = f'(x+h\theta),\ h \neq 0, \\ = f'(x),\ h = 0.$$

In the above comparisons the infinitesimal nature of $small$ quantities are virtually treated as irrelevant, they being either 0 or not 0, as the case may be.

3 Bhāskarācārya's notions of *khaguṇa* (multiple of zero)

It had been the practice to write traditional mathematical works (in Sanskrit) invariably in poetic form. But a poetic form by its very nature could not admit the mathematical symbolism and therefore the author had to resort to algorithmic style. Though Bhāskarācārya has himself written a commentary on the astronomical part of his monumental work *Siddhāntaśiromaṇi*, in the case of *Līlāvatī* and *Bījagaṇita,* being basic mathematics necessary for his work, the demonstrations of proof and the details of workings were left to the wisdom of eminent commentators (such as Gaṇeśadaivajña and Kṛṣṇadaivajña), teachers and gifted students, lest it be too voluminous.[7]

3.1 Ideas of infinitesimals

Bhāskarācārya's *multiple of zero khaguṇa*, $x0$, is the counterpart of *moment of fluxion.* For lack of proper notation, 0 is treated as a symbol denoting an infinitesimal. The commentator of Bhāskarācārya's algebra, i.e., text *Bījagaṇita*, Kṛṣṇadaivajña[8] explains: The idea of infinitesimals by considering a numerical example. The example presented by Kṛṣṇadaivajña is shown in the form of a table below.

Just as Multiplier (यथा गुणकः)	Multiplicand (गुण्यः)	Product (गुणनफलम्)
12	4	48
12	3	36
12	2	24
12	1	12
12	1/2	6
12	1/4	3
12	1/12	1

He further comments:

अनयैव युक्त्या गुण्यस्य परमापचये गुणनफलस्यापि परमापचयेन भाव्यम्। परमापचये शून्यतैव पर्यवस्यतीति शून्ये गुण्ये गुणनफलमपि शून्यमेवेति सिद्धम्।

anayaiva yuktyā guṇyasya paramāpacaye guṇanaphalasyāpi paramāpacayena bhāvyam. paramāpacaye śūnyataiva paryavasyatīti śūnye guṇye guṇanaphalamapi śūnyameveti śiddham.

By the same logic, if the multiplicand (*guṇya*) becomes smaller and smaller so does the product (*guṇanaphala*) which *ultimately* becomes the smallest (i.e., zero) as the multiplicand becomes the smallest.

Similarly this logic is applicable to the multiplier. Thus $x0$ is treated as an *infinitesimal,* ever decreasing quantity, attaining the value zero. Therefore it is natural to *define* $x0$ as zero. In modern notation:

$$\lim_{0 \to zero} x0 = zero, \text{ and } [x0]_{0=zero} = zero.$$

The idea of limit is conveyed by *Buddhivilāsinī* [9] as follows:

[7] *Siddhāntaśiromaṇi*Sastry, *Siddhāntaśiromaṇi of Bhāskarācārya with Vāsanābhāṣya*, p. 39, śloka 9, Somayaji, *Siddhāntaśiromaṇi of Bhāskarācārya*, p. 99, Bhāskarācārya says that he has made his work neither voluminous nor brief, for both the intelligent and the less gifted are to be enlightened.

[8] Bhāskarāchārya *viracita Bījagaṇitīyam*Apte, *Bījagaṇitīyam*, p. 137.

[9] *Līlāvatī*Apte, *Līlāvatī* (*In Sanskrit*) , p. 198.

एवं द्विगुणद्विगुणभुजक्षेत्रकल्पनया यावच्चापासन्नो बाहुः स्यात् तावत्साधयेत्।

evaṃ dviguṇadviguṇabhujakṣetrakalpanayā yāvaccāpāsanno bāhuḥ syāt tāvatsādhayet.

Thus by doubling the number of sides of a regular polygon, [inscribed in a circle], is to be carried out till its side is close enough to the arc containing it.

