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ON THE PROBLEM OF STRETCHED INFINITE PLATE WEAKENED BY A HOLE HAVING ARBITRARY SHAPE

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Complex variable methods have been applied to derive exact expressions for Gaursat's functions for a stretched infinite plate weakened by a hole having arbitrary shape. The edge of the hole is free from stresses. The weakened plate can be mapped conformally on the right half-plane. Closed expressions for Gaursat's functions are obtained for hypotrochoidal holes with four or five rounded corners.

1. INTRODUCTION

For isotropic homogeneous perforated infinite plate stretched at infinity by the application of a uniform tensile stress of intensity P making an angle θ with the x-axis, the stresses can be put in the form (Muskhelishvili 1949)

$$\left.\begin{aligned} \widehat{xx} + \widehat{yy} &= 4 \text{ Re } \{\phi'(z)\} \\ \widehat{yy} - \widehat{xx} + 2i\widehat{xy} &= 2 [\bar{z}\phi''(z) + \psi'(z)] \end{aligned}\right\} \quad \ldots(1.1)$$

where the Gaursat's functions $\phi(z)$ and $\psi(z)$ take the form

$$\phi(z) = \frac{P}{4} z + \phi_0(z), \; \psi(z) = \frac{P}{2} e^{-2i\theta} z + \psi_0(z) \quad \ldots(1.2)$$

$\phi_0(z)$ and $\psi_0(z)$ are single valued analytic functions within the region of the plate and bounded at infinity. When the edge of the hole is free from stresses the boundary condition takes the form

$$\phi_0(t) + t \overline{\phi_0'(t)} + \overline{\psi_0(t)} = - \tfrac{1}{2} P(t - e^{2i\theta}\bar{t}) \quad \ldots(1.3)$$

where t is an arbitrary point on the inner boundary.

In a previous paper (El-Sirafy 1977), the Gaursat's functions were obtained for a stretched infinite plates, weakened by inner curvilinear holes by using the transformation

$$\frac{z}{c} = \frac{\zeta + m\zeta^{-1}}{1 - n\zeta^{-1}}, \; \zeta = \frac{s+1}{s-1}, \; c > 0, \; | n | < 1, \; s = \sigma + i\tau.$$

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This transformation maps the perforated infinite plate onto the area of right half-plane Re $s \geqslant 0$.

In this paper complex variable methods have been applied to solve the problem of stretched infinite plate weakened by arbitrary hole which is free from stresses. The plate, taken in the z-plane can be mapped conformally on the right half-plane Re $s \geqslant 0$ by a rational mapping function bounded at infinity. The case of stretched infinite plates weakened by hypotrochoidal holes with four or five rounded corners are considered and we obtain the functions $\phi(z)$ and $\psi(z)$ in a closed form.

2. Method of Solution

Let

$$z = w(s) \qquad \text{...(2.1)}$$

be a single valued analytic function in the right half-plane Re $s \geqslant 0$, where $w(\infty)$ is bounded and $w'(s) \neq 0$ within Re $s \geqslant 0$, $s = \sigma + i\tau$. We now consider weakened infinite plates mapped on Re $s \geqslant 0$ by the conformal transformation (2.1).

The expression $\overline{w(i\tau)}/w'(i\tau)$ will be assumed in the form

$$\frac{\overline{w(i\tau)}}{w'(i\tau)} = A(i\tau) + B(i\tau), \qquad \text{...(2.2)}$$

where

$$A(s) = \sum_{\gamma=1}^{n} \sum_{l=1}^{r_\gamma} \frac{a_{\gamma l}}{(s - s_\gamma)^l}, \qquad \text{...(2.3)}$$

$a_{\gamma l}$ are complex constants, s_γ are the poles of $A(s)$, Re $s_\gamma > 0$, r_γ are the orders of the poles $s_\gamma (\gamma = 1, \ldots, n)$ and $B(s)$ is a regular function within the right half-plane except at infinity. The boundary condition (1.3) takes the form

$$\overline{\Phi(i\tau)} + A(i\tau)\, \Phi'(i\tau) + \Psi(i\tau) = \overline{f(\tau)}, \qquad \text{...(2.4)}$$

where

$$\Phi(s) = \phi_0(w(s)),$$

$$\Psi(s) = \psi_0(w(s)) + B(s)\, \Phi'(s) + \tfrac{1}{2} P\, [\overline{w(\infty)} - e^{-2i\theta} w(\infty)], \qquad \text{...(2.5)}$$

$$f(\tau) = \tfrac{1}{2} P\, [w(\infty) - w(i\tau) + e^{2i\theta} \{\overline{w(i\tau)} - \overline{w(\infty)}\}],$$

and we assume that $\Phi(\infty) = \Psi(\infty) = 0$.

