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THE FREDHOLM INDEX OF A CLASS OF VECTOR-VALUED SINGULAR INTEGRAL OPERATORS

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In this paper the operators of interest are the singular integral operators T defined on $L_n^2(E)$ by

$$Tf(s) = sf(s) + \frac{B^*(s)}{\pi} \int_E \frac{B(t)f(t)}{s-t}\, dt.$$

In particular it is shown that a real number x is in the essential resolvent of T if and only if there is a neighbourhood $\triangle$ of x such that the Lebesgue measure of $\triangle - E$ equals zero, and $\operatorname*{ess\,inf}_{t \in \triangle} |\det B(t)| > 0$. Moreover, in this case the index of $T - x$ is $-n$.

INTRODUCTION

Let E be a bounded measurable subset of the real line R, and $B \in L^\infty_{M_n}(E)$, where the space $L^\infty_{M_n}(E)$ is the set of all $n \times n$ matrices (ϕ_{ij}) $(1 \leqslant i, j \leqslant n)$, such that each of the functions $\phi_{ij} \in L^\infty(E)$. The notation $L_n^2(E)$ will denote the usual Lebesgue space of $\mathbb{C}^n$ valued square integrable functions on E. The operators of interest are the singular integral operators T defined on $L_n^2(E)$ by

$$Tf(s) = sf(s) + \frac{B^*(s)}{\pi} \int_E \frac{B(t)f(t)}{s-t}\, dt.$$

The singular integral operator T is hyponormal on $L_n^2(E)$, that is, the self-commutator $[T^*, T] = T^*T - TT^*$ of T is a non-negative operator, moreover $[T^*, T]$ is n-dimensional. It should be remarked that a complete description of the Fredholm behaviour of the operator $T - z$, for z complex, has been given for the case $n = 1$ (see Clancey 1974a). In this paper a characterization is given for the Fredholm behaviour of $T - x$, where $x \in R$ and $n > 1$.

Section 1 is concerned with some technical machinery needed for this paper. In section 2 we prove that $T - x$ is Fredholm if and only if there exists a neighbourhood $\triangle$ of x such that the Lebesgue measure of $\triangle - E$ equals zero. In section 3, the definition of the principal function and some of its properties are given. In section 4, we prove that index of $T - x = -n$.

1. Preliminaries

Let z be any non-real complex number and let the function f be an element of $L_n^2(R)$. The Cauchy transform of f, denoted by Cf, is equal to $(Cf_j)_{j=1}^n$ where

$$Cf_j(z) = \frac{1}{2\pi i}\int_R \frac{f_j(t)}{t-z}\,dt, \text{ for } 1 \leqslant j \leqslant n. \qquad \text{...(1.1)}$$

Each function Cf_j ($1 \leqslant j \leqslant n$) is separately holomorphic in the upper half-plane and lower half-plane. Moreover, the functions $f_j^{\pm}$ defined by $f_j^{\pm}(x) = \lim_{y \to 0\pm} Cf_j(x \pm iy)$ exist almost everywhere. The function f_j^{+} ($1 \leqslant j \leqslant n$) is in H^2 and f_j^{-} is in $\bar{H}^2$, where the space H^2 is the usual Hardy space, and $\bar{H}^2$ is its complex conjugate.

For $f \in L_n^2(R)$, write $f^{\pm}$ for $(f_j^{\pm})_{j=1}^n$. The functions $f^{\pm}$ satisfy the Plemelj identities

$$f^+ - f^- = f,\ f^+ + f^- = (1/i)\,Hf. \qquad \text{...(1.2)}$$

Here, $Hf = (Hf_j)_{j=1}^n$ is the Hilbert transform of f. The map $f \to Hf$ is a bounded linear operator on $L_n^2(R)$ and this implies that the maps $f \to f^{\pm}$ are bounded in $L_n^2(R)$. For the boundedness of the Hilbert transform see Neri (1971).

