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A COMPARISON RESULT IN OSCILLATION THEORY

CH. G. PHILOS

Department of Mathematics, University of Ioannina, Ioannina, Greece

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An important problem in Oscillation Theory is to establish comparison results for the oscillatory and asymptotic behaviour of solutions of differential equations with deviating arguments. This problem has been recently the subject of intensive investigations. The purpose of this paper is to present a new such comparison result for differential equations of arbitrary order. Our result generalizes a recent one by Brands (1978) concerning second order linear retarded differential equations.

This paper deals with the comparison of the nth order $(n > 1)$ differential equations with deviating arguments

$$(E_g, \delta) \qquad x^{(n)}(t) + \delta f(t; x\langle g(t)\rangle) = 0, \ t \geqq t_0,$$

$$(E_\sigma, \delta) \qquad x^{(n)}(t) + \delta f(t; x\langle \sigma(t)\rangle) = 0, \ t \geqq t_0$$

with respect to the oscillatory and asymptotic behaviour of their solutions, where $\delta = \pm 1$ and

$$x\langle g(t)\rangle = (x[g_1(t)], \ldots, x[g_m(t)]), \ g = (g_1, \ldots, g_m),$$

$$x\langle \sigma(t)\rangle = (x[\sigma_1(t)], \ldots, x[\sigma_m(t)]), \ \sigma = (\sigma_1, \ldots, \sigma_m).$$

The functions g_j and $\sigma_j (j = 1, \ldots, m)$ are assumed to be continuous on the interval $[t_0, \infty)$ and such that

$$\lim_{t\to\infty} g_j(t) = \lim_{t\to\infty} \sigma_j(t) = \infty \quad (j = 1, \ldots, m).$$

Moreover, f is a continuous function which is defined at least on

$$[t_0, \infty) \times (\mathbf{R}_+^m \cup \mathbf{R}_-^m),$$

where $\mathbf{R}_+ = (0, \infty)$ and $\mathbf{R}_- = (-\infty, 0)$, and has the sign property

$$\begin{cases} t \geqq t_0 \text{ and } y \in \mathbf{R}_+^m \text{ imply } f(t; y) \geqq 0 \\ t \geqq t_0 \text{ and } y \in \mathbf{R}_-^m \text{ imply } f(t; y) \leqq 0. \end{cases}$$

Also, f is supposed to be increasing with respect to the second variable. Note that the increasing character of real-valued functions defined on subsets of the space $\mathbf{R}^m$

is considered with respect to the usual order in $\mathbf{R}^m$ defined by the positive cone $\{y = (y_1, ..., y_m) \in \mathbf{R}^m : (\forall\, j = 1, ..., m)\, y_j \geqq 0\}$, i.e. as follows

$$y \leqq \tilde{y} \Leftrightarrow (\forall\, j = 1, ..., m)\; y_j \leqq \tilde{y}_j.$$

Sufficient smoothness for the existence of such solutions $x(t)$ of the above differential equations which are defined for all large t will be assumed. In what follows, we consider only such solutions $x(t)$ which are defined for all large t. The oscillatory character is considered in the usual sense, i.e. a continuous real-valued function defined on an interval of the form $[T, \infty)$ is said to be 'oscillatory' if the set of its zeros is unbounded above, and otherwise it is said to be 'non-oscillatory'.

Recently, there has been an increasing interest in obtaining comparison results for the oscillatory and asymptotic behavior of solutions of differential equations with deviating arguments. We choose to refer to the recent papers by Kartsatos (1975), Kartsatos and Onose (1976), Onose (1975, 1976) and Staikos and Philos (1978). The comparison results given in this paper are motivated from a recent result by Brands (1978) concerning the comparison of two second order linear delay differential equations with respect to their oscillation.

Now, let $\mathcal{F}$ be a function class of real-valued functions u defined at least on an interval $[T_u, \infty)$, which possesses the property : For every $u \in \mathcal{F}$ there exists a $\tau_u \geqq T_u$ such that all real-valued functions v on $[\tau_u, \infty)$ with $|\, v(t)\,| \leqq |\, u(t)\,|$ for every $t \geqq \tau_u$ belong in $\mathcal{F}$. Such a function class may be that of all real-valued functions u defined at least on an interval $[T_u, \infty)$, or the class of all bounded such functions, or furthermore the class of all such functions u which satisfy an order relation as the following $u(t) = O(t^k)$ as $t \to \infty$, where k is a nonnegative integer. A solution of the differential equation (E_g, δ) or (E_σ, δ) which belongs in the function class $\mathcal{F}$ will be called briefly an $\mathcal{F}$-solution.

