REDUCTION FORMULAE FOR TRIPLE GAUSSIAN
HYPERGEOMETRIC SERIES

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The elementary manipulation of series is employed to give reduction formulae for three classes of triple Gaussian hypergeometric series as consequences of the Euler transform of the hypergeometric function \( _2F_1 \).

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Certain members of Horn's list of double hypergeometric functions of the second order are developed so as to give new reduction formulae for some of the triple Gaussian hypergeometric series listed by Srivastava and Karlsson\(^1\), Table 4. These results are deduced as consequences of Euler's transform.

\[
_2F_1[a, b; c; x] = (1 - x)^{-a} _2F_1[a, c - b; c; x/(x - 1)], \tag{1}
\]

see Erdelyi\(^2\), 109.

The Horn functions under consideration are conveniently listed by Srivastava and Karlsson\(^1\), 24 and are now given. The Pochhammer symbol \((a, n) = \Gamma(a + n)/\Gamma(a)\) is frequently used in the study of hypergeometric functions.

\[
G_1[a, b, b'; x, y] = \sum_{m, n} [(a, m + n) (b, n - m) (b', m - n) x^m y^n]/[m! n!], \tag{2}
\]

\[
G_2[a, a', b, b'; x, y] = \sum_{m, n} [(a, m) (a', n) (b, n - m) (b', m - n) x^m y^n]/[m! n!] \tag{3}
\]

and

\[
H_2[a, b, c, f; g; x, y] = \sum_{m, n} [(a, m - n) (b, m) (c, n) (f, n) x^m y^n]/[(g, m) m! n!]. \tag{4}
\]

The members of Srivastava and Karlsson's\(^1\) table under consideration have the following series representations:
\[ \Phi_{3b}[a_1, a_2, a_3, a_4, b, c; x, y, z] \]
\[ = \sum_{m, n, p} [(a_1, m) (a_2, n) (a_3, p) (b, m + n - p) x^m y^n z^p]/[(c, m)n! p!], \quad \ldots \tag{5} \]

\[ \Phi_{3f}[a_1, a_2, a_3, b_1, b_2; x, y, z] \]
\[ = \sum_{m, n, p} [(a_1, m + n) (a_2, m) (a_3, p) (b_1, n - p) (b_2, p - m - n) x^m y^n z^p]/[m! n! p!], \quad \ldots \tag{6} \]

and

\[ \Phi_{18d}[a_1, a_2, b_1, b_2; x, y, z] \]
\[ = \sum_{m, n, p} [a_1, m + n + p) (a_2, n) (b_1, m - p) (b_2, p - m - n) x^m y^n z^p]/[m! n! p!]. \quad \ldots \tag{7} \]

As a simple consequence of the binomial theorem, we have

\[ G_1[a, b, b'; x, y + z] = \sum_{m, n, p} [(a, m + n + p) \]
\[ (b, n + p - m) (b', m - n - p) x^m y^n z^p]/[m! n! p!]. \quad \ldots \tag{8} \]

The right-hand series can be expanded as a double series of \( _2F_1 \) functions, namely,

\[ \sum_{m, n, p} [(a, m + n) (b, n - m) (b', m - n) x^m y^n z^p]/[m! n!] \]
\[ _2F_1[a + m + n, b + n - m; 1 - b' - m + n; -z]. \quad \ldots \tag{9} \]

On applying (1) to the inner hypergeometric function, this takes the form

\[ (1 + z)^{-a - m - n} _2F_1[a + m + n, 1' b' b'; 1' b' - m + n; z/(1 + z)], \quad \ldots \tag{10} \]

so that (5) becomes

\[ (1 + z)^{-a} \sum [(a, m + n + p) (b, n - m) (b', m - n - p) (1 - b - b', p) \]
\[ x'/y/(1 + z)^m y'/y/(1 + z)^n [-z/(1 + z)]/[(m! n! p!)] \quad \ldots \tag{11} \]

and it follows that

\[ \Phi_{18d}[a, 1 - b - b', b', y/(1 + z), -z/(1 + z), x/(1 + z)] \]
\[ = (1 + z)^a G_1[a, b, b'; x, y + z]. \quad \ldots \tag{12} \]
Similarly,

\[ \Phi_{3f} [a', 1 - b - b', a, b, b'; - z/(z + 1), y/(z + 1), z] \]
\[ = (1 + z)^a G_2[a, a', b, b'; x + y + z] \] ... (13)

and

\[ \Phi_{3b} [g - b, b, c, f, a; g; x/(x - 1), y/(1 - x), z(1 - x)] \]
\[ = (1 - x)^a \mathcal{H}_2[a, b, c, f; g; x + y, z]. \] ... (14)

The function \( \Phi_{3b} \) has previously been encountered by Exton in connection with a study of
the system of partial differential equations associated with the Lauricella function of the fourth type.
Further results now follow from (8), (9) and (10) respectively by putting \( z = -y \) and \( y = -x. \)

\[ \Phi_{18d} [a, 1 - b - b'; b, b'; y/(1 - y), y/(1 - y), x/(1 - y)] = (1 - y)^a \, 2F_1 [a, b'; 1 - b; x] \]
\[ = (1 - y)^a \, 2F_1 [a, b'; 1 - b; x] \] ... (15)

\[ \Phi_{3f} [a', 1 - b - b', a, b, b', y/(1 - y), y/(1 - y), -x] = (1 - y)^a \, 2F_1 [a, b'; 1 - b; x] \]
\[ = (1 - y)^a \, 2F_1 [a, b'; 1 - b; x] \] ... (16)

and

\[ \Phi_{3b} [g - b, b, c, f, a; g; x/(x - 1), x/(x - 1), z(1 - x)] = (1 - x)^a \, 2F_1 [c, f; 1 - a; x] \]
\[ = (1 - x)^a \, 2F_1 [c, f; 1 - a; x] \] ... (17)

REFERENCES