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ON A SUBCLASS OF α-SPIRAL-LIKE FUNCTIONS

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For some real numbers α, $|\alpha| < \pi/2$, and γ, let $S_p(\alpha, \gamma)$ denote the class of functions $f(z) = z + a_2z^2 + \ldots$, which are regular in the unit disc $E = \{z : |z| < 1\}$ and satisfy

$$\mathrm{Re}\left[(e^{i\alpha}zf'(z)/f(z))^{1-\gamma}\left(1 + \frac{zf''(z)}{f'(z)} + i\tan\alpha\,\frac{zf'(z)}{f(z)}\right)^{\gamma}\right] > 0,$$

z in E. In this paper we prove that $S_p(\alpha, \gamma)$ is a subclass of $S_p(\alpha)$, the class of α-spiral-like functions. An inclusion relation is also established.

INTRODUCTION

Let A denote the class of functions $f(z)$ which are regular in the unit disc $E = \{z : |z| < 1\}$ and normalized so that $f(0) = f'(0) - 1 = 0$. Denote by S the subclass of A consisting of functions which are univalent in E. A function $f \in S$ is said to be starlike in E if $f(E)$ is a starlike domain with respect to the point $f(0)$ $(= 0)$ and we denote by S^* the class of all such functions. It is known that a necessary and sufficient condition for $f \in S$ to be in S^* is that

$$\mathrm{Re}\,\frac{zf'(z)}{f(z)} > 0 \quad (z \in E). \qquad \ldots(1)$$

Let γ be a given real number and denote by S_γ the class of functions $f \in A$ for which $zf'(z)/f(z)$, $1 + (zf''(z)/f'(z))$ are non-vanishing in E and which satisfy the condition

$$\mathrm{Re}\left[\left(\frac{zf'(z)}{f(z)}\right)^{1-\gamma}\left(1 + \frac{zf''(z)}{f'(z)}\right)^{\gamma}\right] > 0 \quad (z \in E). \qquad \ldots(2)$$

Lewandowski *et al.* (1974) proved that for each γ, $S_\gamma \subset S_0 \equiv S^*$. A function in S_γ will be referred to as gamma-starlike in E.

For a given real number α, $|\alpha| < \frac{1}{2}\pi$, we denote by $S_p(\alpha)$ the subclass of A whose members satisfy the condition

$$\mathrm{Re}\left(e^{i\alpha}\,\frac{zf'(z)}{f(z)}\right) > 0 \quad (z \in E). \qquad \ldots(3)$$

It is known that every member of $S_p(\alpha)$ is univalent in E (Špaček 1933), and hence $S_p(\alpha) \subset S$. Clearly, we have $S_p(0) \equiv S^*$. A function $f \in S_p(\alpha)$ will be referred to as an alpha-spiral function in E.

We denote by $S_c(\alpha)$, α a given real number, $|\alpha| < \frac{1}{2}\pi$, the subclass of A consisting of functions f for which

$$\text{Re}\left\{1 + \frac{zf''(z)}{f'(z)} + i\tan\alpha\,\frac{zf'(z)}{f(z)}\right\} > 0 \ (z \in E). \qquad \ldots(4)$$

Yoshikawa (1971) proved that $S_c(\alpha)$ is a subclass of $S_p(\alpha)$. Members of $S_c(\alpha)$ will be referred to as alpha-spiral-convex functions in E. We note that $S_c(0) \equiv K$, the subclass of A consisting of functions which map E onto convex domains.

Let γ and α, $|\alpha| < \frac{1}{2}\pi$, be given real numbers, and denote by $S_p(\alpha, \gamma)$ the class of functions $f \in A$ for which $zf'(z)/f(z) \neq 0$.

$$(1 + zf''(z)/f'(z) + i\tan\alpha zf'(z)/f(z)) \neq 0$$

in E and which satisfy the condition

$$\text{Re}\left[\left(e^{i\alpha}\frac{zf'(z)}{f(z)}\right)^{1-\gamma}\left(1 + \frac{zf''(z)}{f'(z)} + i\tan\alpha\,\frac{zf'(z)}{f(z)}\right)^{\gamma}\right] > 0 \ (z \in E). \ \ldots(5)$$

Clearly $S_p(0, \gamma) = S_\gamma$ and $S_p(\alpha, 1) = S_c(\alpha)$.

In this paper we prove that $S_p(\alpha, \gamma)$ is a subclass of $S_p(\alpha)$. Results of Lewandowski *et al.* (1974) and Yoshikawa (1971) follow as special cases (corresponding to $\alpha = 0$ and $\gamma = 1$, respectively) from our theorem.

