LOCAL CONTROLLABILITY OF SEMILINEAR EVOLUTION SYSTEMS IN BANACH SPACES

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In this paper, we shall establish a set of sufficient conditions for the local controllability of semilinear evolution systems in Banach spaces. The results are obtained using the method of analytic semigroups and a fixed point argument. An example is provided to understand the theory.

Key Words: Controllability; Semilinear Systems; Analytic Semigroup

1. INTRODUCTION

Several authors have studied the problem of controllability of linear and nonlinear systems in Banach spaces. Lasieka and Triggiani and Naito have investigated the controllability of semilinear systems in Banach spaces. In Naito has studied the problem for nonlinear Volterra integrodifferential systems. Balachandran et al. have studied the problem for nonlinear integrodifferential systems where as in they have investigated the local null controllability of nonlinear functional differential systems. Dauer et al. obtained sufficient conditions for the semilinear integrodifferential systems with bounded linear operators in Banach space using an asymptotic fixed point theorem. However, in all these cases the family of semigroup of linear operators is either uniformly continuous or compact. Now we consider case where the family of semigroup of linear operators is analytic. In this paper we shall discuss the local controllability of semilinear evolution systems which occur often in application with an operator-A as the infinitesimal generator of an analytic semigroup in a Banach space. The system considered here is an abstract formulation of parabolic partial differential equations discussed in. For motivation and importance of these kinds of semilinear systems and the type of controllability problems one can refer the recent paper by Fabre et al.

2. PRELIMINARIES

Consider the semilinear evolution system

\[ \frac{dx(t)}{dt} + Ax(t) = Bu(t) + f(t, x(t)) \quad t > t_0 \]  \hspace{1cm} (1)

and \[ x(t_0) = x_0. \]

where the state \( x(\cdot) \) takes the values in a Banach space \( X \) and the control function \( u(\cdot) \) is given
in $L^2(J, U)$, a Banach space of admissible control functions with $U$ as a Banach space. Here the linear operator $A$ generates the infinitesimal generator of an analytic semigroup of linear operators $T(t)$ in a Banach space $X$ and $B$ is a bounded linear operator from $U$ into $X$. The nonlinear operator $f \in C(J \times X, X)$ is uniformly bounded and continuous and $J = [t_0, t_1]$. The operator $A^\alpha$ can be defined for $0 \leq \alpha \leq 1$ and $A^\alpha$ is a closed linear invertible operator with domain $D(A^\alpha)$ dense in $X$. The closedness of $A^\alpha$ implies that $D(A^\alpha)$ endowed with the graph norm of $A^\alpha$, that is the norm $\|z\| = \|z\| + \|A^\alpha z\|$, is a Banach space. Since $A^\alpha$ is invertible its graph norm $\|\cdot\|$ is equivalent to the norm $\|z\|_{A^\alpha} = \|A^\alpha z\|$. Thus, $D(A^\alpha)$ equipped with the norm $\|\cdot\|_{A^\alpha}$ is a Banach space which we denote by $X_{A^\alpha}$. From this definition it is clear that $0 < \alpha < \beta$ implies $X_{A^\alpha} \supset X_{A^\beta}$ and that the imbedding of $X_{A^\beta}$ in $X_{A^\alpha}$ is continuous.

**Definition** — The system $(1)$ is said to be locally controllable on the interval $J$ if for every $x_0, x_1 \in Z$, a subset of $X$, there exists a control $u \in L^2(J, U)$ such that a continuously differentiable local solution $x(t)$ of $(1)$ satisfies $x(t_1) = x_1$.

In order to prove our result we shall assume the following conditions.

(i) $A$ is the infinitesimal generator of an analytic semigroup of linear operators $T(t)$ satisfying the condition;

$$\|T(t)\| \leq M_1,$$

where $M_1$ is a positive constant.