In modern ϵ, δ notation it implies: choose the number of sides so large as to make the side δ very small so that the chord-arc difference ϵ is small enough. Bhāskarācārya has not defined the derivative in general but incorporates the idea of derivative in the concept of instantaneous velocity, in his *Siddhāntaśiromaṇi*Sastry, *Siddhāntaśiromaṇi of Bhāskarācārya with Vāsanābhāṣya* as we shall see later. Bhāskarācārya says:[10]

अत्र यावद्यावन्महद्व्यासार्धं बहूनि च खण्डानि तावत् तावत् स्फुटा ज्या स्यात्।

atra yāvadayāvanmahadvyāsārdhaṃ bahūni ca khaṇḍāni tāvat tāvat sphuṭā jyā syāt.

The larger the radius and the more the number of parts into which an arc is divided (and hence the $smaller$ the parts), the better shall be the accuracy of the $rsine$ of the arc.

The concept of infinitesimals is implied here. The manipulations of multiples of zero and zero-divisors (zero as divisors) are based on two rules: Normally $x0$, a multiple of 0, $equals$ 0.

Rule A If the calculations of an expression containing xo and o, on putting $o = 0$, ultimately results in the indeterminate form $\frac{0}{0}$, thereby eliminating the very quantities which are to be determined then treat xo and o as mere symbols, and do not evaluate them to 0 *i.e., xo and o are not equal to 0.*

Rule B The 0 in the numerator of the indeterminate form $\frac{0}{0}$ could be reduced to a multiple $x'0$ of 0 (of the denominator), giving the result $\frac{x'0}{0} = x'$.

The above rules are stated briefly by Bhāskarācāryain his *Līlāvatī*:

......खगुणः खं खगुणश्चिन्त्यश्च शेषविधौ॥ ४५ ॥
शून्ये गुणके जाते खं हारश्चेत्पुनस्तदा राशिः।
अविकृत एव ज्ञेयः ......॥ ४६ ॥

......khaguṇaḥ khaṃ, khaguṇaścintyaśca śeṣavidhau || 45 ||
śūnye guṇake jāte khaṃ hāraścetpunastadā rāśiḥ |
avikṛta eva jñeyaḥ......|| 46 ||

Normally a multiple of zero xo, is zero.

Rule 1 If, however, further (mathematical) calculations are there, multiple of zero, $x0$, should be regarded as not zero. (0 is therefore not zero). *Buddhivilāsinī* Apte, *Līlāvatī* (*In Sanskrit*) explains that it is treated as mere composite symbol (See Appendix).

Rule 2 If multiple of zero be followed by further operation of division by zero, i.e., $\frac{x0}{0}$, it is to be understood that the multiplicand x remains unaltered i.e., $\frac{x0}{0} = x$. (Thus $x0$ is treated as not zero, and the indeterminate form is avoided.) *Buddhivilāsinī* Apte, *Līlāvatī* (*In Sanskrit*) adds:

Rule 3 If, however, division by zero is not there, $x0$ is zero (See Appendix A for complete translation of the Sanskrit text for these rules).

The following example given in śloka 47 *Līlāvatī*Apte, *Līlāvatī* (*In Sanskrit*) by Bhāskarācārya illustrates the $infinitesimal$ nature of multiple of zero.

Example 1 In mathematical terms it amounts to solving the equation:

$$\frac{(x0 + \frac{x0}{2})3}{0} = 63.$$

[10] *Siddhāntaśiromaṇi*Sastry, *Siddhāntaśiromaṇi of Bhāskarācārya with Vāsanābhāṣya*, p. 42.

It is obvious that Bhāskarācārya could not have meant 0 = zero in this case, for, if it were so, x which is to be determined, would itself be eliminated and the problem becomes purposeless. Since there are further calculations, by **Rule 1**, $x0$ is not zero. On simplification we get $\frac{x0}{0}$ = 14. Therefore by **Rule 2** of the above algorithm, x = 14.

It is interesting to see how Newton solves the following similar example.

