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BUCKLING AND VIBRATIONS OF CIRCULAR ANNULAR PLATES OF PARABOLICALLY VARYING THICKNESS

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The natural frequencies of a circular annular plate of parabolically varying thickness under the action of a hydrostatic in-plane force have been studied on the basis of the classical theory of plates. The governing differential equation of motion has been solved by Chebyshev collocation technique. The effect of in-plane force on the frequencies of vibration has been determined for two different boundary conditions for different radii ratio and taper constants. Transverse displacements, moments and the critical buckling loads in compression with thickness variation have also been computed for the first two modes.

1. INTRODUCTION

In the past years there has been growing interest in the study of buckling and vibration of plates of non-uniform thickness because of their applications in various engineering structures. An excellent survey of previous studies has been made by Leissa (1969). Circular annular plates are extensively used as structural elements in the construction of aircrafts, ships, automobiles and other vehicles. Annular plates used in naval and aerospace structures are often subjected to in-plane forces. Recently Soni and Amba-Rao (1975) studied the axisymmetric vibrations of annular plates of variable thickness. Rosen and Libai (1975) analysed the transverse vibrations of uniformly compressed annular plate free at the inner and simply supported at the outer boundary. It appears from the survey of literature that no work has been done to study the effect of an in-plane force on the natural frequencies of vibration of annular plates of variable thickness.

The object of the present work is to investigate the effect of an in-plane force on the frequencies of thin circular annular plates of parabolically varying thickness on the basis of classical theory. For axisymmetric motions, the governing fourth order linear differential equation with variable coefficients has been solved by Chebyshev collocation technique (Soni and Amba-Rao 1975, Snyder 1969). Since this analysis is valid in low modes only, frequencies, mode shapes and moments for the first two modes of vibration have been computed for two different boundary conditions for various values of in-plane force parameter, radii ratio and taper constant. By allowing the frequency to approach zero, the critical buckling loads have also been determined.

2. Equation of Motion and Solution

The small deflection axisymmetric motion of a thin circular plate of radius a and thickness $h(r)$ in the presence of in-plane forces is governed by the equation (Jain 1972)

$$Dw_{,rrrr} + [2(D + rD_{,r})/r]\, w_{,rrr} + [\{-D + (2+\nu)\, rD_{,r} + r^2(D_{,rr} - N)\}/r^2]\, w_{,rr} + [\{D - rD_{,r} + r^2(\nu D_{,rr} - N)\}/r^3]\, w_{,r} + \rho h w_{,tt} = 0 \quad ...(1)$$

where w is the transverse deflection, N the uniform in-plane tensile force, D the flexural rigidity and other symbols have their usual meanings.

The thickness variation can be of any type (Conway 1958, Gupta and Lal 1978, Tomar and Gupta 1976, Gallegojuarez 1973). Quite a large number of papers on non-uniform thickness variation have come to the attention of the authors in which the variation of thickness is parabolic. Considering steady state vibrations and parabolic variation in thickness h, we write $w/a = W(x)\exp(i\omega t)$, $h/a = h_0(1 - \alpha x^2)$, $r/a = x$. By the use of these non-dimensional quantities eqn. (1) now reduces to

$$U_0 \frac{d^4W}{dx^4} + U_1 \frac{d^3W}{dx^3} + U_2 \frac{d^2W}{dx^2} + U_3 \frac{dW}{dx} - (1 - \alpha x^2)\,\Omega^2 W = 0 \quad ...(2)$$

where

$$U_0 = (1 - \alpha x^2)^3,\; U_1 = 2\,[(1/x) - 9\alpha x + 15\alpha^2 x^3 - 7\alpha^3 x^5]$$

$$U_2 = [(-1/x^2) - 3(5 + 2\nu)\,\alpha + 3(19 + 4\nu)\,\alpha^2 x^2 - (41 + 6\nu)\,\alpha^3 x^4 - (N/aD_0)]$$

$$U_3 = [(1/x^3) + \{3(1 - 2\nu)\,\alpha/x\} - 9(1 - 4\nu)\,\alpha^2 x + 5(1 - 6\nu)\,\alpha^3 x^3 - (N/aD_0 x)]$$

$$\Omega^2 = 12(1 - \nu^2)\,\rho a^2 \omega^2 / E h_0^2.$$

Equation (2) together with the boundary conditions at the inner and outer edges constitutes a well-defined two-point boundary value problem in the range $(\epsilon, 1)$, $\epsilon = b/a$, $b (< a)$ being the inner radius of the annular plate, which has been solved by Chebyshev collocation technique.

