Indian J. pure appl. Math., **11**(1) : 128–134, January 1980

A GENERALIZATION OF A THEOREM OF DUPLESSIS ON THE CONVERGENCE OF LAPLACE SERIES

B. K. BEOHAR

Department of Mathematics, Madhav Vigyan Mahavidyalaya, Ujjain 456010

(*Received* 1 *February* 1979)

In the present paper, we prove a new theorem on the convergence of Laplace series. Our theorem generalises an earlier result of Duplessis (1952) and that of Pandey (1968).

§1. A function $f(x)$, $a \leqq x \leqq b$, is said to belong to the class Lip α, $0 < \alpha \leqslant 1$, if

$$f(x \pm t) - f(x) = O(t^{\alpha}),\ t \to 0 \qquad \ldots(1.1)$$

and to the class Lip* α if O is replaced by o. Also if

$$\int_a^b | f(x \pm t) - f(x) |^p \, dz = O(t^{\alpha p}),\ p \geqslant 1 \qquad \ldots(1.2)$$

then $f(x)$ is said to belong to the class Lip (α, p) and to Lip* (α, p), if O is replaced by o.

A function of Lip α belongs to Lip (α, p) for every positive p and the properties of such functions may be regarded as the limiting cases (for $p = \infty$) of these functions of Lip (α, p). It is also well known that if $f(x) \in$ Lip (α, p), $\alpha p > 1$, $p \geqq 1$, $0 < \alpha \leqq 1$, then this is true for all $q > p$ and $f(x)$ is equivalent to a function of Lip $\left(\alpha - \frac{1}{p}\right)$.

§2. Let $f(x) \in L[-1, 1]$; the Jacobi series corresponding to $f(x)$ is

$$f(x) \sim \sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(x) \qquad \ldots(2.1)$$

where

$$a_n = \frac{(2n + \alpha + \beta + 1)\,\Gamma(n + 1)\,\Gamma(n + \alpha + \beta + 1)}{2^{\alpha+\beta+1}\,\Gamma(n + \alpha + 1)\,\Gamma(n + \beta + 1)} \times \int_{-1}^{1} (1 - x)^{\alpha} (1 + x)^{\beta} f(x)\, P_n^{(\alpha,\beta)}(x)\, dx \qquad \ldots(2.2)$$

and $P_n^{(\alpha,\beta)}(x)$ is the nth Jacobi polynomial of order α, β.

The Cesàro summability of the series (2.1) has been studied by Kogbetliantz (1931), Szegö (1959) and Pandey (1968). Pandey proved the following.

Theorem A — The series (2.1) is summable (C, k) for $\alpha - \frac{1}{2} < k < \alpha + \frac{1}{2}$, $-\frac{1}{2} < \alpha < \frac{1}{2}$, $\beta \geqslant \alpha$ at the point $x = \pm 1$ to the sum A provided that

$$f(x) - A \in \text{Lip}^* (\alpha + \tfrac{1}{2} - k),$$

A being a fixed constant.

This theorem is a generalization of an earlier result by Gupta (1958) on Cesàro summability of ultraspherical series which is a special case of the series (2.1) when $\alpha = \beta = \lambda - \frac{1}{2}$. Another important particular case of Theorem A is the well-known theorem of Duplessis on Cesàro summability of Laplace series which is obtained from the series (2.1) by lettiny $\alpha = \beta = 0$.

In an attempt to study the convergence and Cesàro summability of Jacobi series under more general conditions than those of Theorem A, we are at first starting with the famous Laplace series so as to make our calculations simple with the hope that the proof so constructed will give us a strong tool to handle the more difficult problems of Jacobi series.