Example 2 Newton would, for example, solve the equation $\frac{(x+o)^3 - x^3}{o}$ = 12. as follows.

Since there are further calculations, by **Rule 1**, $o \neq 0$. Hence $\frac{(3x^2 + 3xo + o^2)o}{o}$ = 12 and by **Rule 2**, cancelling o, $3x^2 + 3xo + o^2$ = 12.

As no further calculations are there, by **Rule 3**, o = 0. Hence $3x^2$ = 12, giving $x = \pm$ 2.

Bhāskarācārya says that Mathematics of zero manipulations are very useful in Astronomy. We shall consider this now.

3.2 Comparing the algorithms of Newton and Bhāskarācārya

Newton's method	**Bhāskarācārya's method**
Newton's calculation of derivative $\frac{dy}{dx}$ involves the following algorithm. Let $\dot{x}o$ be an infinitesimal increment in x and $\dot{y}o$ the increment in y. Let $o \neq$ 0. **Rule (i)**	If in any expression *multiples of zero* $x0$ occur, 0 is treated as a mere symbol kept by the side of x as a composite symbol $x0 \neq 0$. That is *if there are calculations to be done*, let $x0 \neq zero$ **Rule (i)**. Hence 0 $\neq$ zero.
Newton makes all the calculations till he reaches the form $\frac{f'(x,o)o}{o}$ and $cancels$ o, to get the desired ratio of *fluxions* $$\frac{\dot{y}o}{\dot{x}o} = \frac{f'(x,o)o}{o} = f'(x,o) \qquad \textbf{Rule (ii)}.$$ He, however, finds that there are still some infinitesimals involved in $f'(x,o)$, which cannot be ignored,** errors being unacceptable in mathematics.	Having done the pending operations, if $x0$ *has further operation of division** by zero then* $\frac{x0}{0}$ = x, by cancellation **Rule (ii)**. [Since there is further operation of division by 0, by **Rule (i)**, $x0 \neq$ zero therefore 0 $\neq$ zero. Hence cancellation of the two zeroes is justified].
When all the calculations are made he says let o = 0 **Rule (iii)**. This is same as defining $\left[\frac{\left(\frac{\dot{y}}{\dot{x}}\right)o}{o}\right]_{o=0}$ as $\frac{\dot{y}}{\dot{x}}$. $\left[\frac{(\frac{\dot{y}}{\dot{x}})o}{o}\right]_{o=0}$ as $\frac{\dot{y}}{\dot{x}}$.	*If all the calulations are over, then* $x0$ = 0 (and 0 = zero) **Rule(iii).** This is same as defining $\left[\frac{x0}{0}\right]_{0\,=\,\text{zero}}$ as x. $[\frac{x0}{0}]0{=}zero$ as x.

*The two 0s in $\frac{x0}{0}$ are treated as *same* infinitesimals, whereas $x0$ and 0 are *different* infinitesimals.

** It is useful to consider infinitely small quantities such that when their ratio is sought they may not be considered zero, but which are $rejected$[11] as often as they occur with the quantities incomparably greaterCajori, *A*

[11] virtually treated as zero.

History of Mathematics.

Rule (ii) anticipates the reduction of $\frac{0}{0}$ to the form $\frac{x.0}{0}$.

4 Geometric treatment of infinitesimals

4.1 Newton's method

We illustrate Newton$'s$ geometric method of finding the derivative[12] in which he considers the tangent as a rotating secant KEW in its limiting position KTS. Newton was of the opinion that the *fluxions* of geometric entities such as lines, areas, angles, etc., can be obtained and it is not necessary to introduce them into geometry[13]. But he rightly thought it necessary to demonstrate their role, while considering the tangent as a limit of a secant, as shown below.

