ENUMERATION OF BIPARTITE SELF-COMPLEMENTARY GRAPHS

N. S. BHAVE AND T. T. RAGHUNATHAN

Department of Mathematics, University of Pune, Pune 411 007

(Received 17 February 1997; accepted 15 May 1997)

A bipartite graph is a graph (simple) $G$ where the vertex set $V$ can be decomposed into two subsets $V_1$ and $V_2$ such that each edge of $G$ joins a vertex of $V_1$ to a vertex of $V_2$. Given a bipartite graph $G(V_1, V_2)$ its bipartite — complement is defined as the bipartite graph $\overline{G}(V_1, V_2)$ whose vertex set is $V(G)$ and the edge set is $\{uv | u \in V_1, v \in V_2 \text{ and } uv \notin E(G)\}$. A bipartite graph will be said to be bipartite self-complementary if it is isomorphic to its bipartite — complement. In this paper we obtain generating functions to enumerate the bipartite self-complementary graphs with a given bipartition.

Key Words: Bipartite Graph; Self-Complementary Graph

1. INTRODUCTION

Formulae for the numbers of self-complementary graphs and digraphs were obtained by Read\(^1\) using de Bruin's extension\(^2\) of Polya's theorem. Parthasarathy and Sridharan\(^3\) obtained generating functions for these graphs in terms of their partitions. In this paper, we obtain generating functions for bipartite self-complementary graphs in terms of partitions.

All graphs are assumed to have no multiple edges or loops. We refer the reader to Harary\(^4\) for notations in graph theory. A graph $G$ is said to be bipartite if its vertex set $V$ can be partitioned into two subsets $V_1$ and $V_2$ such that every edge joins a vertex of $V_1$ to a vertex of $V_2$ and is denoted by $G(V_1, V_2)$. Gangopadhyay\(^5\) introduced the notion of bipartite — complement of a graph as follows: Given a bipartite graph $G(V_1, V_2)$ its bipartite — complement is defined to be the bipartite graph $\overline{G}$ whose vertex set is $V(G)$ and the edge set $E(\overline{G})$ is $\{uv | u \in V_1, v \in V_2 \text{ and } uv \notin E(G)\}$. A bipartite graph $G$ is said to be bipartite self-complementary if $G$ is isomorphic to $\overline{G}$. 
2. BIPARTITE GRAPHS $G(m, n)$ WITH $m \neq n$

Let $X$ and $Y$ be two sets. Let $D = X \times Y$ be the cartesian product of $X$ and $Y$ and $R = \{0, 1\}$. Then there is a correspondence between functions from $D$ to $R$ and the bipartite graphs on $X \bigcup Y$. Here we consider the case when $|X| \neq |Y|$. For the permutation group $K$ acting on $D$ we take $S_m \times S_n$, $|X| = m$, $|Y| = n$. Isomorphic bipartite graphs on $X \bigcup Y$ correspond to equivalent functions from $D$ into $R$.

The generating function for the bicoloured graphs is given by Polya's theorem as

$$\sum_{f \in \mathcal{F}} W(f) \equiv S_1(W) = \frac{1}{m! \cdot n!} \sum_{k \in S_m \times S_n} \sum_{f} W(f),$$

where

$$W(f) = 0 \prod_{x_i \in X} \prod_{y_j \in Y} (u_i v_j)^{f(x_i y_j)} \equiv 0W_{0f};$$

the effect of 0 is to replace $W_{0f}(f)$ by the leading term of the symmetric function product $(\pi_m)_u (\pi_n)_v$ corresponding to the bipartition $(\pi_m/\pi_n)$ of the bipartite graph represented by $f$.

Under the weight function isomorphic graphs have equal weights but a graph and its bipartite — complement may have, in general, different weights. To obtain the same weight for a graph and its bipartite — complement we use the modified weight function

$$W_1(f) = \overline{\pi} \prod_{x_i \in X} \prod_{y_j \in Y} (u_i v_j)^{f(x_i y_j)}$$

where $\overline{\pi}$ acts as follows:

$$\overline{\pi} (u_1^{s_1} u_2^{s_2} \ldots u_m^{s_m}/v_1^{t_1} v_2^{t_2} \ldots v_n^{t_n})$$

$$= u_1^{s_1} u_2^{s_2} \ldots u_m^{s_m}/v_1^{\tau_1} v_2^{\tau_2} \ldots v_n^{\tau_n} \text{ if } \pi_v < \overline{\pi}_v$$

$$= u_1^{s_1} \ldots u_m^{s_n}/v_1^{\tau_1} \ldots v_n^{\tau_n} \text{ if } \overline{\pi}_v < \pi_v$$

where $\pi_v = (s_v, \ldots, s_v/t_{\mu}, \ldots, t_{\mu})$ and $\overline{\pi}_v = (S_v, \ldots, S_v/T_{\mu}, \ldots, T_{\mu})$.