To change the interval $(\epsilon, 1)$ into the applicability range $(-1, 1)$ of eqn. (2), let us introduce the new independent variable y given by

$$x = \tfrac{1}{2}\{(1 - \epsilon)\, y + 1 + \epsilon\}. \quad ...(3)$$

The transformation (3) reduces eqn. (2) to

$$V_0 \frac{d^4W}{dy^4} + V_1 \frac{d^3W}{dy^3} + V_2 \frac{d^2W}{dy^2} + V_3 \frac{dW}{dy} + V_4 W = 0 \quad ...(4)$$

where

$$V_0 = \left(\frac{2}{1-\epsilon}\right)^4 U_0,\ V_1 = \left(\frac{2}{1-\epsilon}\right)^3 U_1,\ V_2 = \left(\frac{2}{1-\epsilon}\right)^2 U_2,$$

$$V_3 = \left(\frac{2}{1-\epsilon}\right) U_3 \quad \text{and} \quad V_4 = -(1-\alpha x^2)\,\Omega^2.$$

Assuming

$$\frac{d^4W}{dy^4} = \sum_{k=0}^{m-5} c_{k+5} T_k \qquad \text{...(5)}$$

and successive integrations of eqn. (5) lead to

$$W = c_1 + c_2 T_1 + c_3 T_1^1 + c_4 T_1^2 + \sum_{k=0}^{m-5} c_{k+5} T_k^4 \qquad \text{...(6)}$$

where $c_j (j = 1, 2, ..., m)$ are unknown constants and $T_j (j = 0, ..., m-5)$ are the Chebyshev polynomials.

Substitution of W and its derivatives in eqn. (4) gives

$$V_4 c_1 + (V_4 T_1 + V_3) c_2 + (V_4 T_1^1 + V_3 T_1 + V_2) c_3$$
$$+ (V_4 T_1^2 + V_3 T_1^1 + V_2 T_1 + V_1) c_4 + (V_4 T_1^3 + V_3 T_1^2 + V_2 T_1^1$$
$$+ V_1 T_1 + V_0) c_5 + \sum_{k=1}^{m-5} (V_4 T_k^4 + V_3 T_k^3 + V_2 T_k^2$$
$$+ V_1 T_k^1 + V_0 T_k) c_{k+5} = 0. \qquad \text{...(7)}$$

The satisfaction of this equation at $(m-4)$ collocation points given by equation

$$y_k = \cos\left(\frac{\pi(2k+1)}{2m-8}\right), (k = 0, 1, 2, ..., m-5) \qquad \text{...(8)}$$

provides a set of $(m-4)$ equations in terms of m unknown constants $c_j (j = 1, 2, ..., m)$, which can be written in the matrix form as

$$[B]\,[C] = [0] \qquad \text{...(9)}$$

where B and C are matrices of order $(m-4) \times m$ and $m \times 1$ respectively.

3. Boundary Conditions

The following two sets of boundary conditions have been considered:

(i) C - C : Clamped at both the inner and outer edges.

(ii) C - S : Clamped at the inner and simply supported at the outer edge.

The well-known relations which should be satisfied at a clamped and a simply supported edge are

$$W = \frac{dW}{dy} = 0 \quad \text{and} \quad W = \frac{2}{(1-\epsilon)} \frac{d^2W}{dy^2} + \frac{\nu}{x} \frac{dW}{dy} = 0$$

respectively.

Applying the C - C boundary condition at $y = -1$ and $y = 1$ to the displacement function $W(y)$, one obtains a set of four homogeneous equations. These equations together with field eqns. (9) give a complete set of m equations in m unknowns, which can be denoted by the following matrix equation

$$\left[\frac{B}{B_1}\right][C] = [0] \qquad \text{...(10)}$$

where B_1 is a matrix of order $4 \times m$.

