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ON DETERMINATION OF A COMMON FIXED POINT

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In this paper we have laid down a scheme to determine a fixed point common to $\{T_n\}$, if $\{T_n\}$ is a sequence of M-type self-mappings of X. The scheme involves the introduction of a regularity condition upon the orbit generated by $\{T_n\}$. It has also been shown that such a regularity condition is necessary by citing an example in support of Theorem proved in this respect.

1. INTRODUCTION

The well-known Banach contraction principle states that if X is a nonempty complete metric space then any contraction mapping $T : X \to X$ has a unique fixed point which can be determined as the limit of an iterative sequence $\{T^n(x)\}$, $x \in X$. But the situation starts to be different whenever the mapping is taken to be nonexpansive, Browder (1965) and Kirk (1965) have proved that a nonexpansive self-mapping T of a nonempty bounded closed convex subset K of a uniformly convex Banach space has a fixed point. Goebel (1969) has also arrived at the same conclusion through an alternative proof. In this case Picard's iterative sequence $\{T^n(x)\}$, $x \in K$ does not necessarily converge to a fixed point of T. Krasnoselskü (1955) has set up a scheme of iteration with the help of $(I + T)/2$ ($I =$ identity mapping), and has shown that in a uniformly convex Banach space, Picard's iterative sequence $\{((I + T)/2)^n x\}$, $x \in K$ converges strongly to a fixed point of T, provided T has a compact range. On the other hand Marr (1963) has proved the existence of simultaneous fixed point for a family of nonexpansive mappings over a compact convex subset of a Banach space. In noncompact setting Belluce and Kirk (1967) have also established simultaneous fixed point theorems for a family of nonexpansive mappings using the concept of complete normal structure hypothesis of the space. Again Kannan (1973) has considered another class of mappings, we call them K-type mappings, and has proved fixed point theorems for such a mapping. The determination of a fixed point of such a mapping has been made by use of Krasnoselskü's scheme of iteration (Kannan 1971). Recently Kirk *et al.* (1973) have taken up a class of mappings which includes both nonexpansive mappings and K-type mappings. We call such a mapping as a mixed-type (M-type) mapping. They have proved fixed point theorem for such a mapping towards the existence of a fixed point only. Our aim in this paper is to consider a family of M-type mappings, and give a suitable scheme of iteration through which we can arrive at a common fixed point of a family of such mappings. In case the family consists of a finite number of mappings T_i, then the scheme of iteration through successive application of $(I + T_i)/2$ converges to a common fixed point of $\{T_i\}$. But such a scheme of iteration in general fails for the family whenever the later consists of an infinite number of members. For a countable family of M-type mappings in order that the scheme of iteration works we have given a regularity condition upon the orbit, the necessity of which has later been substantiated by illustrative example. Finally we observe that the family of mappings as considered in Theorem 2 always possesses a common fixed point, whenever the domain set K is taken to be a bounded closed convex subset of a uniformly convex Banach space and the family is supposed to be commutative. This has been proved by Wong (1974), and hence Theorem 2 with the additional hypothesis supplements the result of Wong and also extends the results of Krasnoselskü (1955).

Theorem 1 — Let K be a nonempty convex subsets of a uniformly convex Banach space X. Let $T_1, T_2 : K \to K$ be such that for $x, y \in K$

$$\| T_i(x) - T_i(y) \| \leqslant \alpha [\| x - T_i(x) \| + \| y - T_i(y) \|] + \beta [\| x - T_i(y) \| + \| y - T_i(x) \|] + \gamma \| x - y \|$$

where $2(\alpha + \beta) + \gamma \leqslant 1$, $i = 1, 2$, and one of $T_i(K)$ $(i = 1, 2)$ is compact. If T_1, T_2 have a common fixed point in K, then for any x_0 in K, the sequence $\{x_n\}$ under the following scheme of iteration converges to a common fixed point of T_1, T_2 :

$$x_{2n+1} = \tfrac{1}{2}(x_{2n} + T_1(x_{2n})) \quad \text{and} \quad x_{2n+2} = \tfrac{1}{2}(x_{2n+1} + T_2(x_{2n+1})), \quad n = 0, 1, 2, \ldots .$$

PROOF : If possible, let there exist an $\epsilon > 0$ and a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\| x_{2n_i} - T_1(x_{2n_i}) \| > \epsilon \quad \text{for all } i. \qquad \ldots(1)$$

Let ξ be a common fixed point of T_1 and T_2.