with $S_v = n - s_v$ and $T_{\mu} = m - t_{\mu}$ are monotonic nonincreasing rearrangements of $(s_1, \ldots, s_m/t_{\mu}, \ldots, t_{\mu})$ and $<$ is defined as follows. Let $\pi_1 = (d_1, \ldots, d_m/e_1, \ldots, e_n)$,

$$\pi_2 = (f_1, \ldots, f_m/g_1, \ldots, g_n). \pi_1 < \pi_2 \text{ if there exists } N_1, N_2 \text{ such that } d_i = f_i, i = 1, \ldots, N_1,$$

$$e_j = g_j, j = 1, 2, \ldots, N_2 \text{ and } d_{N_1+1} > f_{N_1+1} \text{ and } e_{N_1+1} > g_{N_1+1}.$$

We now require the general form of a counting theorem due to Harary and Palmer. Take $G = S_m \times S_n$, $H = S_2$ acting on $\{0, 1\}$ and change $K$ to the power
group $H^G$ i.e., $S_{2^m} \times S_n$. We have here the Harary-Palmer situation in which the generating function by equivalence classes is given by

$$S_2(W) = \frac{1}{2} \frac{1}{m! \cdot n!} \sum_{k \in S_m \times S_n} \sum_{f} W(f)^{(k, k')}$$

$$= \frac{1}{2} \frac{1}{m! \cdot n!} \left[ \sum_{f} W(f)^{(k, k_1)} + \sum_{f} W(f)^{(k, k_2)} \right]$$

where $k_1 = (0, 1)$, $k_2 = (0, 1)$ and $W(f)$ is any weight function which gives equal weight to equivalent functions.

With $W_1$ for the weight function, $S_1(W_1)$ counts a bipartite non-self complementary graph and its bipartite—complement as two different graphs with the same weight while $S_2(W_1)$ counts them as one graph. A bipartite—self complementary graph is counted as just one graph with the same weight in $S_1(W_1)$ and $S_2(W_1)$. Hence, the expression $2S_2(W_1) - S_1(W_1)$ enumerates bipartite-selfcomplementary graphs since for such graphs the partitions $\pi_v$ and $\bar{\pi}_v$ are equal. Hence, the generating function is

$$2S_2(W_1) - S_1(W_1) = \frac{1}{m! \cdot n!} \sum_{k \in S_m \times S_n} \sum_{f} W_1(f)^{(k, k_2)}$$

To evaluate the generating function we set

$$\chi(k) = \sum_{f} \prod_{x \in X} \prod_{y \in Y} (u_x v_y)^{k(x, y)}$$

for $k \in S_m \times S_n$. Each $k \in S_m \times S_n$ corresponds to a pair $(g, h)$ where $g \in S_m$, $h \in S_n$ are permutations acting on $X$ and $Y$ respectively. The restriction on the summation requires that $f(d) = (0, 1) f(kd)$ for $d = (x_i, y_i) \in X \times Y$. This implies that

