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REFLECTION AND REFRACTION OF WAVES AT A LIQUID-SOLID INTERFACE IN THE PRESENCE OF A MAGNETIC FIELD

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The reflection and refraction of a plane sound wave at the interface of a liquid-solid boundary has been considered. The results have been discussed for water-aluminium boundary. It is found that the effect of the transverse magnetic field plays an important role.

1. INTRODUCTION

The problems of reflection and refraction of plane waves at the free boundary of an elastic material or at the plane interface between two different materials have been solved by Kolsky (1963) and Brekhovskikh (1960). The corresponding analysis for plane harmonic waves propagating in linear isotropic viscoelastic media has been given by Lockett (1962), Cooper and Reiss (1967) and Cooper (1967). In this case the frequency dependent complex moduli of the materials introduce some new features. Mott (1971) and Lockett (1972) have shown that the amplitude of the reflected wave which is greatly diminished at a critical angle can be explained by a viscoelastic loss mechanism in the reflecting material.

In this paper we have tried to explore other features of the problem viz. the nature of the refracted longitudinal and transverse waves in the solid medium for different values of the incident angles. It is shown that the nature of wave changes with the impressed magnetic field for the same angle of incidence in the liquid. It may be noted that such studies have assumed important applications in electromagnetic wave phenomena.

2. CONSTITUTIVE RELATIONS

Coordinate axes and geometry of the wave system are defined with reference to Fig. 1. The plane boundary between the two media is $z = 0$; region $z > 0$ is occupied by a perfectly conducting compressible inviscid liquid, and $z < 0$ contains an isotropic elastic conductor. The gravity field in the liquid will not be considered. The x-axis coincides with the direction of propagation of the surface wave. The plane of propagation of the incident wave is taken as the (x, z)-plane.

Since the incident medium is supposed to be a compressible inviscid liquid, it cannot support transverse waves, and the incident wave is accordingly chosen to be

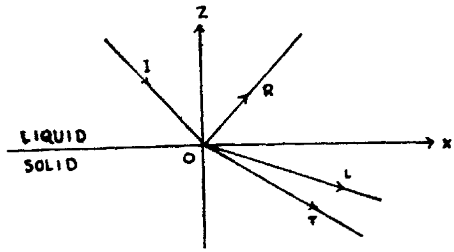

FIG. 1. Wave system at interface.

a longitudinal wave. For the same reason the only reflected wave (R) is also longitudinal. In the transmitting medium both a longitudinal wave (L) and a transverse wave (T) will propagate. It is assumed that the conditions of plane strain exists, that is, there is no strain or displacement component in the y-direction, and the solution will not depend on y. If transverse waves in the fluid are retained when boundary conditions at the interface are applied, it can be shown that their amplitude is several orders of magnitude less than that of the longitudinal waves and waves generated in the solid. The condition for neglecting viscosity and transverse waves in the liquid is

$$\omega\eta \ll \lambda' \quad \text{and} \quad k\omega\eta \ll |Z|^2$$

where η is the viscosity of the liquid, Z a representative value of the modulus of the refracting medium, ω the angular frequency of the wave motion, and λ' the bulk modulus of elasticity for the liquid.

The linearlized set of equations for perfectly conducting liquid and solid (in agreement with Kaliski 1963) are given by

$$\frac{\partial^2 \mathbf{u}'}{\partial t^2} = a_1^2 \text{ grad div } \mathbf{u}' + \frac{\mu'}{4\pi\rho'} [\text{curl curl } (\mathbf{u}' \times \mathbf{H})] \times \mathbf{H}$$

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} = a_2^2 \nabla^2 \mathbf{u} + (a_1^2 - a_2^2) \text{ grad div } \mathbf{u} + \frac{\mu}{4\pi\rho} [\text{curl curl } (\mathbf{u} \times \mathbf{H})] \times \mathbf{H} \qquad \ldots(2.1)$$

where

$$\mathbf{E}' = -\frac{\mu'}{c_0}\left(\frac{\partial \mathbf{u}'}{\partial t} \times \mathbf{H}\right), \quad \mathbf{h}' = \text{curl } (\mathbf{u}' \times \mathbf{H})$$

$$\mathbf{E} = -\frac{\mu}{c_0}\left(\frac{\partial \mathbf{u}}{\partial t} \times \mathbf{H}\right), \quad \mathbf{h} = \text{curl } (\mathbf{u} \times \mathbf{H}) \qquad \ldots(2.2)$$

The symbols with primes refer to the liquid, those without primes to the solid. The equations for the liquid are expressed in displacement not in velocities. In what follows it will be assumed, as is practically justified, that $\mu' = \mu \approx 1$. We also assume that $\mathbf{H} = (0, H, 0)$, and for the plane problem $u_y = u'_y = 0$. In our problem it is useful to express the displacements and stresses in terms of the potentials ϕ and ψ, where $\underset{\sim}{\psi}$ can be chosen such that only its y component, which we will denote by ψ, differs from zero. Then $\mathbf{u}$ will be a vector with the components

$$u_x = \frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial z}, \quad u_y = 0, \quad u_z = \frac{\partial\phi}{\partial z} + \frac{\partial\psi}{\partial x}$$