(ii) $0 \in \rho(-A)$, the resolvent set

(iii) The linear operator $W$ from $L^2(J; U)$ into $X$ defined by

$$Wu = \int_{t_0}^{t_1} T(t_1, s)Bu(s)\, ds$$

has an invertible operator $W^{-1}$ defined on $L^2(J; U)/\ker W$ and there exist positive constants $M_2$ and $M_3$ such that $\|B\| \leq M_2$ and $\|W^{-1}\| \leq M_3$.

(iv) Let $Z$ be an open subset of $R^+ \times X_{A^\alpha}$. The function $f: Z \to X$ satisfies the assumption: if for $(t, x) \in Z$ there is a neighbourhood $V \subset Z$ and constants $K > 0$, $0 < \theta < 1$ such that

$$\|f(t_1, x_1) - f(t_2, x_2)\| \leq K(\|t_1 - t_2\|^\theta + \|x_1 - x_2\|_{A^\alpha})$$

for all $(t_1, x_1) \in V$.

3. MAIN RESULT

**Theorem 1** — If the hypothesis (i)-(iv) are satisfied the system $(1)$ is locally controllable on $J$, that is there exists a unique local solution $x \in C([t_0, t_1] : X) \cap C^1((t_0, t_1) : X)$ such that $x(t_1) = x_1$.

**PROOF** : From our assumption on the operator $A$ it follows that
\[ \| A^\alpha T(t) \| \leq C_\alpha r^{-\alpha} \quad \text{for} \quad t > 0 \]  \quad (3)

Fix \((t_0, x_0) \in Z\) and choose \(t_1^* > t_0, \delta > 0\) such that the estimate (2) holds in the set \(V = \{(t, x) : t_0 \leq t \leq t_1^*, \| x - x_0 \| \leq \delta\}\). Let \(N = \max\| f(t, x_0) \|\) and choose \(t_1\) such that

\[ \| T(t - t_0)A^\alpha x_0 - A^\alpha x_0 \| < \delta/2 \quad \text{for} \quad t_0 \leq t < t_1 \]  \quad (4)

and

\[ 0 < t_1 - t_0 < \min \{ t_1^* - t_0, (\delta/2)(1 - \alpha) C_\alpha^{-1} [K\delta + N + M_1 M_3 (\| x_1 \| + M_1 \| x_0 \| + (N + K\delta) M_1 (t_1 - t_0)]^{-1} \}^{1/(1 - \alpha)} \]  \quad (5)

Let \(Y\) be the Banach space \(C(J, X)\) with the usual supremum norm. For every \(y \in Y\) we define a control

\[ u(t) = W^{-1} [x_1 - T(t_1 - t_0)x_0 - \int_{t_0}^{t_1} T(t_1 - t)f(s, A^{-\alpha} y(s))ds](t) \]  \quad (6)

Using this control now we define a mapping \(F : Y \to Y\) by

\[ Fy(t) = T(t - t_0)A^\alpha x_0 + \int_{t_0}^{t} A^\alpha T(t - s) \left\{ BW^{-1} [x_1 - T(t_1 - t_0)x_0 - \int_{t_0}^{t_1} T(t_1 - \tau)f(s, A^{-\alpha} y(s))d\tau](s) + f(s, A^{-\alpha} y(s)) \right\} ds \]  \quad (7)

Clearly \(Fy(t_0) = A^\alpha x_0\). Let \(S\) be a nonempty closed and bounded subset of \(Y\) defined by

\[ S = \{ y : y \in Y, y(t_0) = A^\alpha x_0, \| y(t) - A^\alpha x_0 \| \leq \delta \} \]  \quad (8)