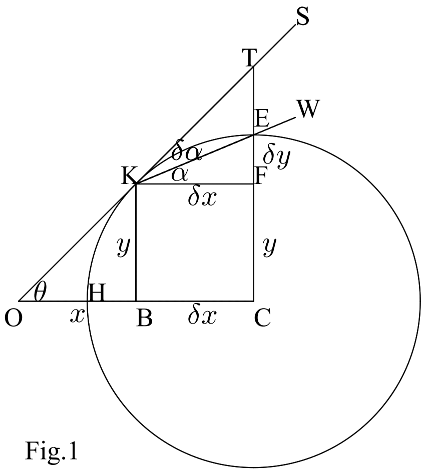

Fig.1

In Fig. 1, consider a point K on the circle, and a secant KEW. Here OKS is the tangent at K, and let the angles. $\angle KOB = \angle TKF = \theta$. The secant KEW is rotated about K till, the two points K and E, ultimately coincide and KEW becomes the tangent KTS, so that in the process the chord KE coincides with the arc KE. In this ultimate position $\delta\alpha$ is 0 and $\alpha = \theta$. Newton states that arc KE is the *fluxion* (increment) of arc HK, BC the *fluxion* of OB, and FE that of CF. These *fluxions* are in fact the infinitesimals of geometric entities.

In modern notation *fluxion* stands for δ so that *fluxion* of $\alpha = \delta\alpha$, *fluxion* of $x = \delta x$ and *fluxion* of $y = \delta y$. As the secant KEW approaches the tangent KTS the line TEF approches the point K and ultimately coincides with it. Also $\delta\alpha$ approaches zero, and α becomes equal to θ.

Therefore the two $dissimilar$ triangles KEF and KTF are $almost$ similar to the finite $\triangle$OKB, for small $\delta\alpha$. So we have

$$\frac{\delta y}{\delta x} = \frac{FE}{KF} \approx \frac{FT}{KF} = m.$$

Thus in the limiting case, the slope of the secant = the slope of the tangent. That is,

$$m = \frac{dy}{dx} = \frac{\dot{y}}{\dot{x}}.$$

This is similar to the following geometric treatment of instantneous quanties by Bhāskara.

4.2 Bhāskarācārya's method

Bhāskara considers *tātkālikagati* or *sūkṣmagati* which means instantaneous motion or displacement BD in a given small interval of time, such as daily motion of slow moving planets, or even smaller interval of time (see Fig.2),

The term *tātkālikabhogyakhaṇḍam* or *tātkālikadorjyayorantaram* which stands for the $resulting$ instantaneous $r$sine-difference TD (See Fig. 4) between two successive $r$sines is given by

$$\delta(r\sin\theta) = (r\cos\theta)\frac{r\delta\theta}{r} = (r\cos\theta)\frac{BD}{r}, \quad (1)$$

where

$$\delta(r\sin\theta) = r\sin\,(\theta+\delta\theta) - r\sin\theta.$$

These are used to find the instantaneous position of planetsJoseph, *The Crest of the Peacock*. Bhāskarācārya proves the relation (1) in two parts.

[12] A History of MathematicsCajori, *A History of Mathematics*, p. 197.

[13] Ibid., p. 198.

Part (i) First he considers sine differences for $various$ arc lengths at $constant$ intervals of $225'$ as given in the $r$sine-table he had prepared, and proves geometrically that

$$\delta'(r\sin\theta) = (r\cos\theta)\frac{225'}{r}. \qquad (2)$$

The prime $'$ on δ indicates *constant* intervals of $225'$.

Part (ii) Next he considers sine differences for a *given* arc length at *variable* intervals of $r\delta\theta$. Using the interpolation formula he proves first for small daily motion of anomaly (*kendragati*), and later for a small displacement $15'$ (the radius (*bimbārdha*) of the Sun), the formula $\delta(r\sin\theta) = (r\cos\theta)\frac{r\delta\theta}{r}$.

तात्कालिकभोग्यखण्डकरणायानुपातः।
tātkālikabhogyakhaṇḍakaraṇāyānupātaḥ.

Use of Proportions to obtain the instantaneous sine difference.

In Figure 2 and 3, $r = 3438'$, $r\delta'\theta = 225' =$ arc $BD \approx BD$, and $TD = r\sin(\delta'\theta) \approx r\delta'\theta = 225'$.