ϕ and ψ are called the potentials of longitudinal and transverse waves. It can be shown from eqn. (2.1) that these potentials will satisfy the wave equations

$$\nabla^2\phi = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}, \quad c^2 = a_1^2 + a^2, \quad \nabla^2\psi = \frac{1}{a_2^2}\frac{\partial^2\psi}{\partial t^2}.$$

The quantities a_1, a_2, a'_1 denote the velocity of longitudinal and transverse waves in the solid, and sound velocity in the liquid and $a^2 = H^2/4\pi\rho$ is the Alfvén velocity. Clearly, all relations obtained for the solid medium can be extended to the liquid medium by setting $\psi = 0$ and G (= modulus of rigidity) = 0.

3. Wave Motions

Let a plane sound wave be incident from a liquid on a liquid-solid interface, and let the sound wave be prescribed by the potential

$$\phi_I = A \exp\,[ik(x \sin\theta - z\cos\theta) + i\omega t] \qquad \text{...(3.1)}$$

where θ is the angle of incidence and A is the amplitude of the wave.

The reflected wave may be written in the form

$$\phi_R = AV \exp\,[ik(x\sin\theta + z\cos\theta) + i\omega t] \qquad \text{...(3.2)}$$

ϕ_I and ϕ_R represent respectively the wave travelling towards the interface, and away from it.

Thus the total sound field in the liquid will be

$$\phi' = A\,[\exp(-ikz\cos\theta) + V\exp(ikz\cos\theta)]\exp(ikx\sin\theta + i\omega t) \qquad \text{...(3.3)}$$

For the transmitting medium, a longitudinal and transverse wave will be present. These waves can be written in the form

$$\phi = AW\exp\,[ik_1(x\sin\theta_1 - z\cos\theta_1) + i\omega t] \qquad \text{...(3.4)}$$

$$\psi = AP\exp\,[id_1(x\sin\gamma_1 - z\cos\gamma_1) + i\omega t] \qquad \text{...(3.5)}$$

where ω is the frequency; k, k_1 and d_1 are wavenumbers:

$$k_1 = \omega/c,\ d_1 = \omega/a_2,\ k = \omega/c',\ c' = (a_1'^2 + a'^2)^{1/2} \qquad \text{...(3.6)}$$

and θ_1 and γ_1 are the angles between the z-axis and the normals to the wave fronts of the longitudinal and transverse waves in the solid.

4. Boundary Conditions

As in Kaliski and Nowacki (1963), the boundary conditions at the interface $z = 0$, for the case when the vector of the initial magnetic field is parallel to the contact plane, are givan by

$$\mathbf{u} = \mathbf{u}',\ \sigma_{ik} + \tau_{ik} = \sigma'_{ik} + \tau'_{ik},\ E_1 = E'_1 \qquad \text{...(4.1)}$$

where σ_{ik} and τ_{ik} are the mechanical and electromagnetic stress tensors in the solid, and σ'_{ik}, τ'_{ik} are the corresponding quantities in the liquid. We have

$$\sigma_{ik} = G(u_{i,k} + u_{k,i}) + \delta_{ik} u_{k,k}$$

$$\tau_{ik} = \frac{1}{4\pi}[H_i h_k + H_k h_i - \delta_{ik}\mathbf{h}\,.\,\mathbf{H}] \qquad \text{...(4.2)}$$

where $\mathbf{h}$ is given by eqn. (2.2).

The boundary conditions (4.1) require only the continuity of u_3, $(\sigma_{33} + \tau_{33})$ and $(\sigma_{31} + \tau_{31})$ since the continuity of u_1 and u_2 are viscous non-slip conditions and would yield negligible amplitudes of the reflected transverse waves — other stresses do not act across the boundary. The condition

$$E_1 = E'_1 \Rightarrow H(u_3 - u'_3) = 0 \text{ i.e., } u_3 = u'_3.$$

Thus the boundary conditions in terms of the potential functions take the form

$$\left(\lambda' + \frac{H^2}{4\pi}\right)\nabla^2\phi' = \left(\lambda + \frac{H^2}{4\pi}\right)\nabla^2\phi + 2G(\phi_{,33} + \psi_{,13}) \qquad \text{...(4.3)}$$

$$G(2\phi_{,13} + \psi_{,11} - \psi_{,33}) = 0 \qquad \text{...(4.4)}$$

$$\phi'_{,3} = \phi_{,3} + \psi_{,1}. \qquad \text{...(4.5)}$$

5. Solution of the Problem

Substituting eqns. (3.3) – (3.5) into eqns. (4.3) – (4.5) and setting $z = 0$, we obtain three equations from which the angles θ_1, γ_1 and the coefficients V, W and P can be found out.