The Riemann-Hilbert barrier operator with symbol $G \in L_{M_n}^{\infty}(R)$ is the bounded linear operator B_G defined on the space $L_n^2(R)$ by

$$B_G f = Gf^+ - f^-. \qquad \text{...(1.3)}$$

The Toeplitz operator with symbol G is the bounded operator T_G on H_n^2 defined by

$$T_G f = P(Gf). \qquad \text{...(1.4)}$$

Here P stands for the orthogonal projection of $L_n^2(R)$ onto H_n^2. The operator P is given by

$$Pf = \tfrac{1}{2}(f - iHf). \qquad \text{...(1.5)}$$

It is known that T_G is Fredholm if and only if B_G is Fredholm. Moreover, they have the same Fredholm index.

Let H be a Hlibert space and let $L(H)$ denote the algebra of all bounded operators on $L(H)$. An operator T in $L(H)$ is called Fredholm in case T has closed range, and dim (ker T), dim (ker T^*) are finite. The index of a Fredholm operator T is defined by

$$\text{ind}(T) = \dim(\ker T) - \dim(\ker T^*). \qquad \ldots(1.6)$$

The essential spectrum of T, denoted by $\sigma_e(T)$, is the set of all λ in the field of complex numbers $\mathbb{C}$ such that $T - \lambda$ is not Fredholm. The essential resolvent of T, denoted by $\rho_e(T)$, is the set of all $\lambda \in \mathbb{C}$, such that $\lambda \notin \sigma_e(T)$.

Let $G \in L^{\infty}_{M_n}(R)$. For λ in R the cluster set of G at λ, denoted by $C(G : \lambda)$, is the set of all $n \times n$ matrices M such that the set

$$\{t \in R : \| G(t) - M \| < \epsilon\} \cap N$$

has positive measure for every $\epsilon > 0$ and every neighbourhood N of λ.

2. The Essential Spectrum

Let E be a bounded measurable subset of the real line R, and $B \in L^{\infty}_{M_n}(E)$ such that

$$\operatorname*{ess\,inf}_{t \in E} |\det B(t)| > 0. \qquad \ldots(2.1)$$

The operators of interest are the singular integral operators T defined on $L^2_n(E)$ by

$$Tf(s) = sf(s) + \frac{B^*(s)}{\pi} \int_E \frac{B(t)f(t)}{s-t}\, dt. \qquad \ldots(2.2)$$

Since $(T^*T - TT^*)f(s) = \frac{2}{\pi} B^*(s) \int_E B(t)f(t)\, dt$, it follows that the operator T is hyponormal with n-dimensional self-commutator.

Note that with the hypothesis (2.1), the operator T defined in (2.2) is similar to the operator S defined on $L^2_n(E)$ by

$$Sf(s) = sf(s) + \frac{B(s)B^*(s)}{\pi} \int_E \frac{f(t)}{s-t}\, dt. \qquad \ldots(2.3)$$

Let x be a real number. From (2.1) we can say that

$$\operatorname*{ess\,inf}_{s \in E} |\det((s-x)I + i(B(s)B^*(s))| > 0. \qquad \ldots(2.4)$$

The symbol of the operator $S - x$ is the function

$$G_x(s) = \begin{cases} \dfrac{(s-x)\,I - i\,BB^*(s)}{(s-x)\,I + i\,BB^*(s)}, & s \in E \\ I, \quad s \notin E \end{cases} \qquad \ldots(2.5)$$

where I is the $n \times n$ identity matrix.

Before proving the main theorem of this section some technical lemmas are needed.

Lemma 2.1 — If $G \in L^{\infty}_{M_n}(R)$ and for some $\epsilon > 0$, $\operatorname{Re} G(t) \geqslant \epsilon I$ almost everywhere, then the Riemann-Hilbert barrier operator B_G is invertible.

PROOF: For proof of the lemma see Clancey (1974b).

There are localization techniques due to Simonenko that will be useful in establishing when certain Riemann-Hilbert barrier operators are Fredholm.

Let G_1 and G_2 be symbols of the Riemann-Hilbert barrier operators B_{G_1} and B_{G_2}, respectively. Let x_0 be a fixed real number. The operators B_{G_1} and B_{G_2} are said to be locally equal at x_0 in case there is a neighbourhood U of x_0 such that $G_1(t) = G_2(t)$, for all $t \in U$. The operator B_{G_1} is said to be locally Fredholm at $x_0 \in R$ in case B_{G_1} is locally equal to a Fredholm barrier operator B_{G_2} at x_0.