$$f(d) = f(k^2 d) = f(k^4 d) = ... , f(kd) = f(k^3 d) = ... , f(d) \neq f(kd).$$

Hence if $k = (g, h)$, then the contribution of $k$ to $\Sigma \chi(k)$ is non-zero only if both $g$ and $h$ do not contain an odd cycle (simultaneously). Thus we must have that either all the cycles of $g$ are even or all the cycles of $h$ are even. This also implies that both $m$ and $n$ cannot be odd. So without loss we assume that $n$ is even. When $m$ is odd we consider only those permutations of $S_m$ which have all even cycles. Thus let $g = 1^4 \cdot 2^4 \cdot ... \cdot m^4$ and $h = 2^m \cdot 4^m \cdot ... \cdot n^m$. We compute $\chi(k)$ for a given pair $(g, h)$ in $S_m \times S_n$ as follows. The elements of $D = X \times Y$ on which $k$ acts may be obtained by considering all cycle pairs $(\alpha, \beta)$ of $k$. Each cycle pair $(\alpha, \beta)$ of $k$ contributes to $\chi(k)$. If lengths of $\alpha$ and $\beta$ are $p$ and $q$ respectively ($p \neq q$, $p$ odd, $q$ even) then $\alpha \times \beta$ forms $(p, q)$ cycles of $k$ of length $[p, q]$ each where $(p, q)$ denotes the gcd and $[p, q]$ the lcm of $p$ and $q$. 
Let $p_i, p_2 \ldots p_k$ be the $i$th cycle of length $p$ in $g$ and $q_j, q_2 \ldots q_k$ be the $j$th cycle of length $q$ in $h$. Let

$$U_{i_p} = u_1 \ldots u_p$$

$$V_{j_q} = v_1 \ldots v_q$$

Then the contribution to $\chi(k)$ from these $[p, q]$-cycles of $(\alpha, \beta)$ is

$$U_{i_p}^{b/2} [V_{j_q}^a + V_{j_q}^b]$$

where $a = p/(p, q), b = q/(p, q)$. There are in all $(p, q)$ such cycles. Thus the contribution from this pair (i.e. $i$th cycle of length $p$ and $j$th cycle of length $q$) is

$$[U_{i_p}^{b/2} \{V_{j_q}^a + V_{j_q}^b\}]^{(p, q)}.$$ 

Now as $g = 1^2 \cdot 2^2 \ldots m^2$, and $h = 2^2 \cdot n^2$, the factor corresponding to odd cycles of $g$ is

$$\prod_{p \text{ odd}} \prod_{q \text{ even}} \prod_{i = 1}^{\lambda_p} \prod_{j = 1}^{\mu_q} [U_{i_p}^{b/2} (V_{1_{i_p}}^a + V_{2_{i_p}}^b)]^{(p, q)}.$$ 

If $p$ and $q$ are both even, the corresponding factor of $\chi(k)$ becomes

$$[(U_{i_p}^b V_{1_{i_p}}^a + U_{2_{i_p}}^b V_{2_{i_p}}^b) (U_{1_{i_p}}^b V_{2_{i_p}}^a + U_{2_{i_p}}^b V_{1_{i_p}}^a)]^{(p, q)/2}.$$ 

[This product covers both $p = q$ and $p \neq q$. If $p = q$ the corresponding term becomes

$$[(U_{i_p}^b V_{1_{i_p}}^a + U_{2_{i_p}}^b V_{2_{i_p}}^a) (U_{1_{i_p}}^b V_{2_{i_p}}^a + U_{2_{i_p}}^b V_{1_{i_p}}^a)]^{(p, q)^2}.$$ 

Hence if $g = 1^k \cdot 2^k \ldots m^k$, $h = 2^k \cdot n^k$, $k = (g, h)$ then

$$\chi(k) = \prod_{p \text{ odd}} \prod_{q \text{ even}} \prod_{i = 1}^{\lambda_p} \prod_{j = 1}^{\mu_q} [U_{i_p}^{b/2} (V_{1_{i_p}}^a + V_{2_{i_p}}^b)]^{(p, q)} \times \prod_{p \text{ even}} \prod_{q \text{ even}} \prod_{i = 1}^{\lambda_p} \prod_{j = 1}^{\mu_q} [(U_{i_p}^b V_{1_{i_p}}^a + U_{2_{i_p}}^b V_{2_{i_p}}^b) (U_{1_{i_p}}^b V_{2_{i_p}}^a + U_{2_{i_p}}^b V_{1_{i_p}}^a)]^{(p, q)/2}$$

Using the above method the following generating functions are obtained for graphs on (3, 2) vertices and (3, 4) vertices respectively.

(3, 2) Vertices :

$$u_1 u_2 u_3 v_1^3 + u_1 u_2 u_3 v_1^2 v_2 + u_1^2 u_2 v_1^2 v_2.$$ 

The corresponding graphs are shown in Figure 1.