Then
$$\| x_{2n+1} - \xi \| = \| \tfrac{1}{2}(x_{2n} + T_1(x_{2n})) - \tfrac{1}{2}(\xi + T_1(\xi)) \| \leqslant \tfrac{1}{2} \| x_{2n} - \xi \| + \tfrac{1}{2} \| T_1(x_{2n}) - T_1(\xi) \|.$$

Now
$$\begin{aligned} \| T_1(x_{2n}) - T_1(\xi) \| &\leqslant \alpha [\| x_{2n} - T_1(x_{2n}) \| + \| \xi - T_1(\xi) \|] \\ &\quad + \beta [\| x_{2n} - T_1(\xi) \| + \| \xi - T_1(x_{2n}) \|] \\ &\quad + \gamma \| x_{2n} - \xi \| \\ &\leqslant \alpha [\| x_{2n} - \xi \| + \| T_1(\xi) - T_1(x_{2n}) \|] \\ &\quad + \beta [\| x_{2n} - \xi \| + \| T_1(\xi) - T_1(x_{2n}) \|] \\ &\quad + \gamma \| x_{2n} - \xi \|. \end{aligned}$$

So, $$\| T_1(x_{2n}) - T_1(\xi) \| \leqslant \frac{\alpha+\beta+\gamma}{1-\alpha-\beta} \| x_{2n} - \xi \| \leqslant \| x_{2n} - \xi \| \qquad \ldots(2)$$

i.e. $$\| x_{2n+1} - \xi \| \leqslant \tfrac{1}{2} \| x_{2n} - \xi \| + \tfrac{1}{2} \| x_{2n} - \xi \| = \| x_{2n} - \xi \|.$$

Similarly, $\| x_{2n+2} - \xi \| \leqslant \| x_{2n+1} - \xi \|$.

Thus $\{\| x_n - \xi \|\}$ is a decreasing sequence of non-negative terms and therefore convergent. From (1) we have $\| (x_{2n_i} - \xi) - (T_1(x_{2n_i}) - \xi) \| > \epsilon$ and by uniform convexity of the space, there is a positive $\delta < 1$ such that

$$\| \tfrac{1}{2}(x_{2n_i} + T_1(x_{2n_i})) - \tfrac{1}{2}(\xi + T_1(\xi)) \| \leqslant \delta \max [\| x_{2n_i} - \xi \|, \| T_1(x_{2n_i}) - T_1(\xi) \|].$$

So we have $\| x_{2n_i+1} - \xi \| \leqslant \delta \| x_{2n_i} - \xi \|$ [from (2)]

$$\leqslant \delta \| x_{2n_{i-1}+1} - \xi \|$$
$$\leqslant \delta^2 \| x_{2n_{i-1}} - \xi \|$$
$$\vdots$$
$$\leqslant \delta^i \| x_0 - \xi \|$$ which tends to zero as i tends to infinity.

By the decreasing property of $\{\| x_n - \xi \|\}$ we have

$$\lim_{n\to\infty} \| x_n - \xi \| = 0.$$

Thus $$\| x_{2n} - T_1(x_{2n}) \| \leqslant \| x_{2n} - \xi \| + \| T_1(\xi) - T_1(x_{2n}) \|$$
$$\leqslant \| x_{2n} - \xi \| + \frac{\alpha+\beta+\gamma}{1-\alpha-\beta} \| x_{2n} - \xi \| \leqslant 2 \| x_{2n} - \xi \|.$$

So, $\lim_{n\to\infty} \| x_{2n} - T_1(x_{2n}) \| = 0$, a contradiction of (1).

Hence $$\lim_{n\to\infty} \| x_{2n} - T_1(x_{2n}) \| = 0. \qquad \ldots(3)$$

Similarly, we obtain $\lim_{n\to\infty} \| x_{2n+1} - T_2(x_{2n+1}) \| = 0$. Let $T_1(K)$ be compact. So there is a subsequence $\{n_j\}$ of $\{n\}$ such that $\{T_1(x_{2n_j})\}$ is convergent. Let

$$\lim_{j\to\infty} T_1(x_{2n_j}) = u \in K.$$

Then from (3) we have $\lim_{j\to\infty} x_{2n_j} = u$.