For \(y \in S\) we have

\[ \| Fy(t) - A^\alpha x_0 \| = \| T(t - t_0)A^\alpha x_0 - A^\alpha x_0 \| + \left\| \int_{t_0}^{t} A^\alpha T(t - s)BW^{-1} \left[ x_1 - T(t_1 - t_0)x_0 - \int_{t_0}^{t_1} T(t_1 - \tau)f(s, A^{-\alpha} y(s))d\tau \right](s) ds \right\| \]
\[
+ \left\| \int_{t_0}^t A^{\alpha}T(t-s) \left[ f(s, A^{-\alpha}y(s)) - f(s, x_0) \right] ds \right\| \\
+ \left\| \int_{t_0}^t A^{\alpha}T(t-s)f(s, x_0)ds \right\| \\
\leq \delta/2 + \int_{t_0}^t C_{\alpha}(t-s)^{-\alpha} \left\| BW^{-1} \left[ x_1 - T(t_1 - t_0)x_0 - \int_{t_0}^{t_1} T(t_1 - \tau) f(\tau, A^{-\alpha}y(\tau)) d\tau \right](s) ds \right\| \\
+ C_{\alpha}(1-\alpha)^{-1} (K\delta + N)(t_1 - t_0)^{1-\alpha} \\
\leq \delta/2 + C_{\alpha}(1-\alpha)^{-1} (t_1 - t_0)^{1-\alpha} [K\delta + N + M_1 M_3 (\| x_1 \| + M_1 \| x_0 \| + (N + K\delta)M_1(t_1 - t_0))] \\
\leq \delta/2 + \delta/2 < \delta.
\]

Therefore, \( F \) maps \( S \) into itself. Let \( y_1, y_2 \in Y \), then

\[
\| Fy_1(t) - Fy_2(t) \| \leq \int_{t_0}^t \| A^{\alpha}T(t-s) \| B \| \| W^{-1} \| \left[ \| x_1 \| + \| T(t_1 - t_0)x_0 \| \\
+ \int_{t_0}^{t_1} \| T(t_1 - \tau) \| \| f(\tau, A^{-\alpha}y_1(\tau)) - f(\tau, A^{-\alpha}y_2(\tau)) \| d\tau \right] ds \\
+ \int_{t_0}^t \| A^{\alpha}T(t-s) \| \| f(s, A^{-\alpha}y_1(s)) - f(s, A^{-\alpha}y_2(s)) \| ds \\
\leq C_\alpha(t_1 - t_0)^{1-\alpha} (1-\alpha)^{-1} \left\{ M_2 M_3 (\| x_1 \| + M_1 \| x_0 \| + KM_1(t_1 - t_0) \| y_1 - y_2 \|) \\
+ K \| y_1 - y_2 \| \right\} \\
\leq (1/2) \| y_1 - y_2 \|.
\]

Hence,

\[
\| Fy_1(t) - Fy_2(t) \| \leq (1/2) \| y_1 - y_2 \|. \quad \ldots \quad (9)
\]

Therefore, \( F \) is a contraction mapping and hence there exists a unique fixed point \( y \in S \) such that \( Fy(t) = y(t) \). This fixed point satisfies the integral equation
\[ y(t) = T(t-t_0)A^\alpha x_0 + \int_{t_0}^{t} A^\alpha T(t-s) \]
\[
\left\{ BW^{-1} \left[ x_1 - T(t_1-t_0)x_0 - \int_{t_0}^{t_1} T(t_1-s)f(t,A^{-\alpha}y(s)) \, ds \right] (s) + f(s,A^{-\alpha}y(s)) \right\} \, ds
\]

Let

\[ f^*(t,A^{-\alpha}y(t)) = BW^{-1} \left[ x_1 - T(t_1-t_0)x_0 - \int_{t_0}^{t_1} T(t_1-s)f(s,A^{-\alpha}y(s)) \, ds \right] (t) + f(t,A^{-\alpha}y(t)) \]

From (2) and the continuity of \( y \) it follows that \( t \to f^*(t,A^{-\alpha}y(t)) \) is continuous on \( J \) and therefore there exists a constant \( N^* \) such that

\[ \| f^*(t,A^{-\alpha}y(t)) \| \leq N^* \text{ for } t_0 \leq t \leq t_1 \]  \[ (11) \]

Now we want to show that \( f^*(t,A^{-\alpha}y(t)) \) is locally Holder continuous on \( (t_0, t_1] \).