Here, we are concerned with oscillatory and asymptotic properties of the $\mathcal{F}$-solutions of the differential equation (E_σ, δ) which are inherited from the same properties of the $\mathcal{F}$-solutions of the equation (E_g, δ).

A non-oscillatory solution x of the differential equation (E_g, δ) or (E_σ, δ) is said to be 'strongly monotone' if and only if

$$\lim_{t \to \infty} x^{(i)}(t) = 0 \text{ monotonically } (i = 0, 1, ..., n-1).$$

To obtain our results we need the following lemma. This lemma is an adaptation of two well-known lemmas due to Kiguradze (1964, 1965).

Lemma — Let h be a positive and n-times differentiable function on an interval $[\tau_0, \infty)$ with its nth derivative $h^{(n)}$ of constant sign. Then there exists an integer

l, $0 \leqq l \leqq n$, with $n+l$ odd for $h^{(n)}$ non-positive or $n+l$ even for $h^{(n)}$ non-negative and such that for some $\tau \geqq \tau_0$

$$\begin{cases} l \leqq n-1 \Rightarrow (-1)^{l+j} h^{(j)} \geqq 0 \text{ on } [\tau, \infty) \quad (j = l, \ldots, n-1) \\ l > 1 \Rightarrow h^{(i)} > 0 \quad \text{on } \quad [\tau, \infty) \quad (i = 1, \ldots, l-1). \end{cases}$$

Now, we state the following theorem which is a special case of a result given by Staikos and Philos (1978) [see Corollary 5]. This theorem will be extended by the main results of this paper.

Theorem 0 — Suppose that

(I) $g_j \leqq \sigma_j$ eventually $(j = 1, \ldots, m)$

in case where $\mathcal{F}$ contains at least one unbounded function. Then we have :

(i) Let n be even [resp. odd]. If all $\mathcal{F}$-solutions of the differential equation $(E_g, +1)$ [resp. of the equation $(E_g, -1)$] are oscillatory, then all $\mathcal{F}$-solutions of the differential equation $(E_\sigma, +1)$ [resp. of the equation $(E_\sigma, -1)$] are also oscillatory.

(ii) Let n be odd [resp. even]. If every $\mathcal{F}$-solution of the differential equation $(E_g, +1)$ [resp. of the equation $(E_g, -1)$] is oscillatory or strongly monotone, then every $\mathcal{F}$-solution of the differential equation $(E_\sigma, +1)$ [resp. of the equation $(E_\sigma, -1)$] is also oscillatory or strongly monotone.

The next two comparison theorem constitute the main results of the present paper.

Theorem 1 — Suppose that :

(II) The functions $g_j - \sigma_j (j = 1, \ldots, m)$ are bounded above

in case where $\mathcal{F}$ contains at least one unbounded function. Then we have :

(i) Let n be even. If all $\mathcal{F}$-solutions of the differential equation $(E_g, +1)$ are oscillatory, then all $\mathcal{F}$-solutions of the equation $(E_\sigma, +1)$ are also oscillatory.

(ii) Let n be odd. If every $\mathcal{F}$-solution of the differential equation $(E_g, +1)$ is oscillatory or strongly monotone, then every $\mathcal{F}$-solution of the equation $(E_\sigma, +1)$ is also oscillatory or strongly monotone.

Theorem 2 — Suppose that (II) is true in case where $\mathcal{F}$ contains at least one unbounded function. Moreover, when (I) fails, suppose that every function u in $\mathcal{F}$ satisfies $u(t) = o(t^{n-1})$ as $t \to \infty$. Then we have :

(i) Let n be odd. If all $\mathcal{F}$-solutions of the differential equation $(E_g, -1)$ are oscillatory, then all $\mathcal{F}$-solutions of the equation $(E_\sigma, -1)$ are also oscillatory.

(ii) Let n be even. If every $\mathcal{G}$-solution of the differential equation $(E_g, -1)$ is oscillatory or strongly monotone, then every $\mathcal{G}$-solution of the equation $(E_\sigma, -1)$ is also oscillatory or strongly monotone.