For a non-trivial solution of eqn. (10), the frequency determinant must vanish and hence

$$\left|\frac{B}{B_1}\right| = 0. \qquad \text{...(11)}$$

Similarly for C - S plate the frequency determinant can be written as

$$\left|\frac{B}{B_2}\right| = 0. \qquad \text{...(12)}$$

4. Numerical Results and Discussion

Equations (11) and (12) are transcendental equations in the frequency parameter Ω and can be numerically solved for a specified plate. In the work, presented here, the effect of taper parameter α, the radii ratio ϵ and the in-plane force parameter $\bar{N} (= N/aD_0)$ on the frequencies have been investigated for $\nu = 0.3$. In addition, the critical values $\bar{N}_c$ of $\bar{N}$, for the critical buckling load in compression have been computed for varying value of taper constant α. Mode shapes and moments have been calculated for the first two modes of vibration, for two different boundary conditions. Results for plates of uniform thickness ($\alpha = 0$) and $\bar{N} = 0$ have been presented in Table I which show good agreement with those of Vogel and Skinner (1965) given by Leissa (1969) in Tables 2.18 and 2.24. IBM 360/44 computer has been used to carry out the numerical results.

In all computations, we have fixed $m = 15$ because a further increase in m does not improve the results even in the fourth place of decimal. Figure 1 shows convergence of Ω with the number of collocation points m.

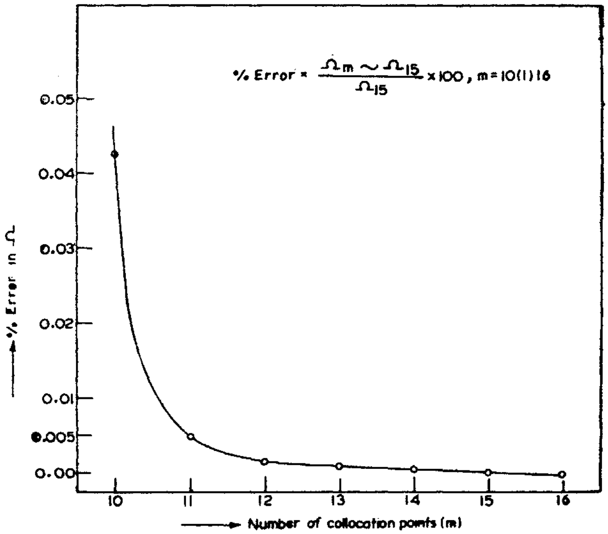

FIG. 1.

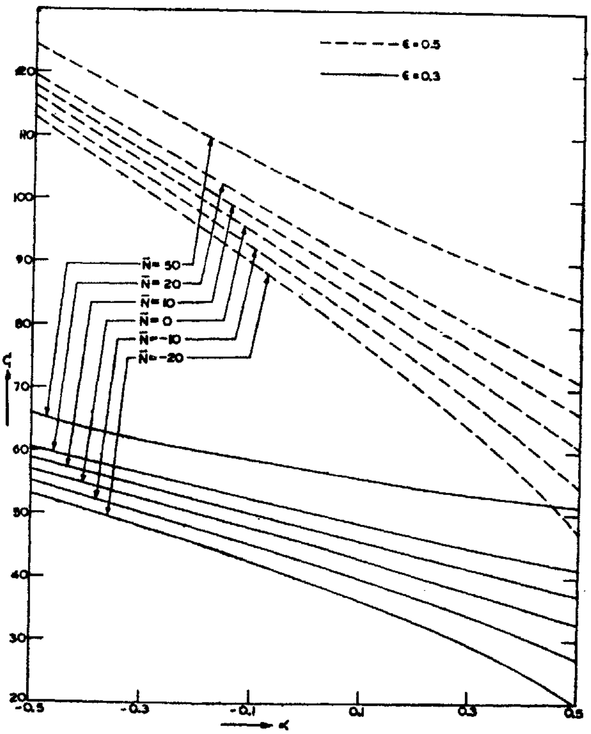

FIG. 2. Natural frequencies of a C - C annular plate for the fundamental mode of vibration.

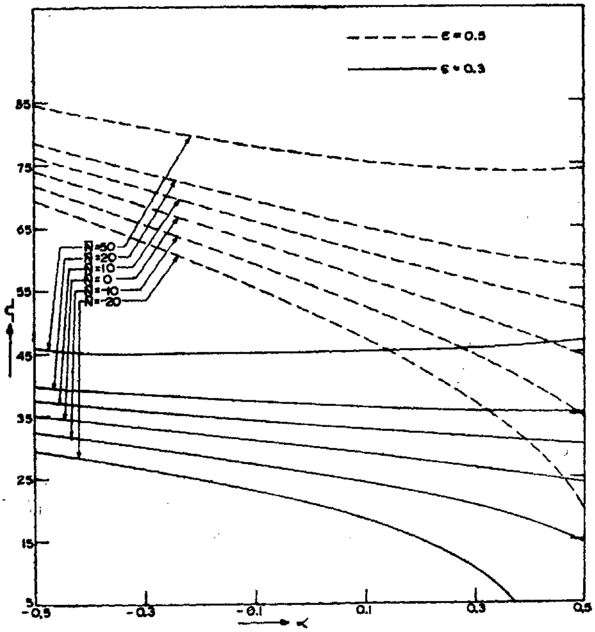

FIG. 3. Natural frequencies of a C - S annular plate for the fundamental mode of vibration.