The following lemma is a special case of a result of Simonenko (1964).

Lemma 2.2 — Let $G \in L^{\infty}_{M_n}(R)$. If the Riemann-Hilbert barrier operator B_G, acting on $L^2_n(R)$, is locally Fredholm at each $x \in R$, then B_G is a Fredholm operator.

The Riemann-Hilbert barrier operators B_{G_1} and B_{G_2} are said to be locally equivalent at $x_0 \in R$, if for every $\epsilon > 0$, there exists a neighbourhood of the point x_0 such that

$$|||\,(B_{G_1} - B_{G_2})\,P_U\,||| = \inf \|\,(B_{G_1} - B_{G_2})\,P_U - K\,\| < \epsilon \qquad \ldots(2.6)$$

where K runs through the ideal of compact operators on $L^2_n(R)$, and the operator P_U is defined on $L^2_n(R)$ by

$$P_U\,f(x) = \chi_U\,f(x) \qquad \ldots(2.7)$$

where χ is the characteristic function of U. The following lemma also appears in Simonenko (1964).

Lemma 2.3 — Let G_1 and G_2 be in $L^{\infty}_{M_n}(R)$ and suppose B_{G_1} and B_{G_2} are locally equivalent at x_0 in R. Then B_{G_1} is locally Fredholm at x_0 if and only if B_{G_2} is locally Fredholm at x_0.

The main theorem of this section is the following:

Theorem 2.1 — If x is a real number, then x is an element of the essential resolvent of S if and only if there exists a neighbourhood $\triangle$ of x such that the Lebesgue measure of $\triangle - E$ equals zero.

PROOF : Let G_x be the symbol of the operator $S - x$ defined by (2.5). Since it is known that $S - x$ is Fredholm if and only if the Riemann-Hilbert barrier operator B_{G_x} is Fredholm, it suffices to prove Theorem 2.1 for the operator B_{G_x}.

Suppose that there is no neighbourhood $\triangle$ of x such that the Lebesgue measure of $\triangle - E$ equals zero. Then it follows that the cluster set $C(G_x : x)$ of G_x at x is $\{I, -I\}$. Since there exists a $0 \leqslant \lambda \leqslant 1$ ($\lambda = \frac{1}{2}$) such that

$$\det((1 - \lambda) I + \lambda(-I)) = 0$$

it follows by a result of Clancey (1974b, Theorem 3.2) that the operator B_{G_x} is not Fredholm.

Suppose for some neighbourhood $\triangle$ of x the Lebesgue measure of $\triangle - E$ equals zero. Since $C(G_x : x) = \{-I\}$, it follows that B_{G_x} and B_{-I} are locally equivalent at x. By Lemma 2.3, B_{G_x} is locally Fredholm at x. Let $t_0 \in R$ and suppose that $t_0 < x$ (a similar argument handles the case in which $t_0 \geqslant x$). It is clear that

$$C(G_x : t_0) \subseteq \{I\} \cup \left\{\frac{(t_0 - x)I - iA}{(t_0 - x)I + iA} : A \in C(BB^* : t_0)\right\}. \qquad \text{...(2.8)}$$

Fix such an A, and let

$$C = \frac{(t_0 - x)I - iA}{(t_0 - x)I + iA}. \qquad \text{...(2.9)}$$

It can be concluded from (2.1) and (2.9) that there is a $\beta > 0$ (independent of the choice of A in $C(BB^* : t_0)$) such that $\operatorname{Im}(C) > \beta I$. Let $M \in C(-i\,G_x : t_0)$, then $M = -iI$ or $\operatorname{Re} M > \alpha I$. For θ small, there is a $\alpha' = \alpha(\theta)$ such that $\operatorname{Re} Q > \alpha' I$, where $Q \in C(-ie^{i\theta}\,G_x : t_0)$. Fix such a θ. Let $0 < \epsilon < \frac{1}{2}\alpha'$, and choose a neighbourhood N_ϵ of t_0 such that for $t \in N_\epsilon(t_0)$,

$$\operatorname{dist}(-ie^{i\theta}\,G_x(t),\ C(-ie^{i\theta}\,G_x : t_0)) < \epsilon. \qquad \text{...(2.10)}$$