Multiplying both sides of (2.4) by $\frac{1}{2}[\pi(\bar{s} + i\tau)]^{-1}$ and integrating with respect to τ from $-\infty$ to ∞, we get

$$\overline{\Phi(s)} + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{A(i\tau)\,\Phi'(i\tau)}{\bar{s} + i\tau}\, d\tau = \overline{G(s)} \qquad \text{...(2.6)}$$

where

$$G(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{f(\tau)}{s - i\tau}\, d\tau. \qquad \text{...(2.7)}$$

The integral of the formula (2.6) can be calculated by using (2.3), we get

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{A(i\tau)\,\Phi'(i\tau)}{\bar{s} + i\tau}\, d\tau = -\sum_{\gamma=1}^{n}\sum_{l=1}^{r_\gamma} \frac{a_{\gamma l}}{(l-1)!}\cdot\frac{d^{(l-1)}}{d\lambda^{(l-1)}}\left[\frac{\Phi'(\lambda)}{\lambda + \bar{s}}\right]_{\lambda = s_\gamma}$$

$$= \sum_{\gamma=1}^{n}\sum_{l=1}^{r_\gamma}\sum_{\mu=1}^{l} \frac{(-1)^{\mu}\,\alpha_{\mu l\gamma}}{(s_\gamma + \bar{s})^{\mu}}$$

$$\alpha_{l\mu\gamma} = a_{\gamma l}\,\frac{(\mu-1)!}{(l-1)!}\, C_{l-1}^{\mu-1}\, \Phi^{(l-\mu+1)}(s_\gamma). \qquad \text{...(2.8)}$$

Hence we have

$$\Phi(s) = G(s) + \sum_{\gamma=1}^{n}\sum_{l=1}^{r_\gamma}\sum_{\mu=1}^{l} (-1)^{\mu+1}\, \frac{\bar{\alpha}_{\mu l\gamma}}{(\bar{s}_\gamma + s)^{\mu}}. \qquad \text{...(2.9)}$$

To calculate the constants $\alpha_{\mu l\gamma}$, inserting in (2.8) the differential values $\Phi^{(l-\mu+1)}(s_\gamma)$ given by (2.9), we get the following system of a linear algebraic equations:

$$(l-1)!\,\alpha_{\mu l\gamma} + (\mu-1)!\,a_{\gamma l}C_{l-1}^{\mu-1}\sum_{q=1}^{n}\sum_{h=1}^{r_q}\sum_{p=1}^{h} E_{(q,p+l-\mu)}(s_\gamma)\cdot\frac{\bar{\alpha}_{phq}}{(p-1)!}$$

$$= (\mu-1)!\,a_{\gamma l}C_{l-1}^{\mu-1}\, G^{(l-\mu+1)}(s_\gamma),\ \mu = 1, 2, \ldots, l;\ l = 1, \ldots, r_\gamma;$$

$$\gamma = 1, \ldots, n, \qquad \text{...(2.10)}$$

where

$$E_{(q,j)}(s) = \frac{(-1)^{j+1}\, j!}{(\bar{s}_q + s)^{j+1}}. \qquad \text{...(2.11)}$$

Using the formula (2.3), (2.9) and the boundary condition (2.4), we obtain $\Psi(s)$ in the form

$$\Psi(s) = H(s) + L(s) + \sum_{\gamma=1}^{n} \sum_{l=1}^{r_\gamma} \sum_{q=1}^{n} \sum_{h=1}^{r_q} \sum_{p=1}^{h} \sum_{j=0}^{p} p a_{\gamma l} \bar{\alpha}_{phq} \times \frac{E_{(q,l+j-1)}(s_\gamma) \, . \, E_{(q,p-j)}(s)}{j! \, . \, (p-j)! \, . \, (l-1)!} \quad ...(2.12)$$

where

$$H(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\overline{f(\tau)}}{s - i\tau} \, d\tau \quad ...(2.13)$$

$$L(s) = -\frac{1}{2\pi} \sum_{\gamma=1}^{n} \sum_{l=1}^{r_\gamma} a_{\gamma l} \int_{-\infty}^{\infty} \frac{G'(i\tau)}{(s - i\tau)(i\tau - s_\gamma)^l} \, d\tau. \quad ...(2.14)$$

3. Stretched Infinite Plates Weakened by Hypotrochoidal Holes

We now turn our attention to derive closed expressions for $\phi(z)$ and $\psi(z)$ in two special cases of the mapping function (2.1). Consider an infinite plate weakened by hypotrochoidal hole Γ, conformally mapped on the right half-plane Re $s \geqslant 0$ by one of the transformations

$$z = w(s) = c\left[\frac{s+1}{s-1} + m\left(\frac{s-1}{s+1}\right)^k\right], \; c > 0, \; |m| \leqslant \frac{1}{k}, \; (k = 3, 4). \quad ...(3.1)$$

Case (*a*) : $k = 3$ — Here we have a hypotrochoidal hole with four round corners (see Fig. 1).