§3. Let $f(\theta, \phi)$ be defined for the range $0 \leqq \theta \leqq \pi$, $0 \leqq \phi \leqq 2\pi$, and integrable (L) on the surface of a sphere S. Given the point P on the sphere we choose co-ordinates so that P is the pole. The Laplace series for $f(\theta, \phi)$ at the point P is given by

$$\frac{1}{4\pi} \sum_{m} (2m + 1) \int_0^{\pi} \int_0^{2\pi} f(\theta, \phi)\, P_m (\cos \theta) \sin \theta \, d\theta \, d\phi \qquad \text{...(3.1)}$$

where $P_m(t)$ are the Legendre polynomials of the first kind.

We write

$$F_P(\theta) = \int_0^{\pi} f(\theta, \phi)\, d\phi. \qquad \text{...(3.2)}$$

The nth partial sum of the series (3.1) is given by

$$S_n(P) = \frac{1}{4\pi} \int_0^{\pi} \int_0^{2\pi} f(\theta, \phi) \sum_{k=0}^{n} (2k + 1)\, P_k (\cos \theta) \sin \theta \, d\theta.$$

Therefore

$$S_n(P) = \frac{1}{4\pi} \int_0^{\pi} F(\theta) \sin \theta \sum_{k=0}^{n} (2k + 1)\, P_k (\cos \theta)\, d\theta.$$

Thus

$$S_n(P) - f(P) = \frac{1}{4\pi} \int_0^\pi \{F(\theta) - F_P(\theta)\} \sin\theta \sum_{k=0}^{n} (2k+1) P_k (\cos\theta)\, d\theta.$$

Without loss of generality we may set

$$f(P) = 0;\; F_P(\theta) = F(\theta) = 0.$$

Then

$$S_n(P) = \frac{1}{4\pi} \int_0^\pi F(\theta)\, L_n(\theta)\, d\theta \qquad \text{...(3.3)}$$

where

$$L_n(\theta) = \sum_{k=0}^{n} (2k+1) P_k (\cos\theta)$$

$$= \frac{d}{d\theta} \{P_{n+1} (\cos\theta) + P_n (\cos\theta)\}. \qquad \text{...(3.4)}$$

For the Cesàro summability of the series (3.1) Duplessis (1952) has proved the following:

Theorem B — If $f(P)$ is integrable over the unit sphere and $F_P(\theta) \in \text{Lip}^* (\frac{1}{2} - k)$, ($P$ is the pole) then for $-\frac{1}{2} < k < \frac{1}{2}$ the series (3.1) is (C, k) summable at the point P to the value $f(P)$.

The object of this paper is to generalize Theorem B for $k = 0$. The following theorem can be proved.

Theorem — If $F(\theta) \in \text{Lip}^* (\frac{1}{2}, p)$, $p > 2$, then the series (3.1) is convergent at the point P (pole) to the sum $f(P)$.

§4. For the proof of the theorem we need the following lemmas.

Lemma 1 — For $0 \leqq \theta \leqq \pi$

$$L_n(\theta) = O(n^2\theta). \qquad \text{...(4.1)}$$

The proof follows from the well-known (see Sansone 1959) relation,

$$| P_m (\cos\theta) | \leqslant 1.$$

Lemma 2 — For $\pi - (1/n) \leqslant \theta \leqslant \pi$

$$L_n(\theta) = O(n \sin\theta). \qquad \text{...(4.2)}$$

For the proof see Duplesis (1952).

Lemma 3 — For $(C/n) \leqslant \theta \leqslant \pi - (1/n)$ and $\alpha > -1, \beta > -1,$

$$\frac{d}{d\theta}\{P_n^{(\alpha,\beta)}(\cos\theta)\} = n^{1/2}k(\theta)\{-\sin(N\theta+\gamma) + (n\sin\theta)^{-1}O(1)\}$$

$$k(\theta) = \pi^{-1/2}(\sin\theta/2)^{-\alpha-(1/2)}(\cos\theta/2)^{-\beta-(1/2)},$$

$$N = n + \tfrac{1}{2}(\alpha+\beta+1), \gamma = -(\alpha+\tfrac{1}{2})\pi/2. \quad \ldots(4.3)$$

For the proof see Szegö (1959).