Now $$\| u - T_1(u) \| \leqslant \| u - x_{2n_j} \| + \| x_{2n_j} - T_1(x_{2n_j}) \| + \| T_1(x_{2n_j}) - T_1(u) \|$$
$$\leqslant \| u - x_{2n_j} \| + \|x_{2n_j} - T_1(x_{2n_j}) \| + \alpha \,[\| x_{2n_j} - T_1(x_{2n_j})\| + \| u - T_1(u) \|] + \beta\, [\| x_{2n_j} - T_1(u) \| + \| u - T_1(x_{2n_j}) \|] + \gamma \| x_{2n_j} - u \|.$$

$$\therefore \quad \| u - T_1(u) \| \leqslant \| u - x_{2n_j} \| + (1 + \alpha) \| x_{2n_j} - T_1(x_{2n_j}) \| + \alpha \| u - T_1(u) \| + \beta\, [\| x_{2n_j} - u \| + \| u - T_1(u) \| + \| u - x_{2n_j} \| + \| x_{2n_j} - T_1(x_{2n_j}) \|] + \gamma \| x_{2n_j} - u \|.$$

This gives
$$\| u - T_1(u) \| \leqslant \left(1 + \frac{2\beta + \gamma}{1 - \alpha - \beta}\right) \| x_{2n_j} - u \| + \left(1 + \frac{\alpha + \beta}{1 - \alpha - \beta}\right) \| x_{2n_j} - T_1(x_{2n_j}) \|$$
which tends to zero as j tends to infinity. Thus $u = T_1(u)$.

Again $$\| x_{2n_j+1} - u \| \leqslant \| u - x_{2n_j} \| + \| x_{2n_j} - x_{2n_j+1} \|$$
$$= \| u - x_{2n_j} \| + \| x_{2n_j} - \tfrac{1}{2} (x_{2n_j} + T_1(x_{2n_j})) \|$$
$$= \| u - x_{2n_j} \| + \tfrac{1}{2} \| x_{2n_j} - T_1(x_{2n_j}) \|$$
which tends to zero as j tends to infinity.

Now proceeding as above we deduce that $u = T_2(u)$. Thus u has been shown to be a common fixed point of T_1 and T_2. Since $\{\| x_n - u \|\}$ decreases, and a subsequence $\{\| x_{2n_j+1} - u \|\}$ converges to zero, we have $\lim\limits_{n\to\infty} \| x_n - u \| = 0$, i.e. $\lim\limits_{n\to\infty} x_n = u$. This completes the proof of the theorem.

Remark 1 : If there is a family containing finite number of self-mappings as considered in Theorem 1 above, say $T_1, T_2, \ldots, T_n$, following scheme of iteration gives us a common fixed point of $\{T_i\}$: $x_0 \in K$ and
$$x_{pn+i+1} = \tfrac{1}{2} (x_{pn+i} + T_{i+1}(x_{pn+i})),$$
where $i = 0, 1, \ldots, (n - 1)$ and $p = 0, 1, \ldots$.

In case the family containing a countable infinite number of members say, $T_1, T_2, \ldots$, then the scheme of iteration $x_{n+1} = \frac{1}{2} (x_n + T_n(x_n))$, $x_0 \in K$, in general, fails to give us a common fixed point for $\{T_i\}$. The following example supports our contention.

Example 1 — Let $X = l_2$ and $K = B$ where B is the closed unit ball in l_2. Define a family $\{T_i\}$ of self-mappings on K as follows

for any $x = (x_1, x_2, x_3, \ldots) \in K$, let $T_n(x) = (x_{n+1}, x_2, x_3, \ldots, x_n, x_1, 0, 0, \ldots)$,

$$n = 1, 2, \ldots .$$

Then each T_n satisfies the conditions of Theorem 1 and each $T_n(K)$ is compact and $(0, 0, 0, \ldots)$ is the common fixed point of the family $\{T_n\}$. Next, let us consider the point $x_0 = (1, 0, 0, \ldots) \in K$. Then

$$x_{n+1} = \tfrac{1}{2}(x_n + T_n(x_n)) = \left(\frac{1}{2^{n+1}}, \frac{1}{2}, \frac{1}{2^2}, \ldots, \frac{1}{2^{n+1}}, 0, 0, \ldots\right)$$

so that the sequence $\{x_n\}$ converges to $\left(0, \frac{1}{2}, \frac{1}{2^2}, \ldots, \frac{1}{2^n}, \ldots\right)$ which is not a common fixed point of $\{T_n\}$.