Observe that for every \( \beta \) satisfying \( 0 < \beta < 1 - \alpha \) and every \( 0 < h < 1 \) we have by Theorem 2.6.13 of\( ^{12} \)

\[ \| (T(h) - I) A^\alpha T(t-s) \| \leq C h^\beta \| A^{\alpha+\beta} T(t-s) \| \leq C h^\beta (t-s)^{-(\alpha+\beta)}, \]  \[ (12) \]

where \( C \) is some positive constant.

If \( t_0 < t < t+h < t_1 \), then

\[ \| y(t+h) - y(t) \| \leq \| T(h) - I \| A^\alpha T(t-t_0)x_0 \| + \int_{t_0}^{t} \| T(h) - I \| A^\alpha T(t-s) \]

\[ f^*(s,A^{-\alpha}y(s)) \| ds + \int_{t}^{t+h} \| A^\alpha T(t+h-s)f^*(s,A^{-\alpha}y(s)) \| ds \]

\[ \leq I_1 + I_2 + I_3 \]  \[ (13) \]

Using (11) and (12) we estimate each of the terms of (13) separately,

\[ I_1 \leq C(t-t_0)^{-(\alpha+\beta)} h^\beta \| x_0 \| \leq M_4 h^\beta \]

\[ I_2 \leq CN^* h^\beta \int_{t_0}^{t} (t-s)^{-(\alpha+\beta)} ds \leq M_3 h^\beta \]

Using the above estimates, we can conclude that \( f^*(t,A^{-\alpha}y(t)) \) is Holder continuous on \( (t_0, t_1] \) with constant \( N^* \).
\[ I_3 \leq N^\alpha C_\alpha \int_{t_0}^{t+h} (t+h-s)^{-\alpha} ds \leq N^\alpha C_\alpha h^{(1-\alpha)/(1-\alpha)} \leq M_0 h^\beta \]

Combining (13) with these estimates it follows that for every \( t_0^* > t_0 \) there is a constant \( C \) such that \( \| y(t) - y(s) \| \leq C | t - s |^{1-\beta} \) for \( t_0 < t_0^* \leq t, s \leq t_1 \) and therefore \( y \) is locally Holder continuous on \((t_0, t_1] \). The local Holder continuity of \( t \rightarrow f^\alpha(t, A^{-\alpha} y(t)) \) follows from

\[ \| f^\alpha(t, A^{-\alpha} y(t)) - f^\alpha(t, A^{-\alpha} y(s)) \| < C^\alpha (| t - s |^\theta + | t - s |^\beta). \]

Let \( y \) be the solution of (10) and consider the initial value problem

\[ \frac{dx(t)}{dt} + Ax(t) = f^\alpha(t, A^{-\alpha} y(t)) \]

\[ x(t_0) = x_0 \]

... (14)

By Corollary 4.3.3 of 12 this problem has a unique solution \( x \in C^1(t_0, t_1] : X \). The solution of (14) is given by

\[ x(t) = T(t - t_0) x_0 + \int_{t_0}^{t} T(t - s) \left\{ BW^{-1} \left[ x_1 + T(t_1 - t_0) x_0 - \int_{t_0}^{t_1} T(t_1 - \tau) f(\tau, A^{-\alpha} y(\tau)) d\tau \right] (s) + f(s, A^{-\alpha} y(s)) \right\} ds \]

... (15)

For \( t > t_0 \) each term of (15) is in \( D(A) \) and a fortiori is in \( D(A^\alpha) \). Operating on both sides of (15) with \( A^\alpha \) we find

\[ A^\alpha x(t) = T(t - t_0) A^\alpha x_0 + \int_{t_0}^{t} A^\alpha T(t - s) \left\{ BW^{-1} \left[ x_1 + T(t_1 - t_0) x_0 - \int_{t_0}^{t_1} T(t_1 - \tau) f(\tau, A^{-\alpha} y(\tau)) d\tau \right] (s) + f(s, A^{-\alpha} y(s)) \right\} ds \]

... (16)