∠ BOD = $\delta'\theta$
∠TDB = $\theta + \frac{\delta'\theta}{2} \approx \theta$

Initially $\theta = 0$,

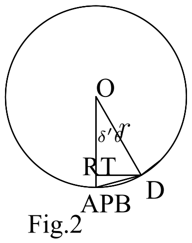

Fig.2

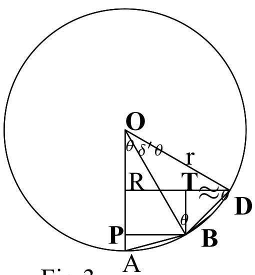

Fig.3

Bhāskara's proof of Part (i)

The proof of (2) as given in Bhāskarācārya's *vāsanābhāṣya* (commentary) proceeds as follows. He commences the proof by difining *kendragati*:[14]

अद्यतनश्वस्तनकेन्द्रयोरन्तरं केन्द्रगतिः।

adyatanaśvastanakendrayorantaraṃ kendragatiḥ

The arc-distance, $r\delta\theta$, between the positions of (centre of) the planet, say, at Sun rise of today and tomorrow, is the daily motion (displacement) of planet.

In the previous paragraph Bhāskarācārya says:

तस्य कालस्य मध्येऽनया गत्या ग्रहश्चालयितुं-युज्यत इति। इयं किल स्थूला गतिः।

tasya kālasya madhye'nayā gatyā grahaścālayituṃyujyata iti. iyaṃ kila sthūlā gatiḥ.

During this interval this daily motion can be used to find planetary position at any instant. This indeed is approximate.

भुजज्याकरणे यद्भोग्यखण्डं तेन सा गुण्या शरद्विदस्रैर्भाज्या।

bhujajyākaraṇe yadbhogykhaṇḍaṃ tena sā guṇyā śaradvidasrairbhājyā.

This (arc $r\delta\theta$) be multiplied by the $r$sine-difference δ' $(r\sin\theta)$ (for the arc $r\theta$), obtained from the $r$sine-differences[15] prepared (at intervals of $\delta'\theta = 225'$) and divided by 225.

This prescription essentially gives the $r$sine-difference $\delta'(r\sin\theta)\dfrac{r\delta\theta}{225}$ for the arc $r\theta$ and the $small$ interval $r\delta\theta$. Then it is stated:

तत्र तावत् तात्कालिकभोग्यखण्डकरणायानुपातः।

tatra tāvat tātkālikabhogyakhaṇḍakaraṇāyānupātaḥ.

To get this instantaneous $r$sine-difference, here is the method of direct proportions.

Now let's consider the similar triangles BDT and BOP (see Figs 3 and 4). With respect to these similar triangles Bhāskara lays down the rule of proportion as follows:

[14] *Siddhāntaśiromaṇi*Sastry, *Siddhāntaśiromaṇi of Bhāskarācārya with Vāsanābhāṣya*, pp. 52-53, verses 36-37.
[15] Bhāskarācārya's *Siddhāntaśiromaṇi*Sastry, *Siddhāntaśiromaṇi of Bhāskarācārya with Vāsanābhāṣya*, slokas 2-9, p. 40.

यदि त्रिज्यातुल्यया कोटिज्ययाद्यं भोग्यख-ण्डं शरद्विदस्रतुल्यं लभ्यते तदेष्टया किमित्य-त्र कोटिज्यायाः शरद्विदस्रा २२५ गुणस्त्रिज्या हरः।

yadi trijyātulyayā koṭijyayādyaṃ bhogyakhaṇḍaṃ śaradvidasratulyaṃ labhyate tadeṣṭayā kimityatra koṭijyāyāḥ śaradvidasrā 225 guṇastrijyā haraḥ.

It may noted that triangles BDT and BOP are almost similar for all θ and the given small interval of $225'$.

If initial $r$sine-difference (225') is obtained from an altitude equal to the radius r, then what is the $r$sine-difference for any desired altitude ($r\cos\theta$)? See Fig. 3.