Theorems 1 and 2 can be obtained by combining Theorem 0 and the following proposition.

Proposition — Let (II) be satisfied. If z is a positive [resp. negative] unbounded solution of the differential equation (E_σ, δ) with $z(t) = o(t^{n-1})$ as $t \to \infty$ in case where $\delta = -1$, then there exists a positive [resp. negative] solution x of the equation (E_g, δ) with $| x(t) | \leqq | z(t) |$ for all large t and $\lim_{t\to\infty} x(t) \neq 0$.

PROOF OF PROPOSITION : The substitution $\tilde{x} = -x$ transforms the differential equations (E_g, δ) and (E_σ, δ) into the equations

$$\tilde{x}^{(n)}(t) + \delta \tilde{f}(t; \tilde{x} \langle g(t) \rangle) = 0$$

and

$$\tilde{x}^{(n)}(t) + \delta \tilde{f}(t; \tilde{x} \langle \sigma(t) \rangle) = 0$$

respectively, where $\tilde{f}(t; y) = -f(t; -y)$ for every (t, y) in the domain of f. The function $\tilde{f}$ is subject to the same assumptions possed on f. Hence, with respect to the non-oscillatory solutions of the equation (E_g, δ) or (E_σ, δ) we can restrict our attention only to the positive ones.

Let z be a positive unbounded solution on an interval $[T_0, \infty)$, $T_0 \geqq t_0$, of the differential equation (E_σ, δ) with $z(t) = o(t^{n-1})$ as $t \to \infty$ in case where $\delta = -1$. Moreover, let $\tau_0 \geqq T_0$ be chosen so that

$$\sigma_j(t) \geqq T_0 \quad \text{for every} \quad t \geqq \tau_0 \quad (j = 1, \ldots, m).$$

Then, because of the sign property of the function f, from (E_σ, δ) it follows that $z^{(n)}$ is non-positive on $[\tau_0, \infty)$ for $\delta = +1$ or non-negative on $[\tau_0, \infty)$ for $\delta = -1$. Thus, Lemma ensures that there exist a $\tau \geqq \tau_0$ and an integer l, $0 \leqq l \leqq n$, with $n + l$ odd for $\delta = +1$ or $n + l$ even for $\delta = -1$ so that, when $l \leqq n - 1$,

$$(-1)^{l+j} z^{(j)}(t) \geqq 0 \quad \text{for every } t \geqq \tau \quad (j = l, \ldots, n - 1)$$

and, provided that $l > 1$,

$$z^{(i)}(t) > 0 \quad \text{for every} \quad t \geqq \tau \quad (i = 1, \ldots, l - 1).$$

Since z is unbounded, we always have $l > 0$. Furthermore, we must have $l \leqq n - 1$. Indeed, for $l = n$ we immediately obtain that

$$\lim_{t\to\infty} [z(t)/t^{n-1}] = \frac{1}{(n-1)!} \lim_{t\to\infty} z^{(n-1)}(t) > 0.$$

But, $l = n$ occurs only for $\delta = -1$. This proves our assertion. It is easy to verify that for every $t \geqq \tau$

$$z^{(l)}(t) \geqq \int_t^\infty \frac{(s-t)^{n-1-l}}{(n-1-l)!} f(s; z\langle \sigma(s)\rangle)\, ds.$$

Because of (II), we can consider a non-negative constant K so that

$$g_j(t) - K \leqq \sigma_j(t) \quad \text{for every} \quad t \geqq t_0 \quad (j = 1, \ldots, m).$$

Furthermore, we choose a $T \geqq \tau + K$ such that

$$g_j(t) \geqq \tau + K \quad \text{for all} \quad t \geqq T \quad (j = 1, \ldots, m).$$

Then, because of the fact that z is increasing on $[\tau, \infty)$ and f is increasing with respect to the second variable, we obtain that for $t \geqq T$

$$z^{(l)}(t) \geqq \int_t^\infty \frac{(s-t)^{n-1-\rho}}{(n-1-l)!} f(s; z\langle g(s) - K\rangle)\, ds.$$

Next, we set

$$w(t) = z(t-K),\ t \geqq \tau + K$$

and, by the fact that $z^{(l)}$ is decreasing on $[\tau, \infty)$, we have

$$w^{(l)}(t) = z^{(l)}(t-K) \geqq z^{(l)}(t) \quad \text{for} \quad t \geqq \tau + K.$$