But by (10) the right hand side of (16) equals \( y(t) \) and therefore \( x(t) = A^{-\alpha} y(t) \) and by (15), \( x \) is a \( C^1 \) solution of (1). The uniqueness of \( x \) follows from the uniqueness of the solution of (10) and (14). Thus
\[ x(t) = U(t - t_0)x_0 + \int_{t_0}^{t} U(t - s)ds \]

\[
\left\{ \begin{array}{l}
B W^{-1} \left[ x_1 + T(t_1 - t_0)x_0 - \int_{t_0}^{t_1} T(t_1 - \tau)f(\tau, x(\tau))d\tau \right] (s) + f(s, x(s)) \\
\end{array} \right\} ds
\]

It is easy to see that \( x(t_1) = x_1 \). Hence the system (1) is locally controllable.

4. QUASILINEAR EVOLUTION SYSTEMS

Consider the quasilinear evolution system

\[
\frac{dx(t)}{dt} + A(t, x)x(t) = B(t)u(t) + f(t, x(t), x(t))x(t_0) = x_0
\]

(17)

The problem of controllability of (17) can be studied with the help of the Schauder fixed point theorem. For that we assume the following conditions:

(A1) The operator \( A_0 = A(t_0, x_0) \) is a closed operator with a domain \( D_0 \) dense in \( X \) and

\[
\| (\lambda I - A_0)^{-1} \| \leq C/(1 + |\lambda|) \text{ for all } \lambda \text{ with } Re \lambda \leq 0 \text{ and } C > 0:
\]

(A2) \( A_0^{-1} \) is completely continuous operator.

(A3) For some \( \alpha \in [0, 1) \) and \( R > 0 \) and for any \( y \in X \) with \( y \| < R \) the operator \( A(t, A_0^{-\alpha}y) \) is well defined on \( D_0 \), for all \( t_0 \leq t \leq t_1 \). Further more, for any \( t, \tau \in [t_0, t_1] \) and \( y, z \in X \) with \( y \| < R, z \| < R \)

\[
\| [A(t, A_0^{-\alpha}y) - A(\tau, A_0^{-\alpha}z)]A^{-1} (\tau, A_0^{-\alpha}z) \| \leq C(R) (|t - \tau|^\sigma + y \| z \| \theta)
\]

where \( 0 < \sigma \leq 1 \) and \( 0 < \theta \leq 1 \).

(A4) For every \( t, \tau \in [t_0, t_1] \), \( \| B(t) - B(\tau) \| \leq C |t - \tau| \sigma \)

(A5) For every \( t, \tau \in [t_0, t_1] \) and \( y, z \in X \) with \( y \| < R, z \| < R \)

\[
\| [f(t, A_0^{-\alpha}y) - f(\tau, A_0^{-\alpha}z)] \| \leq C(R) (|t - \tau|^\sigma + y \| z \| \theta)
\]

(A6) \( x_0 \in D(A_0^\beta) \) for some \( \beta > \alpha \) and \( \| A_0^\beta x_0 \| < R \)

Under the assumption (A1) to (A6), there exists a continuously differentiable local solution of (17) as (see Friedman)

\[
x(t) = T(t, t_0; x(t_0)x_0 + \int_{t_0}^{t} T(t, s; x(s)) [B(s)u(s) + f(s, x(s))] ds
\]

... (18)
where \( \{ T(t, s; x) : t_0 \leq s \leq t \leq t_1 \} \) is the linear evolution system generated by \( A(t, x) \) for each fixed \( x \in C(J, X) \).

\((A_2)\) For fixed \( x \in C(J, X) \) the linear operator \( W \) from \( L^2(J; U) \) in to \( X \) defined by

\[
Wu = \int_{t_0}^{t_1} T(t_1, s; x) B(s)u(s) \, ds
\]

has an invertible operator \( W^{-1} \) defined on \( L^2(J; U) \)/ker\( W \) and there exist positive constants \( N_1, N_2 \) such that \( \| B \| \leq N_1 \) and \( \| W^{-1} \| \leq N_2 \).

**Theorem 2** — If the conditions \((A_1)\) to \((A_2)\) hold, then the system \((17)\) is locally controllable on \( J \).

