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A NOTE ON THE STEIN-WATERMAN SEQUENCES

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The present note describes a method of finding the values of $S_n(m)$ for any given $m \geqslant 0$ directly.

1. INTRODUCTION

Stein and Waterman (1978) have recently introduced a new sequence $\{S_n(m), n = 0, 1, 2, 3, \ldots; m \text{ an integer} \geqslant 0\}$ of integers which generalizes the Catalan and Motzkin numbers. We can conveniently define it by the relations:

$$S_n = S_n(m) = 0 \text{ for each } n < m;$$
$$= 1 \text{ for } n = m;$$

and for $n > m$,

$$S_n = \sum_{j=n-m-1}^{n-1} S_j + \sum_{j=m}^{n-m-2} S_j S_{n-j-2}. \qquad \ldots(1)$$

The generating function of S_n is defined as usual by the relation

$$y = \sum_{n \geqslant m} S_n x^n. \qquad \ldots(2)$$

We write

$$A_n \text{ for } \sum_{j=n-m-1}^{n-1} S_j; \text{ and } B_n \text{ for } \sum_{j=m}^{n-m-2} S_j S_{n-j-2}. \qquad \ldots(3)$$

The object of this note is to describe a method of finding the values of S_n for any given $m \geqslant 0$ directly. Stein and Waterman have made use of generalized Fibonacci numbers and the Catalan numbers for the purpose.

2. THE FUNCTIONAL EQUATION

It would be readily seen that $S_n - A_n$ is the coefficient of x^n in the expansion of

$$(1 - x - x^2 - \ldots - x^{m+1})\, y - x^m$$

in ascending powers of x.

On the other hand, B_n is the coefficient of x^n in $(xy)^2$. Hence

$$(xy)^2 = (1 - x - x^2 - \ldots - x^{m+1})\, y - x^m.$$

This is the sams as

$$(1 - x)(xy)^2 = (1 - 2x + x^{m+2})\,y - x^m(1 - x). \qquad \ldots(4)$$

Now (4) is a quadratic in y and can be solved in the usual way. The discriminant of (4) is given by

$$\begin{aligned} D_m &= (1 - 2x + x^{m+2})^2 - 4x^{m+2}(1 - x) \\ &= (1 - 2x - x^{m+2})^2 - 4x^{m+4}. \end{aligned} \qquad \ldots(5)$$

What we need is the square-root of D_m. We can assume that

$$D_m^{1/2} = 1 - 2x - x^{m+2} - 2x^{m+4} - \sum_{j \geqslant m+5} a_j x^j. \qquad \ldots(6)$$

The constants a_j can be determined by comparing the coefficients in the square of the expression on the right of (6) with those in (5). Thus for $m = 3$, we have

$$\begin{aligned} D_m^{1/2} = 1 &- 2x - x^5 - 2x^7 - 4x^8 - 8x^9 - 16x^{10} - 32x^{11} - 66x^{12} \\ &- 136x^{13} - 282x^{14} - 588x^{15} - 1232x^{16} - \ldots . \end{aligned}$$

This gives

$$\begin{aligned} (1 - x)\,y &= \{(1 - 2x + x^5) - D_m^{1/2}\}/2x^2 \\ &= x^3 + 0x^4 + x^5 + 2x^6 + 4x^7 + 8x^8 + 16x^9 + 33x^{10} \\ &\quad + 68x^{11} + 141x^{12} + 294x^{13} + 616x^{14} + \ldots . \end{aligned}$$

Multiplying on the left by $(1 - x)^{-1}$ and by $(1 + x + x^2 + x^3 + \ldots)$ on the right, we get

$$\begin{aligned} y = x^3 &+ x^4 + 2x^5 + 4x^6 + 8x^7 + 16x^8 + 32x^9 + 65x^{10} + 133x^{11} \\ &+ 274x^{12} + 568x^{13} + 1184x^{14} + \ldots . \end{aligned}$$

Reference

Stein, P. R., and Waterman, M. S. (1978). On some new sequences generalizing the Catalan and Motzkin numbers. *Discrete Math.*, **26**, 261–72.