फलं तात्कालिकं स्फुटभोग्यखण्डम्....

phalaṃ tātkālikaṃ sphuṭabhogyakhaṇḍaṃ ...

The result is the instantaneous $r$sine-difference.

That is, $\delta'(r\sin\theta) = (r\cos\theta)\frac{225'}{r}$.

Bhāskara's proof of Part (ii)

The proof of the result given by (3) commences with the following argument.

तेन केन्द्रगतिर्गुणनीया शरद्विदस्रैर्भाज्या। अत्र शरद्विदस्रमितयोर्गुणकभाजकयोस्तुल्यत्वा-न्नाशे कृते केन्द्रगतेः कोटिज्यागुणस्त्रिज्या हरः स्यात्। फलं अद्यतनश्वस्तनकेन्द्रदोर्ज्ययोरन्तरं भवति।

tenakendragatirguṇanīyā śaradvidasrairbhājyā atra śaradvidasramitayorguṇakabhājakayostulyatvānnāśe kṛte kendragateḥ koṭijyāguṇastrijyā haraḥ syāt phalaṃ adyatanaśvastanakendradorjyayorantaraṃ bhavati.

By this $r$sine-difference be multiplied the daily motion[16] and divided by $225'$. When we cancel $225'$ from the numerator and the denominator what we obtain is the product $r\cos\theta$ BC divided by r. The result would be the difference in the daily motion.

This is just interpolation which follows from the Rule of Three or from the almost similar triangles BDT and BCSSharma, *Siddhāntaśiromaṇi of Bhāskarācārya* similar to the finite triangle POB, see Fig. 4.

The infinitesimal triangles BDT and BCS are (*almost*) $\sim$ to the finite $\triangle$ BOP, for $small\ \delta\theta$ and for all θ including the initial value $\theta = 0$, Fig. 2.

It may be noted that

$$\triangle TDB = \theta + \frac{\delta'\theta}{2} \approx \theta.$$

Also,

$$\begin{aligned} TD &= RD - PB \\ &= r\sin\,(\theta + \delta'\theta) - r\sin\theta. \end{aligned}$$

Morevoer,

$$D\ \angle SCB = \theta + \frac{\delta\theta}{2} \approx \theta$$

$$SC = r\sin\,(\theta + \delta\theta) - r\sin\theta = r\cos\theta\left(\frac{BC}{r}\right).$$

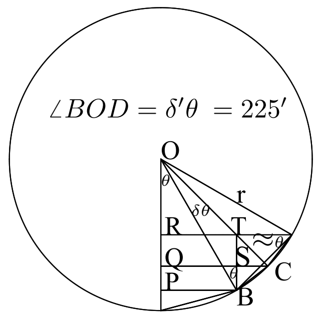

Fig.4

Bhāskarācārya applies the above method to find $r$sine-difference for the radius of Sun's disk, as explained in *Siddhāntaśiromaṇi*Sastry, *Siddhāntaśiromaṇi of Bhāskarācārya with Vāsanābhāṣya* given below:

[16] The daily motion denored by $BC\ (= r\delta\theta)$, will be less than $225'$ minutes for all planets except in the case of the moon.

यदि त्रिज्यातुल्यायां कोटौ प्रथमं ज्यार्धं शरद्विदस्रा भोग्यखण्डं तदाभिमतायामस्यां किमिति फलं स्फुटं भोग्यखण्डम्। तेन गुणितं बिम्बार्धं शरद्विदस्रैर्भाज्यं। एवं स्थिते शरद्विदस्रमितयोर्गुणहरयोर्नाशे कृते बिम्बार्धस्य कोटिज्यागुणस्त्रिज्याहरः फलं दोर्ज्ययोरन्तरम्।

yadi trijyātulyāyāṃ koṭau prathamaṃ jyārdhaṃ śaradvidasrā bhogyakhaṇḍaṃ tadābhimatāyāmasyāṃ kimiti phalaṃ sphuṭaṃ bhogyakhaṇḍam. tena guṇitaṃ bimbārdhaṃ śaradvidasrairbhājyaṃ□ evaṃ sthite śaradvidasramitayorguṇaharayornāśe kṛte bimbārdhasya koṭijyāguṇastrijyāharaḥ phalam dorjyayorantaram.