**Outline of the Proof:** Define the control function \( u \) as

\[
u(t) = W^{-1} \left[ x_1 - T(t_1, t_0; x)x_0 - \int_{t_0}^{t_1} T(t_1, s; x)f(s, x)ds \right] (t) \quad \ldots \quad (19)
\]

We can show that, when using this control, the operator defined by

\[
\Phi(x)(t) = T(t, t_0; x(t))x_0 + \int_{t_0}^{t} T(t, s; x(s)) B(s)W^{-1} [x_1 - T(t_1, t_0; x)x_0

- \int_{t_0}^{t_1} T(t_1, \tau; x)f(\tau, x)\, d\tau] (s)ds + \int_{t_0}^{t} T(t, s; x)f(s, x(s))ds \quad \ldots \quad (20)
\]

has a fixed point. This fixed point is then a solution of eq. \((17)\). Clearly \( \Phi(x)(t_1) = x_1 \), which means that the control \( u \) steers the quasilinear system \((17)\) from the initial state \( x_0 \) to \( x_1 \). If we take a set \( S \) in \( Y \) then it is easy to see that the set \( \Phi S \) is contained in a compact subset of \( Y \). Indeed, the functions \( z(t) \) of \( S \) are uniformly bounded and equicontinuous. Further for each \( t \) the set \( \{ \Phi z(t) : z \in S \} \) is contained in a compact subset of \( X \), and by Arzela-Ascoli’s theorem, \( \Phi S \) is contained in a compact subset of \( Y \). Hence by the Schauder fixed point theorem the mapping \( \Phi \) has a fixed point. This fixed point is the solution of the controllability problem \((17)\).

5. Example

Consider the nonlinear heat equation \(^{11,12,15}\) with a control term

\[
z_t(x, t) = z_{xx} + u(t) + f(z(x, t), z_x(x, t)), \quad 0 \leq x < \pi, \quad 0 \leq t \leq a, \quad z(0, t) = z(\pi, t) = 0 \quad \ldots \quad (21)
\]

and

\[
z(x, 0) = z_0,
\]

where \( f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) with \( f(0, 0) = 0 \) and \( f \) continuous in its first variable and Lipschitz continuous in its second variable. Now we have to prove that there exists a control \( u \in L^2 \) which steers the
system (21) from any specified initial state to the final state in a subspace. Observe that the solution for (21) exists locally, that is in a subspace.

Let $X=L^2(0, \pi)$ and let $A: X \to X$ be defined by $Az = -z''$ with domain

$$D(A) = \{ z \in X : z, z' \text{ are absolutely continuous } z'' \in X, z(0) = z(\pi) = 0 \}$$

Then

$$Az = \sum_{n=1}^{\infty} n^2(z_n(s))z_n, \quad z \in D(A),$$

where $z_n(s) = (\sqrt{2}/\pi) \sin(ns), \quad n = 1, 2, ...$ is the orthogonal set of eigenvectors of $A$. It is well known that $-A$ is the infinitesimal generator of an analytic semigroup $T(t), t \geq 0$ in $X$ and is given by

$$T(t)z = \sum_{n=1}^{\infty} \exp(-n^2t)(z, z_n)z_n \quad z \in X$$

The analytic semigroup $T(t)$ is compact if and only if $A^{-1/2}$ is compact. Further $T(t)$ satisfies the conditions (i)-(iv) of Theorem 1. The compactness of $A^{-1/2}$ follows from the fact that the eigenvalues of $A^{-1/2}$ are $\lambda_n = 1/n, \quad n = 1, 2, ...$ Let us assume that there exists a linear operator $W$ from $L^2$ into $X$ defined by

$$Wu = a \int_{0}^{a} T(a,s)u(s) \; ds$$

such that $W^{-1}$ exists in $L^2(J; U)/\ker W$ and is uniformly bounded. Define the function $f: X \to X$ by

$$(fz)(s) = f(z(s)(x), z(s)'(x)).$$

Observe that $f$ is well defined and continuous. Hence, by Theorem 1, the system (21) is locally controllable on $J$.

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