(3, 4) Vertices :

$$(u_1 u_2 u_3)^2 v_1^3 v_2^3 + (u_1 u_2 u_3)^2 v_1^2 v_2^2 v_3 + 2(u_1 u_2 u_3)^2 v_1^2 v_2^2 v_3 v_4$$

$$+ u_1^2 u_2^2 v_1^2 v_2^2 v_3 v_4 + u_1^2 u_2^2 u_3 v_1^2 v_3 v_4 + u_1^2 u_2^2 u_3 v_1^2 v_2 v_3 v_4.$$ 

The corresponding graphs are shown in Figure 2.
Fig. 1. Bipartite self-complementary graphs of type (3, 2)

Fig. 2. Bipartite self-complementary graphs of type (3, 4)
3. Bipartite Graphs $G_{m,n}$ with $m = n$

The changes required for this case from the previous one are in the weight function and the permutation group acting on the domain set $D = X \times Y$. To count the number of bipartite graphs with given degree sequence we define the weight function by

$$W(f) = \tilde{0} \ W_0(f) = \tilde{0} \ \prod_{x_i \in X} \ \prod_{y_j \in Y} (u_{ij})^{x_i y_j}$$

where the effect of $\tilde{0}$ is to replace $W_0(f)$ by the leading term of the symmetric function product $(\pi_1^a)^{a} \times (\pi_1^b)^{b}$, corresponding to the bipartition $(\pi_1^a, \pi_1^b)$ of the graph represented by $f$ if $\pi_1^a > \pi_1^b$; if however $\pi_1^a < \pi_1^b$, $\tilde{0}$ replaces $W_0(f)$ by the leading term $(\pi_1^a)^{a} \times (\pi_1^b)^{b}$. Here $\pi_1^a$ and $\pi_1^b$ denote two $n$-part partitions of the same number of lines.

Under the weight function $\tilde{0}$, isomorphic bipartite graphs have equal weights but a graph and its bipartite complement have in general different weights. To obtain the same weight for a graph and its bipartite-complement, we use the modified weight function

$$W_1(f) = \tilde{0} \left[ \tilde{0}\left( \prod_{x_i \in X} \ \prod_{y_j \in Y} (u_{ij})^{x_i y_j} \right) \right]$$

where $\tilde{0}$ (\tilde{0}) is as follows:

Given $\pi_1^a/\pi_1^b$ two $n$-part partitions of the same number of lines, if $\pi_1^a > \pi_1^b$ attach $\pi_1^a$ to the $X$'s and $\pi_1^b$ to the $Y$'s. If, however, $\pi_1^b > \pi_1^a$ attach $\pi_1^a$ to the $Y$'s and $\pi_1^b$ to the $X$'s, after having fixed these, call the resulting sequence as $\pi^* = (s_1, ..., s_n/t_1, ..., t_n)$. Then operate $\tilde{0}$, i.e. consider

$$\tilde{0}(u^{s_1}_1, ..., u^{s_n}_n/v^{t_1}_1, ..., v^{t_n}_n)$$

$$= u^{s_1}_1 \cdot u^{s_2}_2 \cdot ... \cdot u^{s_n}_n \times v^{t_1}_1 \cdot v^{t_2}_2 \cdot ... \cdot v^{t_n}_n \quad \text{if} \quad \pi^*_v \geq \pi^*_v$$

$$= u^{s_1}_1 \cdot u^{s_2}_2 \cdot ... \cdot u^{s_n}_n \times v^{t_1}_1 \cdot v^{t_2}_2 \cdot ... \cdot v^{t_n}_n \quad \text{if} \quad \pi^*_v \geq \pi^*_v$$

where

$$\pi^*_v = (s_v, s_v, ..., s_v/t_v, t_v, ..., t_v)$$

and

$$\pi^*_v = (S_v, S_v, ..., S_v/T_{\mu_v}, T_{\mu_v}, ..., T_{\mu_v})$$

$S_v = n - s_v$, $T_{\mu_v} = n - t_v$
are monotonic nonincreasing rearrangements of \((s_1, s_2, \ldots, s_n/t_1, t_2, \ldots, t_n)\) and \(\succ\) is defined as in section 2.

Harary\(^7\) has shown that the permutation group \(K\) appropriate for this case is the exponentiation group \(S_n^2\) which is the line group of the complete bipartite graph \(K_{nn}\). The point-group or automorphism group of \(K_{nn}\) is the composition group \(S_2[S_n]\).