From above it follows that $\operatorname{Re}(-ie^{i\theta}\,G_x(t)) \geqslant \frac{1}{2}\alpha' I$ for $t \in N_\epsilon(t_0)$. Define the function $H(t)$ as follows:

$$H(t) = \begin{cases} -ie^{i\theta}\,G_x(t), & t \in N_\epsilon(t_0) \\ Q_0, & t \notin N_\epsilon(t_0) \end{cases} \qquad \text{...(2.11)}$$

where Q_0 is any fixed element in $C(-ie^{i\theta} G_x : t_0)$. Since $H \in L^{\infty}_{M_n}(R)$ and $\operatorname{Re} H(t) > \frac{1}{2}\alpha' I$, by Lemma 2.1 the Riemann-Hilbert barrier operator B_H on $L^2_n(R)$ is Fredholm. Since $H(t)$ is equal to $-ie^{i\theta} G_x(t)$ for $t \in N_\epsilon(t_0)$ it follows that B_{G_x} is locally Fredholm at t_0. Therefore, B_{G_x} is locally Fredholm at every $t \in R$. Hence, by Lemma 2.2, the operator B_{G_x} is Fredholm, and this ends the proof of the theorem.

Corollary 2.1 — Let T be the singular integral operator defined by (2.2). Then a real number x is in the essential resolvent of T if and only if there exists a neighbourhood $\triangle$ of x such that the Lebesgue measure of $\triangle - E$ equals zero.

PROOF : The operators S and T are similar.

3. The Principal Function

In order to show that the Fredholm index of $T - x$ is $-n$, some properties of the principal function of T are needed. For the sake of completeness the definition of the principal function of T and some of its basic properties are presented in this section.

Let H be a separable complex Hilbert space. An operator J on H is said to be completely non-normal if there exists no reducing sub-space of J on which J is normal. The operator J is a trace class operator if $\sum_j ((J^*J)^{1/2} \phi_i, \phi_i) < \infty$ for an orthonormal basis $\{\phi_i\}$ of H. The trace of the operator J, denoted by

$$tr(J) = \sum_j (J\phi_i, \phi_i).$$

Let $A = X + iY$ be the Cartesian decomposition of a bounded operator on H, such that A is completely non-normal hyponormal operator with trace class self-commutator. Helton and Howe (1973) have associated with the operator A, a set function $\tilde{\mu}$ defined on the collection σ of semi-closed rectangles in the following manner:

Let $\alpha = [a, b)$ and $\beta = [c, d)$ be half-open intervals such that $\alpha \times \beta \in \sigma$. Denote by $\int_R \lambda \, dE(\lambda)$, and $\int_R \lambda \, dF(\lambda)$ the spectral resolutions of the operators X and $E(\alpha)Y E(\alpha)$, respectively. The set function $\tilde{\mu}$ is defined on $\alpha \times \beta$ by

$$\tilde{\mu}(\alpha \times \beta) = tr(E(\alpha)\, F(\beta)\, E(\alpha)\, [A^*, A]). \qquad \text{...(3.1)}$$

These authors established that $\tilde{\mu}$ extends to a non-negative regular Borel measure μ of bounded total variation on the plane. Pincus (1979) (see also Carey and Pincus 1974) has established that μ is absolutely continuous with respect to planar Lebesgue measure. The derivative

$$g = \frac{\pi \, d\mu}{d \times dy} \qquad \text{...(3.2)}$$

is called the principal function of the operator A. The following is a summary of some of the principal functions associated with the operator A.

(i) On a component of the complement of the essential spectrum of A, $g(\lambda) = \text{ind}\,(A - \lambda)$. For a proof, see Helton and Howe (1973).

We will need some further notation before describing the next property of the principal function. Let $\int_R \lambda \, dE\,(\lambda)$ be the spectral resolution of X, and let $\triangle$ be a Borel set in the real line R. Denote the Hilbert space $E(\triangle)H$ by $H_\triangle$. The operator $A_\triangle$ on $H_\triangle$ is defined by $A_\triangle f = E(\triangle)Af$. It is known that $A_\triangle$ is completely non-normal hyponormal operator with trace class self-commutator. Let $g_\triangle$ be the principal function of $A_\triangle$.