We shall use the lemma for the special case $\alpha = \beta = 0.$

§5. *Proof of the Theorem* — In order to prove the theorem we have to show that under the hypothesis of our theorem

$$S_n(P) = o(1), \text{ as } n \to \infty.$$

We have from (3.3)

$$S_n(P) = \frac{1}{4\pi}\int_0^{\pi} F(\theta)\, L_n(\theta)\, d\theta$$

$$= \frac{1}{4\pi}\left\{\int_0^{\lambda_n} + \int_{\lambda_n}^{\pi-(1/n)} + \int_{\pi-(1/n)}^{\pi}\right\} \quad (\lambda_n = n^{-4p/(5p-2)})$$

$$= \frac{1}{4\pi}\{I_1 + I_2 + I_3 + I_4\}, \text{ say.} \quad \ldots(5.1)$$

Now

$$I_1 = O(n^2)\int_0^{\lambda_n} |F(\theta)|\,\theta\, d\theta$$

$$= O(n^2)\int_0^{\lambda_n} o(\theta^{(p-2)/2p})\,\theta\, d\theta.$$

$$= o\,(n^2\lambda_n^{(5p-2)/2p})$$

$$= o(1), \text{ as } n \to \infty. \quad \ldots(5.2)$$

Also

$$I_3 = O(n)\int_{\pi-(1/n)}^{\pi} \sin\theta\, d\theta$$

$$= o(1), \text{ as } n \to \infty. \quad \ldots(5.3)$$

And finally

$$I_2 = \int_{\lambda_n}^{\pi-(1/n)} F(\theta) \frac{d}{d\theta} \{P_{n+1}(\cos\theta) + P_n(\cos\theta)\}\, d\theta$$

$$= I_{2\cdot 1} + I_{2\cdot 2}, \text{ say} \qquad \ldots(5.4)$$

where $$I_{2\cdot 1} = \int_{\lambda_n}^{\pi-(1/n)} F(\theta) \frac{d}{d\theta} \{P_{n+1}(\cos\theta)\}\, d\theta.$$

Now by an application of Lemma 3 we obtain

$$I_{2\cdot 1} = \sqrt{\frac{n+1}{\pi}} \int_{\lambda_n}^{\pi-(1/n)} (\sin\tfrac{1}{2}\theta)^{-1/2} (\cos\tfrac{1}{2}\theta)^{-1/2} [-\sin\{(n+\tfrac{3}{2})\theta - \tfrac{1}{4}\pi\}] \times F(\theta)\, d\theta + o(1)$$

$$= \sqrt{\frac{n+1}{2\pi}} \int_{\lambda_n}^{\pi-(1/n)} (\sin\tfrac{1}{2}\theta)^{-1/2} (\cos\tfrac{1}{2}\theta)^{-1/2} \{\cos(n+\tfrac{3}{2})\theta - \sin(n+\tfrac{3}{2})\theta\} F(\theta)\, d\theta$$

$$= \sqrt{\frac{n+1}{2\pi}} \{i_1 + i_2\}, \text{ say}$$

where

$$i_1 = \int_{\lambda_n}^{\pi-(1/n)} (\sin\tfrac{1}{2}\theta)^{-1/2} (\cos\tfrac{1}{2}\theta)^{-1/2} \cos(n+\tfrac{3}{2})\theta\, F(\theta)\, d\theta$$

$$= \tfrac{1}{2} [\int_{\lambda_n}^{\pi-(1/n)} (\sin\tfrac{1}{2}\theta)^{-1/2} (\cos\tfrac{1}{2}\theta)^{-1/2} \cos(n+\tfrac{3}{2})\theta\, F(\theta)\, d\theta$$

$$= \tfrac{1}{2} \Big[\int_{\lambda_n}^{\pi-(1/n)} (\sin\tfrac{1}{2}\theta)^{-1/2} (\cos\tfrac{1}{2}\theta)^{-1/2} \cos(n+\tfrac{3}{2})\theta\, F(\theta)\, d\theta$$