Here Bhāskarācārya uses the rationale exactly as given above.

The only difference here is that $r\delta\theta$ stands for the arc-difference $15'$, corresponding to half the width of Sun's disk, giving the r sine-difference relation: $\delta(r\sin\theta) = r\cos(\theta)(\frac{r\delta\theta}{r})$.

4.3 Trigonometric method

Bhāskarācārya gives a hint as to how the above results may be obtained using the expansion[17] of $r\sin(x+1)^\circ$. The relation given by Bhāskara is:

$$r\sin(x+1)^\circ = r\sin(x)^\circ - \frac{r\sin(x)^\circ}{6569} + \frac{10r\cos(x^\circ)}{573},$$

The rationale behind the above expression can be understand as follows. The sine addition formule was well known. Hence,

$$r\sin\,(x+1)^\circ = r\sin x^\circ \cos 1^\circ + r\cos x^\circ \sin 1^\circ.$$

Here,

$$\begin{aligned} r\cos 1^\circ = r\left(\frac{6568}{6569}\right) &\approx r \text{ and } r\sin\,(1^\circ) \\ = r\,\left(\frac{10}{573}\right) &\approx 60'. \end{aligned}$$

$$\begin{aligned} \delta(r\sin x^\circ) &= r\sin(x+1)^\circ - r\sin x^\circ \\ &= r\cos x^\circ \frac{60}{r}. \end{aligned}$$

For arbitrarily small interval $r\delta x$, by rule of three

$$\delta(r\sin(x)) = r\cos(x)\frac{r\delta x}{r}.$$

4.4 Infinitesimals and zero-manipulations, involved in the geometric treatment by Bhāskarācārya

In Fig. 5 consider the arc BD($= r\delta\theta$). This arc represents the *tātkālikagati or infinitesimal increment of arc* AB ($= r\theta$). TD which represents the *tātkālikabhogyakhaṇḍam* is the infinitesimal increment of PB ($= r\sin\delta\theta$). As the line OD approaches and reaches OB, the triangle TDB becomes a point right triangle and $\dfrac{TD}{BD}$ assumes the form $\frac{0}{0}$. For this to have any meaning the infinitesimal TD must be, a multiple of 0, say, $x0$. Hence

$$\frac{TD}{BD} = \frac{x0}{0} = x,$$

by Bhāskara's Rule (ii). This point right triangle TDB, is similar to finite trianlge POB. Thus we have $\dfrac{TD}{BD} = \dfrac{OP}{OB}$, from which we get

$$\delta\;(r\sin\theta) = r\cos\theta\;\left(\frac{BD}{r}\right).$$

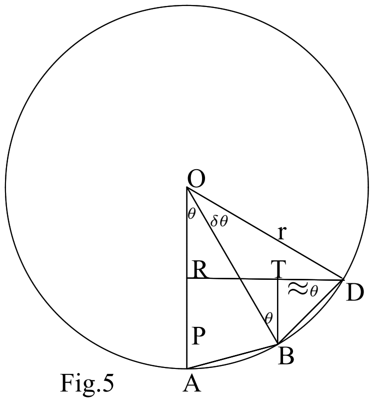

Fig.5

Bhāskarācārya was indeed referring to these manipulations of infinitesimals Apte, *Līlāvatī* (*In Sanskrit*) when he said that these zero manipulations are very useful in astronomy.

[17]English translation of *Siddhāntaśiromaṇi*Wilkinson, *Siddhāntaśiromaṇi, An English Translation*, *ślokas* 16 and 17, p. 267.