(ii) The principal functions g and $g_\triangle$ of the operators A and $A_\triangle$ are related by

$$g_\triangle = g\chi_{\triangle \times R} \qquad \text{...(3.3)}$$

here, $\chi_{\triangle \times R}$ denotes the characteristic function of $\triangle \times R$. For a proof, see Carey and Pincus (1977).

4. The Fredholm Index

Let $x \in R$ be in the essential resolvent of T, and in the spectrum of T, where the operator T has been defined by (2.2). In this section it is shown that

$$\text{ind}\,(T - x) = -n.$$

Note that $C(C_x : x) = \{-I\}$. Choose a small open ball D centered at $-I$ so that D is contained in $Gl(n : ₵)$. From this, it follows that there exists a small neighbourhood $\triangle = (x - \xi, x + \xi)$ contained in E such that the closed convex hull of the essential range of G_x restricted to $\triangle$ is contained in D. Define the function $G_x^\triangle$ as follows:

$$G_x^\triangle(s) = \begin{cases} \dfrac{(s-x)I - iB(s)B^*(s)}{(s-x)I + iB(s)B^*(s)}, & s \in \triangle \\ I, & s \notin \triangle. \end{cases} \qquad \text{...(4.1)}$$

This is the symbol of the singular integral operator $T_\triangle - x$ defined on $L_n^2(\triangle)$.

Theorem 4.1 — The operator $T - x$ is Fredholm if and only if the operator $T_\Delta - x$ is Fredholm. Moreover,

$$\text{ind}\,(T_\Delta - x) = \text{ind}\,(T - x).$$

PROOF: Putnam (1970) has shown that x is in the essential resolvent of T if and only if x is in the essential resolvent of T_Δ. Also, using properties (i) and (ii) of the principal function, it follows easily that ind $(T - x) = \text{ind}\,(T_\Delta - x)$, and this ends the proof.

From Theorem 4.1, it suffices to show that ind $(T_\Delta - x) = -n$. Since $T_\Delta - x$ is Fredholm if and only if $B_{G_x^\Delta}$ is Fredholm, and ind $(T_\Delta - x) = \text{ind}\,(B_{G_x^\Delta})$, it suffices to show that ind $(B_{G_x^\Delta}) = -n$. To establish our result, we proceed as follows:

Let η be a function from $[x - \xi, x + \xi]$ into $[0, 1]$ such that η is continuous, $\eta(x - \xi) = 0$ and $\eta(x + \xi) = 1$. Define the function $\tilde{G}$ as follows:

$$\tilde{G}(t) = \begin{cases} (1 - \eta(t))\, U^+ + \eta(t)\, U^-, & t \in \Delta \\ I, & t \notin \Delta \end{cases} \qquad \ldots(4.2)$$

where $U^\pm$ are different from the identity $n \times n$ matrix I, and $U^\pm \in C(G_x^\Delta : x \mp \xi)$. Any element in $C(G_x^\Delta : x \mp \xi)$ is either the matrix I, or a matrix of the form

$$((s - x)\, I + iA^\pm)^{-1}\, ((s - x)\, I - iA^\pm) \qquad \ldots(4.3)$$

where A^+ is a cluster value from the right of BB^* at $x - \xi$, and A^- is a cluster value from the left of BB^* at $x + \xi$. Note that $U^\pm$ are unitary. The function $\tilde{G}$ is piecewise continuous, and it is clear from (Theorem 2.1, see Clancey 1974b), that the Riemann-Hilbert barrier operator $B_{\tilde{G}}$ is a Fredholm operator on $L_n^2(R)$.

For $0 \leqslant s \leqslant 1$ and t a real number, the function H will be defined in the following way:

$$H(s, t) = \begin{cases} (1 - s)\, G_x^\Delta(t) + s\, \tilde{G}(t), & t \in \Delta \\ I, & t \notin \Delta. \end{cases} \qquad \ldots(4.4)$$

Theorem 4.2 — The Riemann-Hilbert barrier operator $B_{H(s,\cdot)}$ is a Fredholm operator on $L_n^2(R)$, and

$$\text{ind}\,(B_{G_x^\Delta}) = \text{ind}\,(B_{\tilde{G}}).$$

PROOF : It has already been observed that the operators $B_{H(0,.)} = B_{G_x^\Delta}$, and $B_{H(1,.)} = B_{\tilde{G}}$ are Fredholm operators.