$$- \int_{\lambda_n-\mu_n}^{\pi-(1/n)-\mu_n} \left(\sin\frac{\theta+\mu_n}{2}\right)^{-1/2} \left(\cos\frac{\theta+\mu_n}{2}\right)^{-1/2} F(\theta+\mu_n) \times \cos(n+\tfrac{3}{2})\theta\, d\theta \Big]$$

$$\left(\mu_n = \frac{\pi}{n+\frac{3}{2}}\right)$$

$$\leqslant \tfrac{1}{2}(J_1 + J_2 + J_3 + J_4), \text{ say.}$$

where

$$J_1 = \int_{\lambda_n-\mu_n}^{\lambda_n} |\,(\sin \tfrac{1}{2}\,\theta)^{-1/2}\,(\cos \tfrac{1}{2}\,\theta)^{-1/2}\,F(\theta+\mu_n)\,|\,d\theta$$

$$= o \int_{\lambda_n-\mu_n}^{\lambda_n} \theta^{-1/2}\,(\theta+\mu_n)^{(1/2)-(1/p)}\,d\theta$$

$$= o\,(\lambda_n)^{1-(1/p)} \qquad \ldots(5.5)$$

and

$$J_2 = \int_{\pi-(1/n)-\mu_n}^{\mu-(1/n)} |\,(\sin \tfrac{1}{2}\,\theta)^{-1/2}\,(\cos \tfrac{1}{2}\,\theta)^{-1/2}\,F(\theta)\,|\,d\theta$$

$$= O\,(\mu_n). \qquad \ldots(5.6)$$

$$J_3 = \int_{\lambda_n}^{\pi-(1/n)-\mu_n} |\,F(\theta+\mu_n)-F(\theta)\,|\,(\sin \tfrac{1}{2}\,\theta)^{-1/2}\,(\cos \tfrac{1}{2}\,\theta)^{-1/2}\,d\theta.$$

Applying Hölder's inequality we get

$$J_3 = \int_{\lambda_n}^{\pi-(1/n)-\mu_n} |\,F(\theta+\mu_n)-F(\theta)\,|^{\,p})^{1/p}\,\Big(\int_{\lambda_n}^{\pi-(1/n)-\mu_n} \theta^{-q/2}\Big)^{1/q}$$

$$= o\,(\mu_n^{1/2}). \qquad \ldots(5.7)$$

Finally, we have

$$J_4 = \int_{\lambda_n}^{\pi-(1/n)-\mu_n} \left|\left(\sin\frac{\theta+\mu_n}{2}\right)^{-1/2}\left(\cos\frac{\theta+\mu_n}{2}\right)^{-1/2} - (\sin \tfrac{1}{2}\,\theta)^{-1/2}\,(\cos \tfrac{1}{2}\,\theta)^{-1/2}\right|\,|\,F(\theta)\,|\,d\theta$$

$$= O\,(\mu_n) \int_{\lambda_n}^{\pi-(1/n)-\mu_n} \theta^{-3/2}\,\theta^{(1/2)-(1/p)}\,d\theta$$

$$= O\,(\mu_n). \qquad \ldots(5.8)$$

Combining (5.5) to (5.8), we get

$$i_1 = O\,(\mu_n). \qquad \ldots(5.9)$$

Similarly, we can show that

$$i_2 = O\,(\mu_n). \qquad \ldots(5.10)$$

Thus

$$I_{2\cdot 1},\, I_{2\cdot 2} = O\,(\mu_n) \qquad \ldots(5.11)$$

Therefore,

$$I_2 = o(1), \text{ as } n \to \infty. \qquad ...(5.12)$$

By virtue of the relations (5.2), (5.3) and (5.12), the theorem is completely established.

Acknowledgement

The author is highly grateful to Dr. G. S. Pandey for his valuable suggestions during the preparation of this paper.

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