5 Conclusion

Though there are strking similarities between the algorithms of Newton and Bhāskarācārya there are differences in perspective. Newton, Leibnz and Euler and Bhāskarācārya treat the infinitesimal nature of *small* quantities virtually as irrelevant. These quantities are either 0 or not 0, as the case may be. This zero-nonzero dichotomy was there which was implicit in the case of Newton, Leibnz and Euler and was rather explicit in Bhāskarācārya's case.

Unlike Newton Bhāskarācārya does not explicitly state the infinitesimal nature of multiples of zero. The idea of infinitesimals is implied in his treatment of multiples of zero and zero divisors. This was restrcted to the $r$sine and $r$cosine functions, and he found it handy to use the Rule of Three and the proportional properties of similar right triangles.

He uses geometric method ingeniously thereby avoiding the indeterminate form $\frac{0}{0}$ and replaces Rule of Proporions by the Rule of Three by identifying the *almost* proportional quantities involved, in a way difficult to imagine. In this geometric process are involved dissimilar right angled triangles tending to similarity to a finite triangle, as they converge to a point. This avoids direct encounter with the indeterminate form $\frac{0}{0}$ and gives us an impression that we are dealing with finite quantities. Newton also says in his geometric treatment of slope of a tangent as the derivative, infinitesimals are not necessary as they can be denoted by finite lengths.

Bhāskarācārya's use of the concept of infinitesimals was restricted to applications in planetary calculations unlike that of Newton field of application was wide and varied. In Bhāskarācārya's times due to lack of proper notations and motivation there was no development of Calculus, as such, though clearly, there was development of Calculus by Mādhava and NīlakaṇṭhaDunham, *The Calculus Gallery, Masterpieces from Newton to Lebsegue* of Kerala School of Mathematics and others, during 15th and 16th centuries leading to the expansion of sine and inverse tangent functions etc.

These developments by Mādhava were again a continuation of geometric treatment of infinitesimals, whereas Newton used the method of term by term differenciation of infinite series unmindful of the convergence problem involved in it. Considering these facts Bhāskarācārya's bold attempt to give an algorithm to deal with infinitesimals and use these ideas in astronomy is commendable. Highlighting these facts is appropriate in his 900th birth year which is being presently celebrated.

Appendix

Buddhivilāsinī [18] makes this more clear, as given below.

> शेषस्य विधौ कर्तव्ये सति खगुणश्चिन्त्यः। तथा हि राशेः शून्ये गुणके प्राप्ते तस्यान्यो विधिश्चेदस्ति, तदा खगुणो राशिः खं स्यादिति न कार्यम्। किन्तु शून्यं तत्पार्श्वे गुणकस्थाने स्थाप्यम्।
>
> *śeṣasya vidhau kartavye sati khaguṇaścintyaḥ. tathā hi rāśeḥ śūnye guṇake prāpte tasyānyo vidhiścedasti tadā khaguṇo rāśiḥ khaṃ syāditi na kāryam .kintu śūnyaṃ tatpārśve guṇakasthāne sthāpyam.*
>
> Further operations pending multiple of zero be given a second thought. That is, if a number is multiplied by zero and there are further operations remaining then the multiple of zero is not to be construed as zero, but 0 is to be kept by the side of the number (as a mere symbol)
>
> ततः शेषविधाने कृते पुनः खं हरश्चेत्तदा तयोः शून्यगुणकहरयोस्तुल्यत्वेन नाशः कार्यः। नो चेत् खं हरः, तदा खगुणो राशिः खं स्यात्...
>
> *tataḥ śeṣavidhāne kṛte punaḥ khaṃ haraścettadā tayoḥ śūnyaguṇakaharayostulyatvena nāśaḥ kāryaḥ . no cet khaṃ haraḥ, tadā khaguṇo rāśiḥ khaṃ syāt.*
>
> If, having done all the operations, there is further operation of division by zero then the denominator and the numerator

[18] *Līlāvatī*Apte, *Līlāvatī* (*In Sanskrit*) pp. 39, 40.

being equal, they be cancelled (0 being treated as a mere symbol). If, however, there is no division by zero then the multiple of zero is to be treated as zero.

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