From eqn. (4.5) we get

$$k \sin\theta = k_1 \sin\theta_1 = d_1 \sin\gamma_1 \qquad \text{...(5.1)}$$

and $$k \cos\theta(V-1) = -k_1 \cos\theta_1\, W + d_1 \sin\gamma_1 P. \qquad ...(5.2)$$

Similarly, from eqns. (4.3) and (4.4) we obtain

$$\frac{\rho'}{\rho'}(1+V) = \left(1 - \frac{2k_1^2}{d_1^2}\sin^2\theta_1\right) W - \sin 2\gamma_1 P \qquad ...(5.3)$$

$$k_1^2\, W \sin 2\theta_1 + d_1^2\, P \cos 2\gamma_1 = 0. \qquad ...(5.4)$$

Solving the system of eqns. (5.2) — (5.4) by using (5.1) we get the reflection and refraction coefficients

$$V = \frac{Z_1 \cos^2 2\gamma_1 + Z_t \sin^2 2\gamma_1 - Z_2}{Z_1 \cos^2 2\gamma_1 + Z_t \sin^2 2\gamma_1 + Z_2} \qquad ...(5.5)$$

$$W = \frac{2Z_1 \cos 2\gamma_1}{Z_1 \cos^2 2\gamma_1 + Z_t \sin^2 2\gamma_1 + Z_2} \cdot \frac{\rho'}{\rho} \qquad ...(5.6)$$

$$P = -\frac{2Z_t \sin 2\gamma_1}{Z_1 \cos^2 2\gamma_1 + Z_t \sin^2 2\gamma_1 + Z_2} \cdot \frac{\rho'}{\rho} \qquad ...(5.7)$$

where Z_2, Z_1 and Z_t denote respectively the impedences of sound waves in the liquid and longitudinal and transverse waves in the solid in presence of the magnetic field :

$$Z_2 = \frac{\rho' c'}{\cos\theta}, \quad Z_1 = \frac{\rho c}{\cos\theta_1}, \quad Z_t = \frac{\rho a_2}{\cos\gamma_1}.$$

6. Analysis of the Result for Water-Aluminium Interface

(i) When $H \neq 0$ i.e. in absence of the magnetic field, we calculate the velocities of the longitudinal and transverse waves for water as follows :

Water : $\rho' = 1$ gm/cm^3, $\lambda' = 2.04 \times 10^{10}$ dynes/sq cm

Aluminium : $\rho = 2.7$ gms/cm^3 $\lambda = 5.6 \times 10^{11}$ dynes/sq cm

$G = 2.6 \times 10^{11}$ dynes/sq cm.

$$c' = (a_1'^2 + a'^2)^{1/2} = 1.428 \times 10^5 \text{ (approx.)}$$

$$a_2 = 3.103 \times 10^5, \quad c = 6.32 \times 10^5$$

Thus

$$c' < a_2 < c. \qquad ...(6.1)$$

From eqns. (5.1) we have

$$\sin\theta_1 = \frac{c}{c'}\sin\theta, \quad \sin\gamma_1 = \frac{a_2}{c'}\sin\theta. \qquad ...(6.2)$$

Hence it is clear that for $0 < \sin\theta < c'/c$ the angle θ_1 and γ_1 are real, i.e. we shall have the usual case of reflection from the boundary. This is possible for $\theta < (13.058)^\circ$ (approx.). If now $H > 0$ and $0 < H \leqslant 9.76 \times 10^5$ oe (approx.) the relation (6.1) still holds, and in this case the range of θ for usual reflection from the boundary is increased and the maximum value is given by $\theta = (28.3103)^\circ$ (approx.).

When $$\frac{c'}{c} < \sin\theta < \frac{c'}{a_2} \qquad ...(6.3)$$

it is clear from (6.2) that γ_1 will be real but θ_1 will be complex. Thus, the longitudinal wave in the solid will be an inhomogenous wave, gliding along the boundary, while the transverse wave will be an ordinary plane wave. As c' and c increases with the magnetic field the range of values of θ from (6.3) also changes with H.

(ii) When 9.76×10^5 oe $< H \leqslant 27.518 \times 10^5$ oe (approx.) we have

$$a_2 < c' < c. \qquad ...(6.4)$$

Hence, for $\sin\theta > c'/c$ the angle θ_1 will be complex, but the angle γ_1 will be real for all θ i.e. we obtain the case considered above. Since c, c' increases with the increase of the magnetic field, the range of values of θ also changes with H.

(iii) For $H > 27.518 \times 10^5$ oe we have $a_2 < c < c'$, and in this case θ_1 and γ_1 are real for all values of θ. Thus the plane longitudinal and shear waves will be excited in the solid.

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