Let s be a fixed real number between 0 and 1. If $t_0 \notin \bar{\Delta}$, then the Riemann Hilbert barrier operator $B_{H(s,.)}$ is locally equal to the Fredholm Riemann-Hilbert barrier operator B_I at t_0. All that has to be shown now is that $B_{H(s,.)}$ is locally Fredholm at t_0 in $\bar{\Delta}$.

For $t = x$, the Riemann-Hilbert barrier operator $B_{H(s,.)}$ is locally equivalent to the Riemann-Hilbert barrier operator with symbol $(1 - s)(-I) + s\,\tilde{G}(x)$, which is an element of D. It follows by Lemma 2.3 that $B_{H(s,.)}$ is locally Fredholm at $t = x$. If $t \in \bar{\Delta}$, t is different from x, then to show that the operator $B_{H(s,.)}$ is locally Fredholm at t, we follow the argument of Theorem 2.1. For s fixed the operator $B_{H(s,.)}$ is locally Fredholm at every $t \in R$. Hence, by Lemma 2.2, the operator $B_{H(s,.)}$ is Fredholm.

Since the function $s \to H(s,.)$ from $[0, 1]$ to $L^\infty_{M_n}(R)$ is continuous, it follows that $\text{ind}\,(B_{H(0,.)}) = \text{ind}\,(B_{H(1,.)})$, in other words $\text{ind}\,(B_{G_x^\Delta}) = \text{ind}\,(B_{\tilde{G}})$, and this ends the proof,

It is known that the Toeplitz operator $T_{\tilde{G}}$ is Fredholm if and only if the Riemann-Hilbert barrier operator $B_{\tilde{G}}$ is Fredholm, moreover they have the same Fredholm index. Gohberg and Krupnik (1968) have shown that the Fredholm index of $T_{\tilde{G}}$ is equal to the negative of the winding number of $\det(\tilde{G}^*)$ where $\tilde{G}^*$ is the curve obtained from the piecewise continuous function $\tilde{G}$ by joining the left and right hand limits by a line segment at points of discontinuity. From above it suffices to show that the winding number of $\det(\tilde{G}^*)$ around zero is equal to n. In order to show that the winding number of $\det(\tilde{G}^*)$ is equal to n, it is needed to be shown that the jump in the argument of $\det(\tilde{G}^*)$ on sufficiently small Δ is close to n.

Let $U^\pm$ be the unitary $n \times n$ matrices defined by (4.3). Since A^+ is positive, the eigenvalues $e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n}$, $0 \leqslant \theta_j < 2\pi$, $1 \leqslant j \leqslant n$ of U^+ are in the upper half-plane and close to -1. Similarly, the eigenvalues $e^{i\phi_1}, \ldots, e^{i\phi_n}$, $1 \leqslant j \leqslant n$, of U^- are in the lower half-plane and close to -1. Since the matrices $U^\pm$ are unitary, it follows that there exist unitary matrices $V_\pm$, such that

$$D^{+} = V_{+}\, U^{+}\, V_{+}^{*} = \begin{bmatrix} e^{i\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\theta_n} \end{bmatrix},$$

and

$$D^{-} = V_{-}\, U^{-}\, V_{-}^{*} = \begin{bmatrix} e^{i\phi_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\phi_n} \end{bmatrix}.$$

Since $\tilde{G}(t)$ was defined in (4.2) by

$$\tilde{G}(t) = (1 - \eta(t))\, U^{+} + \eta(t)\, U^{-}, \text{ for } t \in \Delta$$

it follows that

$$\begin{aligned} \det \tilde{G}(t) &= \det ((1 - \eta(t))\, V_{+}^{*} D^{+} V_{+} + \eta(t)\, V_{-}^{*} D^{-} V_{-}) \\ &= \det (V_{+}^{*} V_{-}) \,.\, \det ((1 - \eta(t))\, D^{+} W + \eta(t)\, W D^{-}) \end{aligned}$$

where W is the unitary matrix $V_{+} V_{-}^{*}$.