Thus, for every $t \geqq T$ we get

$$w^{(l)}(t) \geqq \int_t^\infty \frac{(s-t)^{n-1-l}}{(n-1-l)!} f(s; w\langle g(s)\rangle)\, ds.$$

From this inequality it is easy to obtain that for every $t \geqq T$

$$w(t) \geqq M + \int_T^t \frac{(t-s)^{l-1}}{(l-1)!} \int_s^\infty \frac{(r-s)^{n-1-l}}{(n-1-l)!} f(r; w\langle g(r)\rangle)\, dr\, ds$$

where $M = w(T) > 0$.

Now, let X be the set of all continuous functions x on the interval $[T, \infty)$ with

$$M \leqq x(t) \leqq w(t) \quad \text{for every} \quad t \geqq T.$$

Moreover, for any function x in X let $\hat{x}$ be defined by

$$\hat{x}(t) = \begin{cases} x(t), & \text{if} \quad t \geqq T \\ \dfrac{x(T)}{w(T)} w(t), & \text{if} \quad \tau + K \leqq t \leqq T. \end{cases}$$

Then, by taking into account the fact that f is increasing with respect to the second variable, we can easily verify that the formula

$$(Sx)(t) = M + \int_T^t \frac{(t-s)^{l-1}}{(l-1)!} \int_s^\infty \frac{(r-s)^{n-1-l}}{(n-1-l)!} f(r; \hat{x} \langle g(r) \rangle)\, dr\, ds$$

makes sense for any function x in X and every $t \geqq T$ and, moreover, that this formula defines an increasing mapping S of X into itself. Note here that the set X is considered endowed with the usual pointwise ordering. Next, we consider the decreasing sequence $(x_\nu)_{\nu=0,1,\ldots}$ of functions in X, where x_0 is the restriction of w on $[T, \infty)$ and $x_\nu = Sx_{\nu-1}$ ($\nu = 1, 2, \ldots$), and we set $x = \lim x_\nu$ pointwise on $[T, \infty)$. Then we can apply the Lebesque dominated convergence theorem to obtain $x = Sx$. Thus, the limit function x is a positive solution on the interval $[T, \infty)$ of the differential equation (E_g, δ). Moreover, this solution is the required one, since $\lim_{t\to\infty} x(t) > 0$ and $x(t) \leqq w(t) \leqq z(t)$ for every $t \geqq T$.

Now, in order to demonstrate the significance of our results, let us consider the special case of the ordinary differential equation

$$(D, \delta) \qquad x^{(n)}(t) + \delta f(t; x(t)) = 0$$

and the delay equation

$$(D^*, \delta) \qquad x^{(n)}(t) + \delta f(t; x(t-d)) = 0$$

where d is a positive constant. From Theorems 1 and 2 we immediately obtain the following corollaries.

Corollary 1 — We have :

(i) Let n be even. All solutions of the delay equation $(D^*, +1)$ are oscillatory, if and only if all solutions of the ordinary equation $(D, +1)$ are also oscillatory.

(ii) Let n be odd. Every solution of the delay equation $(D^*, +1)$ is oscillatory or strongly monotone, if and only if every solution of the ordinary equation $(D, +1)$ is also oscillatory or strongly monotone.

Corollary 2 — We have :

(i) Let n be odd. All $o(t^{n-1})$ as $t \to \infty$ solutions of the delay equation $(D^*, -1)$ are oscillatory, if and only if all $o(t^{n-1})$ as $t \to \infty$ solutions of the ordinary equation $(D, -1)$ are also oscillatory.

(ii) Let n be even. Every $o(t^{n-1})$ as $t \to \infty$ solution of the delay equation $(D^*, -1)$ is oscillatory or strongly monotone, if and only if every $o(t^{n-1})$ as $t \to \infty$ solution of the ordinary equation $(D, -1)$ is also oscillatory or strongly monotone.

Remark: Consider the special case where $n = 2$, $m = 1$, $f(t; y) = p(t)y$ for $(t, y) \in [t_0, \infty) \times \mathbf{R}$ and $\mathcal{F}$ is the class of all real-valued functions u defined at least on an interval of the form $[T_u, \infty)$. In this case Theorem 1 leads to a recent result by Brands (1978) [see Theorem 2].

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