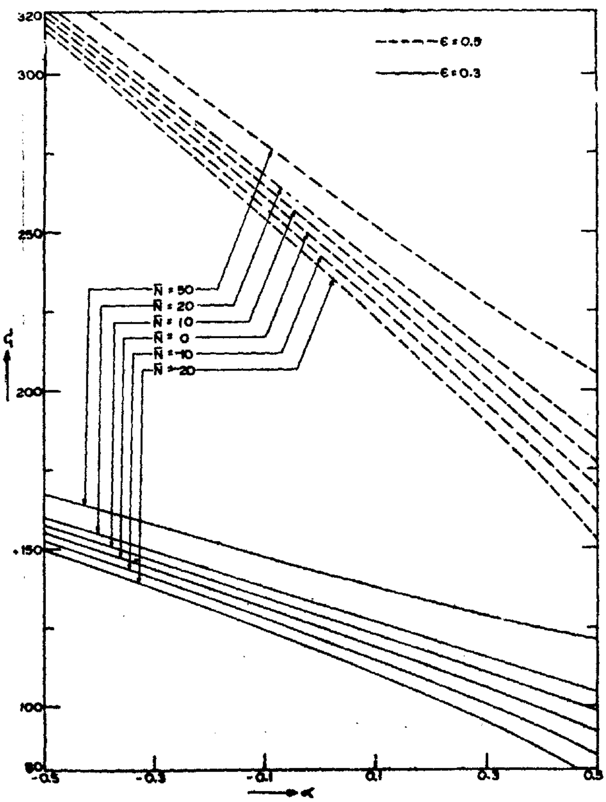

FIG. 4. Natural frequencies of a C - C annular plate for the second mode of vibration.

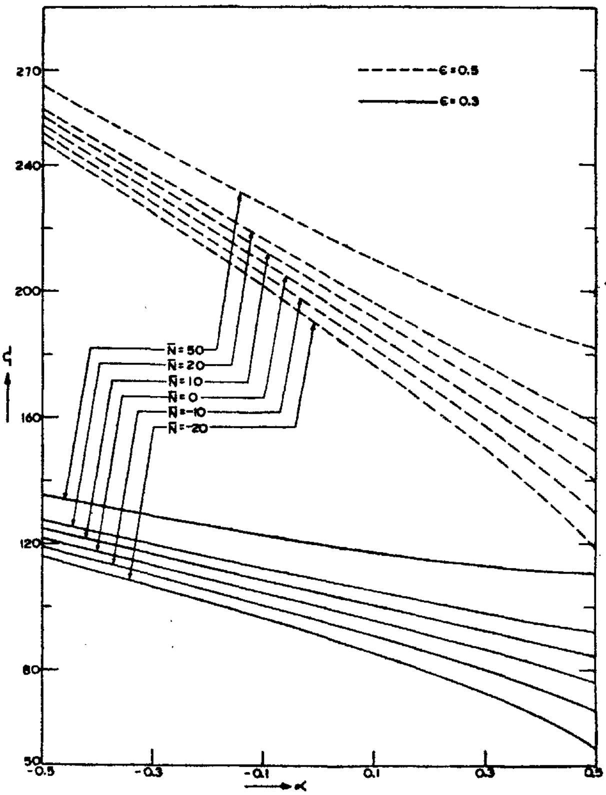

FIG. 5. Natural frequencies of a C - S annular plate for the second mode of vibration.

The results are presented in Figs. 2–8 and Table II. It is found that the frequency parameter for a C - C plate is greater than the corresponding frequency parameter for a C - S plate. Figs. 2 and 3 show the effect of in-plane force parameter $\bar{N}$ on frequencies for C - C and C - S plates vibrating in the fundamental mode for $\epsilon = 0.3, 0.5$ and $\alpha = -0.5(0.2)\, 0.5$. It is clear that the frequency of the plate increases with the increase in $\bar{N}$ for both the radii ratio and for each value of α. The frequency parameter is found to increase continuously for moderate values of $\bar{N}$ with the decrease in α, the rate of increase for C - C plate being appreciably greater than for C - S plate. For higher values of $\bar{N}$, the rate of increase of Ω with α is considerably reduced for both the boundary conditions.