From the definition of the determinant, we see that $\det ((1 - \eta(t))\, D^{+} W + \eta(t) W D^{-}$ can be written as follows:

$$\sum_{\sigma \in P} (-1)^{\epsilon(\sigma)} \prod_{i=1}^{n} ((1 - \eta(t))\, w_{i\sigma(i)}\, e^{i\theta_i} + \eta(t)\, w_{i\sigma(i)}\, e^{i\phi_{\sigma(i)}} \qquad \ldots(4.5)$$

where P is the set of all permutations on $\{1, 2, \ldots, n\}$, and $\epsilon(\sigma)$ is the sign of the permutation. The last expression (4.5) can be written in the following form

$$\begin{aligned} &\prod_{i=1}^{n} ((1 - \eta(t))\, w_{ii}\, e^{i\theta_i} + \eta(t)\, w_{ii}\, e^{i\phi_i}) \\ &+ \sum_{P-\{\tau\}} (-1)^{\epsilon(\sigma)} \prod_{i=1}^{n} ((1 - \eta(t))\, w_{i\sigma(i)}\, e^{i\theta_i} + \eta(t)\, w_{i\sigma(i)}\, e^{i\phi_{\sigma(i)}} \\ &+ \sum_{P-\{\tau\}} (-1)^{\epsilon(\sigma)} \prod_{i=1}^{n} ((1 - \eta(t))\, w_{i\sigma(i)}\, e^{i\theta_i} + \eta(t)\, w_{i\sigma(i)}\, e^{i\phi_i} \\ &- \sum_{P-\{\tau\}} (-1)^{\epsilon(\sigma)} \prod_{i=1}^{n} ((1 - \eta(t))\, w_{i\sigma(i)}\, e^{i\theta_i} + \eta(t)\, w_{i\sigma(i)}\, e^{i\phi_i}), \end{aligned}$$

where τ is the identity permutation. By adding the first term to the third, the following expression is obtained,

$$\det W \prod_{i=1}^{n} ((1 - \eta(t)) e^{i\theta_i} + \eta(t) e^{i\phi_i}$$

$$+ \sum_{P-\{\tau\}} (-1)^{\epsilon(\sigma)} \prod_{i=1}^{n} ((1 - \eta(t)) w_{i\sigma(i)} e^{i\theta_i} + \eta(t) w_{i\sigma(i)} e^{i\phi_{\sigma(i)}})$$

$$- \sum_{P-\{\tau\}} (-1)^{\epsilon(\sigma)} \prod_{i=1}^{n} ((1 - \eta(t)) w_{i\sigma(i)} e^{i\theta_i} + \eta(t) w_{i\sigma(i)} e^{i\phi_i}).$$

Since $e^{i\phi_i}$ can be chosen arbitrarily close to -1, the difference of the second and third terms in the last expression does not contribute significantly to the jump in the argument of $\det ((1 - \eta(t)) D^+W + \eta(t) WD^-)$ on $\triangle$. The jump of the argument of

$$\det W \prod_{i=1}^{n} ((1 - \eta(t)) e^{i\theta_i} + \eta(t) e^{i\phi_i})$$

can be made arbitrarily close to n by taking $\triangle$ sufficiently small. So we have established that the winding number of $\det (\tilde{G}^*)$ is n. The main result of this section is the following theorem.

Theorem 4.3 — Let T be the singular integral operator defined by (2.2), and assume that x is a real number such that some interval $\triangle$ containing x satisfies

$$\text{(i) } \triangle \subset E \text{ and (ii) } \operatorname*{ess\,inf}_{t \in \triangle} | \det B(t) | > 0.$$

The operator $T - x$ is Fredholm, and $\operatorname{ind} (T - x) = -n$.

Proof: The fact that $T - x$ is Fredholm is a direct consequence of Theorem 2.1. The arguments given above established that if $\triangle$ is a sufficiently small neighbourhood of x, then the operator $T_\triangle - x$ has index $-n$. By Theorem 4.1, we conclude that

$$\operatorname{ind} (T - x) = \operatorname{ind} (T_\triangle - x) = -n$$

and that ends the proof.

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