The expression $\overline{w(i\tau)}/w'(i\tau)$ will be assumed as

$$\frac{\overline{w(i\tau)}}{w'(i\tau)} = \frac{-4m}{i\tau - 1} + B(i\tau) \quad ...(3.2)$$

where

$$B(s) = \frac{1}{2(1-s)} \left\{ \frac{m(1+s)^7 + (1-s)(1-s^2)^3}{(1+s)^4 - 3m(1-s)^4} - 8m \right\}. \quad ...(3.3)$$

In this case the boundary condition (1.3) takes the form (2.4) with

$$\left.\begin{aligned} \Phi(s) &= \phi_0(w(s)), \\ \Psi(s) &= \psi_0(w(s)) + B(s)\,\Phi'(s) + \tfrac{1}{2} Pc(1+m)(1 - e^{-2i\theta}) \\ f(\tau) &= -\tfrac{1}{2} Pc\,[w(i\tau) - e^{2i\theta}\,\overline{w(i\tau)}] + \tfrac{1}{2} Pc(1+m)(1 - e^{2i\theta}) \end{aligned}\right\} \quad ...(3.4)$$

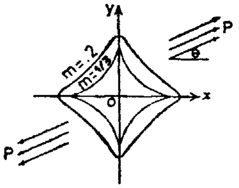

FIG. 1. $z = c\left[\frac{s+1}{s-1} + m\left(\frac{s-1}{s+1}\right)^3\right]$

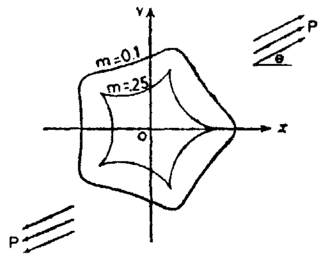

FIG. 2. $z = c\left[\frac{s+1}{s-1} + m\left(\frac{s-1}{s+1}\right)^4\right]$

by using the formulae (2.9) – (2.14) the functions $\Phi(s)$ and $\Psi(s)$ may be represented by

$$\Phi(s) = G(s) + \frac{\bar{\alpha}_{111}}{s+1}, \qquad \text{...(3.5)}$$

$$\Psi(s) = H(s) + L(s) - 4m\bar{\alpha}_{111}\sum_{j=0}^{1} E_{(1,j)}(1)\,.\,E_{(1,1-j)}(s)$$

where

$$G(s) = \frac{Pc}{s+1}\left[3m - e^{2i\theta} - \frac{6m}{s+1} + \frac{4m}{(s+1)^2}\right]$$

$$H(s) = \frac{Pc}{s+1}\left[1 - me^{-2i\theta}\left\{3 - \frac{6}{s+1} + \frac{4}{(s+1)^2}\right\}\right]$$

$$L(s) = -\frac{Pcm}{s+1}\left[e^{2i\varphi} + \frac{2e^{2i\theta}}{s+1} + \frac{12m}{(s+1)^2} - \frac{24m}{(s+1)^3}\right]$$

$$\alpha_{111} = -Pcm\beta,\ \beta = \frac{\cos 2\theta}{1-m} + i\,\frac{\sin 2\theta}{1+m}.$$

With these values of $G(s)$, $H(s)$, $L(s)$ and α_{111} inserted in (3.5), the final closed expressions for $\Phi(s)$ and $\Psi(s)$ are

$$\Phi(s) = \frac{Pc}{s+1}\left[3m - \beta - \frac{6m}{s+1} + \frac{4m}{(s+1)^2}\right]$$

$$\Psi(s) = \frac{Pc}{s+1}\left[1 - m\beta - 3me^{-2i\theta} + 2m\,\frac{3e^{-2i\theta} - \beta}{s+1} - 4m\,\frac{3m + e^{-2i\theta}}{(s+1)^2} + \frac{24m^2}{(s+1)^3}\right]. \qquad \text{...(3.6)}$$

For $m = \frac{1}{3}$, we get the mapping function

$$z = c\left[\frac{s+1}{s-1} + \frac{1}{3}\left(\frac{s-1}{s-1}\right)^3\right],\ c > 0$$

and the corresponding formulae for $\Phi(s)$ and $\Psi(s)$ become

$$\Phi(s) = \frac{Pc}{s+1}\left[1 - \alpha - \frac{2}{s+1} + \frac{4}{3(s+1)^2}\right]$$

$$\Psi(s) = \frac{Pc}{s+1}\left[1 - \tfrac{1}{3}\alpha - e^{-2i\theta} + \frac{2(3e^{-2i\theta} - \alpha)}{3(s+1)} - \frac{4(1+e^{-2i\theta})}{3(s+1)^2} + \frac{8}{3(s+1)^3}\right]$$

$$\alpha = \tfrac{3}{4}(2\cos 2\theta + i \sin 2\theta)$$

which are in agreement with Belonosov's result (1962) obtained for the hypotrochoidal hole with four cusps.