The values of frequency parameter Ω for $\epsilon = 0.5$ are greater than the corresponding values of Ω for $\epsilon = 0.3$ and this difference is found to increase with the decrease in taper constant α. Thus the radii ratio plays an important role on the natural frequencies. The desired frequency can be achieved for a specified plate by a suitable choice of in-plane force parameter $\bar{N}$. When the plate is vibrating in the

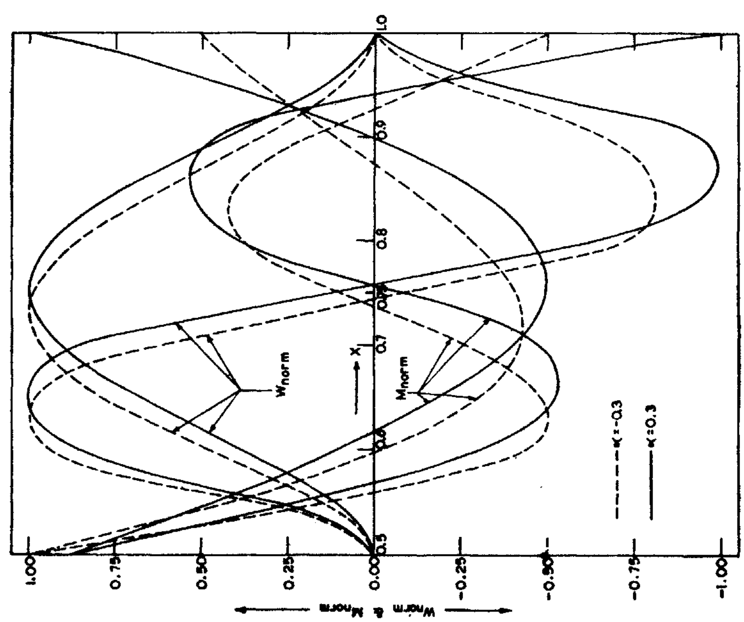

FIG. 6. Normalised displacements and moments of a C - C plate for the first two modes of vibration for $\varepsilon = 0.5$, $\bar{N} = 10$.

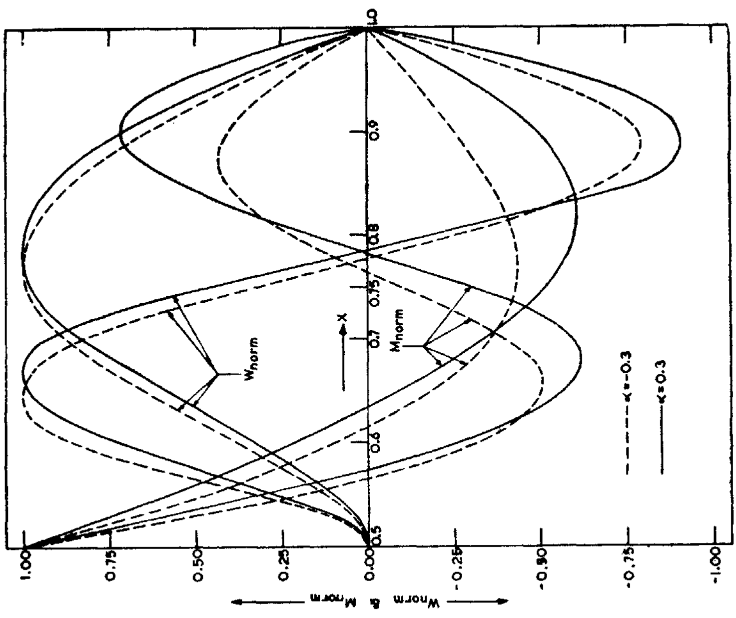

FIG. 7. Normalised displacements and moments of a C - S plate for the first two modes of vibration for $\varepsilon = 0.5$, $\bar{N} = 10$.