Definition 1 — A sequence $\{x_n : x_0 = x\}$ generated by successive application of members of $\{T_n\}$ in any order at x is called an orbit of $\{T_n\}$ at x.

Remark 2 : In view of Example 1 above we remark that given any orbit of $\{\frac{1}{2}(I + T_n)\}$ at x_0, the same does not necessarily give us a common fixed point for $\{T_n\}$. We make the following definition.

Definition 2 — An orbit $\{x_n; x_0 \in K\}$ is said to be a regular orbit of $\{T_i\}$ at x_0 if there is a sequence $\{p_i\}$ of positive integers satisfying

$$x_{p_i+1} = T_1(x_{p_i}),\ x_{p_i+2} = T_2(x_{p_i+1}),\ \ldots,\ x_{p_{i+1}} = T_{p_{i+1}-p_i}(x_{p_{i+1}-1}),$$
$$i = 1, 2, \ldots, \text{ with } x_{p_1} = x_0.$$

Theorem 2 — Let K be a nonempty convex subset of a uniformly convex Banach space X and $\{T_n\}$ be a countable family of self-mappings on K such that for $x, y \in K$

$$\|T_i(x) - T_i(y)\| \leqslant \alpha[\|x - T_i(x)\| + \|y - T_i(y)\|] + \beta[\|x - T_i(y)\| + \|y - T_i(x)\|] + \gamma\|x - y\|$$

where

$$2(\alpha + \beta) + \gamma \leqslant 1,\ i = 1, 2, \ldots .$$

If the family $\{T_i\}$ has a common fixed point in K, and at least one of $T_i(K)$ is compact, then there is a regular orbit of $\{\frac{1}{2}(I + T_i)\}$ at $x_0 \in K$ which converges to a common fixed point of $\{T_i\}$.

PROOF : Without loss of generality let $T_1(K)$ be compact. Put $S_i = \frac{1}{2}(I + T_i)$, $i = 1, 2, \ldots$. Taking $p_i = \frac{1}{2}(i(i + 1)) - 1$ and following Definition 2 above we construct a regular orbit $\{x_n\}$ of $\{S_i\}$ at x_0. If possible, let there exist an $\epsilon > 0$ and a subsequence $\{y_{n_i} = x_{(n_i(n_i+1)/2)-1}\}$ of $\{x_n\}$ such that

$$\| y_{n_i} - T_1(y_{n_i}) \| > \epsilon \quad \text{for all } i. \qquad \ldots(4)$$

Let ξ be a common fixed point of $\{T_i\}$. Then proceeding as in the proof of Theorem 1 we have $\{\| x_n - \xi \|\}$ as a decreasing sequence of nonnegative reals and therefore $\{\| x_n - \xi \|\}$ is convergent. Now from (4) we have,

$$\|(y_{n_i} - \xi) - (T_1(y_{n_i}) - \xi)\| > \epsilon.$$

Hence by uniform convexity of X, there is a positive $\delta < 1$ such that

$$\| \tfrac{1}{2}(y_{n_i} + T_1(y_{n_i})) - \tfrac{1}{2}(\xi + T_1(\xi)) \| \leqslant \delta \max [\| y_{n_i} - \xi \|, \| T_1(y_{n_i}) - T_1(\xi) \|]$$

i.e. $$\| S_1(y_{n_i}) - \xi \| \leqslant \delta \max [\| y_{n_i} - \xi \|, \| T_1(y_{n_i}) - T_1(\xi) \|].$$