In Harary's notation these groups can be written as

\[
S_2[S_n] = (S_n \cdot S_n) \cup r(S_n \cdot S_n)
\]

\[
S_n^2 = (S_n \times S_n) \cup \rho(S_n \times S_n)
\]

Here \(S_n \cdot S_n\) and \(S_n \times S_n\) are the direct product and Cartesian product of two copies of \(S_n\) acting on \(X\) and \(Y\). The set \(r(S_n \cdot S_n)\) consists of \((n!)^2\) permutations on \(X \cup Y\) of the form \(r(g, h)\) where \(g, h \in S_n\) and the effect of \(r\) is to interchange corresponding elements \(x_i\) and \(y_i\) of \(X\) and \(Y\). Each of these permutations is obtained by interposing the elements of two permutations of the two copies of \(S_n\). Thus, corresponding to each permutation \(g \in S_n\) with cycle structure \((\lambda) = 1^{d_1} 2^{d_2} \ldots n^{d_n}\), there are \(n!\) permutations of \(r(S_n \cdot S_n)\) with cycle structure \((2\lambda) = 2^{d_1} 4^{d_2} \ldots (2n)^{d_n}\). The set of permutations \(\rho(S_n \times S_n)\) are those induced on the elements of \(D = X \times Y\) by the members of \(r(S_n \cdot S_n)\).

Using Polya's theorem with the above specifications we obtain the required generating function as

\[
2S_2(W_i) - S_1(W_i) = \frac{1}{2(n!)^2} \left[ \sum_{k \in S_n \times S_n} \sum_{f} (k, k_f) \sum_{f} W_i(f) + \sum_{k \in \rho(S_n \times S_n)} \sum_{f} (k, k_f) \sum_{f} W_i(f) \right]
\]

The first term is as in section 2. For the second sum we set

\[
\chi(k) = \sum_{(k, k_f)} \prod_{x_i \in X} \prod_{y_j \in Y} \left(u_i v_j y^{k, y_j}\right) \text{ for } k \in \rho(S_n \times S_n).
\]

Here also, as in section 2, the restriction

\[
f(kd) = f(k^2d), \ldots, f(k^2d) = f(k^4d) = \ldots, f(d) \neq f(kd)
\]

implies that in \(r(g, h)\) \(g\) cannot have any odd cycle. Thus the cycle structure of \(g\) is of the form \((\lambda) = 2^{d_1} 4^{d_2} \ldots\) and hence that of \(r(g, h)\) is \((2\lambda) = 4^{d_1} 8^{d_2} \ldots\).

Straightforward computations give the expression for \(\chi(k)\) as
\[ \chi(k) = \prod_{p=1}^{n} \prod_{q=1}^{n} \prod_{i=1}^{\lambda_p} \prod_{j=1}^{\lambda_q} (U_{pq}^a V_{ij}^a + U_{pq}^b V_{ij}^b)^{(p,q)} \]

\[ \times \prod_{p=1}^{n} \prod_{i=1}^{\lambda_p} (U_{pi} V_{pi} + U_{pi} V_{pi})^p \]

\[ \times \prod_{i=1}^{n} \prod_{i=1}^{\lambda_p} (2U_{pi} V_{pi})^{p/2}. \]

Using the above method the following generating function is obtained for graphs on (4, 4) vertices.

\[ u_1^4 u_2^4 (v_1 v_2 v_3 v_4)^2 + u_1^3 u_2 u_3 u_4 v_1^3 v_2 v_3 v_4 + 2(u_1 u_2 u_3 u_4)^2 (v_1 v_2 v_3 v_4)^2 \]

\[ + u_1^4 u_2 u_3 (v_1 v_2 v_3 v_4)^2 + u_1^4 u_2 u_3 (v_1 v_2 v_3 v_4)^2 + 2u_1^3 u_2 u_3 u_4 (v_1 v_2 v_3 v_4)^2 \]

\[ + 2u_1^3 u_2 u_3 u_4 (v_1 v_2 v_3 v_4)^2 + u_1^4 u_2^3 u_3 v_1^3 v_2 v_3 v_4 + u_1^4 u_2^3 u_3 v_1^3 v_2 v_3 v_4 \]

\[ + 2u_1^3 u_2^3 u_3 u_4 v_1^3 v_2^2 v_3 v_4 + u_1^4 u_2^3 u_3^3 v_1^3 v_2 v_3 v_4 + u_1^4 u_2^3 u_3^3 v_1^3 v_2 v_3 v_4. \]

The corresponding graphs are shown in Figure 3.

---

**Fig. 3. Bipartite self complementary graphs of type (4; 4)**
REFERENCES