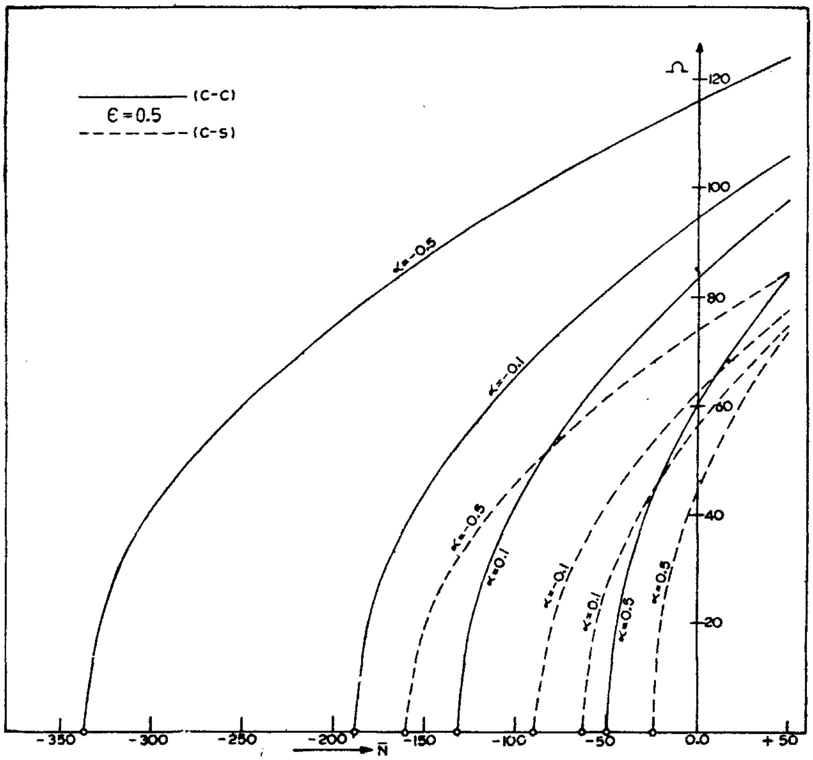

Fig. 8. Buckling loads for C - C and C - S annular plates for fundamental mode.

second mode (Figs. 4 and 5) a similar behaviour of the frequency parameter is seen as in the fundamental mode except that the rate of increase of Ω with decreasing α for various values of $\bar{N}$ is increased for both the plates.

Figures 6 and 7 show the plots for the normalised displacements ($W_{norm} = W/W_{max}$) and moments ($M_{norm} = M/M_{max}$) for the first two modes of vibrations for $\epsilon = 0.5$, $\bar{N} = 10$ and $\alpha = \pm 0.3$ for both the boundary conditions. The transverse deflection for $\alpha = -0.3$ is greater towards the inner boundary and smaller towards the outer boundary than the corresponding deflection for $\alpha = 0.3$. The nodal circles are seen to shift towards the inner boundary as taper constant α decreases. This may be attributed to the thickening towards the outer boundary. As regards the lines along which moments vanish, these are shifted towards the inner boundary for both the plates.

The critical values $\bar{N}_c$ of $\bar{N}$ corresponding to the critical buckling load in compression, for $\epsilon = 0.5$ and various values of taper constants α, for both the plates are reported in Table II. Figure 8 shows the plots of Ω versus $\bar{N}$ for various values of the taper constant α for $\epsilon = 0.5$ for the fundamental mode. It is found that the buckling loads for C - C plate are higher as compared to that for C - S plate whatever be the taper constant α.

TABLE I

Comparison of frequency parameter Ω, for $\alpha = 0$, $\bar{N} = 0$ and $\nu = 0.3$

Boundary condition	$\epsilon = 0.3$		$\epsilon = 0.5$	
	Fundamental mode	Second mode	Fundamental mode	Second mode
C - C	45.3457	125.3502	89.2507	246.3235
	45.2*	125*	89.2*	246*
C - S	29.9783	100.4211	59.8200	198.0512
	29.9*	100*	59.8*	198*

*Values taken from Leissa (1969).

TABLE II

Values of $\bar{N}$ for the critical buckling load in compression for $\epsilon = 0.5$ and $\nu = 0.3$

Boundary condition	Mode	values of taper constant α					
		−0.5	−0.3	−0.1	0.1	0.3	0.5
C - C	Fundamental	−337.126	−256.383	−188.125	−131.502	−85.673	−49.958
	Second	−680.936	−519.459	−382.610	−268.767	−176.285	−103.514
C - S	Fundamental	−160.256	−122.310	−90.166	−63.421	−41.676	−24.543
	Second	−498.198	−380.195	−280.166	−196.894	−129.175	−75.856

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