Case (*b*) : $k = 4$ — We get the transformation (see Fig. 2)

$$z = w(s) = c\left[\frac{s+1}{s-1} + m\left(\frac{s-1}{s+1}\right)^4\right],\quad c > 0,\ |m| \leqslant \tfrac{1}{4} \qquad \text{...(3.7)}$$

and the formula (2.2) simplifies to

$$\frac{\overline{w(i\tau)}}{w'(i\tau)} = -\frac{16m}{i\tau - 1} - \frac{8m}{(i\tau-1)^2} + B(i\tau)$$

where

$$B(s) = \frac{1}{2(s-1)^2}\left\{\frac{(1-s)^5 - m(1+s)^5}{(1+s)^5 + 4m(1-s)^5}(1+s)^4 + 16m(2s-1)\right\}. \qquad \text{...(3.8)}$$

Following the usual procedure of determining the constants α_{111}, α_{121} and α_{221} involved in (2.10) with $n = 1$, $r_1 = 2$, it is found after considerable calculations that

$$\Phi(s) = G(s) + \frac{\bar{\alpha}_{111} + \bar{\alpha}_{121}}{s+1} - \frac{\bar{\alpha}_{212}}{(s+1)^2}, \qquad \text{...(3.9)}$$

$$\Psi(s) = H(s) + L(s) + 2m(\bar{\alpha}_{111} + \bar{\alpha}_{121})\left[\frac{1}{s+1} + \frac{3}{(s+1)^2}\right] - m\bar{\alpha}_{221}\left[\frac{1}{s+1} + \frac{4}{(s+1)^2} + \frac{12}{(s+1)^3}\right], \qquad \text{...(3.10)}$$

where

$$G(s) = \frac{Pc}{s+1}\left[4m - e^{2i\theta} - \frac{12m}{s+1} + \frac{16m}{(s+1)^2} - \frac{8m}{(s+1)^3}\right],$$

$$H(s) = \frac{Pc}{s+1}\left[1 - 4me^{-2i\theta}\left\{1 - \frac{3}{s+1} + \frac{4}{(s+1)^2} - \frac{2}{(s+1)^3}\right\}\right],$$

$$L(s) = -\frac{2mcP}{s+1}\left[e^{2i\theta} + \frac{3e^{2i\theta}}{s+1} + \frac{24m}{(s+1)^2} - \frac{36m}{(s+1)^3}\right]$$

$$\alpha_{111} = 2\alpha_{221} = -\frac{4Pcm}{1-2m^2}e^{2i\theta}$$

$$\alpha_{121} = \frac{2Pcm}{1-2m^2}(e^{2i\theta} - me^{-2i\theta}).$$

Inserting these values in (2.5) the functions $\phi_0(z)$ and $\psi_0(z)$ are given by

$$\phi(z) = \Phi(s) = \frac{Pc}{s+1}\left[4m - \frac{e^{2i\theta} + 2me^{-2i\theta}}{1-2m^2} + \frac{2m}{s+1}\left(\frac{e^{-2i\theta}}{1-2m^2} - 6\right) + \frac{16m}{(s+1)^2} - \frac{8m}{(s+1)^3}\right] \quad \ldots(3.11)$$

$$\psi_0(z) = \Psi(s) - B(s)\,\Phi'(s) - \tfrac{1}{2}Pc(1+m)(1-e^{-2i\theta}) \quad \ldots(3.12)$$

where

$$\Psi(s) = \frac{Pc}{s+1}\left[1 + \frac{2m^2}{1-2m^2}e^{-2i\theta} - \frac{2m}{1-2m^2} \times \left(e^{2i\theta} + 2m\frac{s-1}{s+1}e^{-2i\theta}\right)\frac{s+4}{s+1} - \frac{4m}{(s+1)^3}\{6m(2s-1) + s(s^2+1)e^{-2i\theta}\}\right]. \quad \ldots(3.13)$$

For $m = 0$, we have the case of an infinite plate weakened by a circular hole $|z| = c$, and (3.11), (3.13) reduce to

$$\Phi(s) = -\frac{Pc}{s+1}e^{2i\theta}, \quad \Psi(s) = \frac{Pc}{s+1}$$

which coincide with (21) of El-Sirafy (1977).

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