But $$\| T_1(y_{n_i}) - T_1(\xi) \| \leqslant \| y_{n_i} - \xi \|.$$

Therefore,

$$\| S_1(y_{n_i}) - \xi \| \leqslant \delta \| y_{n_i} - \xi \|.$$

Since $\{\| x_n - \xi \|\}$ is decreasing, we have,

$$\| S_1(y_{n_i}) - \xi \| \leqslant \delta \| S_1(y_{n_{i-1}}) - \xi \| \leqslant \delta^2 \| y_{n_{i-1}} - \xi \|$$

$$\leqslant \ldots \leqslant \delta^i \| y_{n_1} - \xi \| \quad \text{which tends to zero as } i \to \infty.$$

Thus $\| x_{n_i(n_i+1)/2} - \xi \| \to 0$ as $i \to \infty$. By the decreasing property of $\{\| x_n - \xi \|\}$, we have $\| x_n - \xi \| \to 0$ as $n \to \infty$, i.e. $\lim_{n\to\infty} \| y_n - T_1(y_n) \| = 0$, a contradiction of (4).

Therefore

$$\lim_{n\to\infty} \| y_n - T_1(y_n) \| = 0$$

i.e. $$\lim_{n\to\infty} \| x_{(n(n+1)/2)-1} - T_1(x_{(n(n+1)/2)-1}) \| = 0.$$

Similarly, $\| (x_{(n(n+2i-1)+i(i+1))/2)-2}) - T_i(x_{(n(n+2i-1)+i(i+1))/2)-2}) \| \to 0$ as $n \to \infty$ for each i.

Since $T_1(K)$ is compact, there is a subsequence $\{n_j\}$ of $\{n\}$ such that

$$\lim_{\to\infty} T_1(x_{(n_j(n_j+1)/2)-1}) = u \text{ (say)} = \lim_{j\to\infty} (x_{(n_j(n_j+1)/2)-1}).$$

Then it is easy to verify that $u = T_1(u)$. Next let $T_i \in \{T_i\}$, $i > 1$ be arbitrary.

Put $N_j = \frac{1}{2}(n_j - i + 1)\overline{(n_j - i + 1} + 2i - 1) - 2.$

Then
$$\| x_{N_j} - u \| \leqslant \| x_{N_j} - x_{N_j-1} \| + \| x_{N_j-1} - x_{N_j-2} \| + \ldots$$
$$+ \| x_{n_j(n_j+1)/2} - x_{(n_j(n_j+1)/2)-1} \| + \| x_{(n_j(n_j+1)/2)-1} - u \| \to 0$$

as $j \to \infty$, since the number of terms on the right-hand side is finite and each of them tends to zero as $j \to \infty$. Hence $u = T_i(u)$. Since T_i is an arbitrary member of $\{T_j\}$, u is a common fixed point of $\{T_j\}$. Then $\{\| x_n - u \|\}$ is decreasing. Since a subsequence of $\{\| x_n - u \|\}$ converges to zero. We have $\{x_n\}$ converges to u. This completes the proof of the theorem.

If in Theorem 2 one assumes further that K is bounded closed convex and $\{T_n\}$ to be a commuting family of self-mappings of K then we can apply a result proved by Wong (1974) to conclude that $\{T_n\}$ possesses a common fixed point. Hence we have the following theorem.

Theorem 3 — If K is a nonempty bounded closed convex subset of a uniformly convex Banach space, and $\{T_n\}_{n=1}^{\infty}$ is a commuting family of self-mappings of K such that for $x, y \in K$

$$\| T_i(x) - T_i(y) \| \leqslant \alpha [\| x - T_i(x) \| + \| y - T_i(y) \|]$$
$$+ \beta [\| x - T_i(y) \| + \| y - T_i(x) \|] + \gamma \| x - y \|,$$
$$i = 1, 2, \ldots \quad \text{and} \quad 2(\alpha + \beta) + \gamma \leqslant 1$$

and that $T_i(K)$ is compact for some i, then for any point $x \in K$ there is a regular orbit of $\{(I + T_n)/2\}$ at x which converges to a common fixed point of $\{T_n\}$.

Corollary 1 (Theorem of Krasnoselskü (1955) — If K is a nonempty bounded closed convex subset of a uniformly convex Banach space and $T : K \to K$ is non-expansive, such that $T(K)$ is compact, then for any point x in K, the sequence $\{x_n, x_0 = x\}$ converges to a fixed point of T, where

$$x_{n+1} = (x_n + T(x_n))/2, \quad n = 0, 1, \ldots .$$

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