Theorem 1 — If $f \in S_p(\alpha, \gamma)$, then

$$\text{Re}\left(e^{i\alpha}\frac{zf'(z)}{f(z)}\right) > 0 \ (z \in E)$$

and hence $S_p(\alpha, \gamma) \subset S_p(\alpha)$.

PROOF: Since $f \in S_p(\alpha, \gamma)$, we have, by definition

$$\text{Re}\left[\left(e^{i\alpha}\frac{zf'(z)}{f(z)}\right)^{1-\gamma}\left(1 + \frac{zf''(z)}{f'(z)} + i\tan\alpha\,\frac{zf'(z)}{f(z)}\right)^{\gamma}\right] > 0 \ (z \in E). \ \ldots(6)$$

Let

$$P(z) = e^{i\alpha}\frac{zf'(z)}{f(z)}.$$

To prove our theorem, we have to show that Re $P(z) > 0$ in E.

Define $w(z)$ in E as follows:

$$e^{i\alpha}\frac{zf'(z)}{f(z)} = P(z) = i\sin\alpha + \cos\alpha\,\frac{1+w(z)}{1-w(z)}$$

$$= \frac{e^{i\alpha} + e^{-i\alpha}w(z)}{1-w(z)}. \qquad \ldots(7)$$

Then clearly $w(z)$ is regular in E, $w(0) = 0$ and $w(z) \neq 1$ in E. To prove that Re $P(z) > 0$ it suffices to show that $|w(z)| < 1$ in E. Let us suppose that there exists a point $z_0 = re^{i\theta} \in E$ such that $\max_{|z|<r} |w(z)| = |w(z_0)| = 1$. Then by Jack's (1971) Lemma, we have $z_0 w'(z_0) = kw(z_0)$ where $k \geqslant 1$.

Differentiating (7) logarithmically and simplifying, we get

$$\left(e^{i\alpha}\frac{zf'(z)}{f(z)}\right)^{1-\gamma}\left(1+\frac{zf''(z)}{f'(z)} + i\tan\alpha\,\frac{zf'(z)}{f(z)}\right)^{\gamma}$$

$$= \left(\frac{e^{i\alpha}+e^{-i\alpha}w(z)}{1-w(z)}\right)\left(\sec\alpha + \frac{2\cos\alpha zw'(z)}{(e^{i\alpha}+e^{-i\alpha}w(z))^2}\right)^{\gamma}.$$

At the point z_0 we have, in view of Jack's Lemma with $w(z_0) = e^{i\phi}$,

$$\text{Re}\left[\left(e^{i\alpha}\frac{zf'(z)}{f(z)}\right)^{1-\gamma}\left(1+\frac{zf''(z)}{f'(z)} + i\tan\alpha\,\frac{zf'(z)}{f(z)}\right)^{\gamma}\Bigg|_{z=z_0}\right]$$

$$= \text{Re}\left[\left(\frac{e^{i\alpha}+e^{-i\alpha}e^{i\phi}}{(1-e^{i\phi})}\right)\left(\sec\alpha + \frac{2k\cos\alpha e^{i\phi}}{(e^{i\alpha}+e^{-i\alpha}e^{i\phi})^2}\right)^{\gamma}\right]$$

$$= 0$$

since the first factor is pure imaginary and the second one is real and positive. This is a contradiction to the hypothesis (6). We have thus proved that $|w(z)| < 1$ and so Re $P(z) > 0$ in E. This completes the proof of Theorem 1.

Theorem 2 — Let $f \in S_p(\alpha, \gamma)$ and $0 \leqslant \delta \leqslant \gamma$ or $\gamma \leqslant \delta \leqslant 0$. Then

(i) $S_p(\alpha, \gamma) \subset S_p(\alpha, \delta)$ and (ii) $S_p(\alpha, \gamma) \subset S_c(\alpha)$ for all $\gamma \geqslant 1$.

PROOF: Let $f \in S_p(\alpha, \gamma)$, we have

$$\text{Re}\left[\left(e^{i\alpha}\frac{zf'(z)}{f(z)}\right)^{1-\gamma}\left(1+\frac{zf''(z)}{f'(z)} + i\tan\alpha\,\frac{zf'(z)}{f(z)}\right)^{\gamma}\right] > 0 \ (z \in E)$$

or equivalently

$$\left|\arg\left[\left(e^{i\alpha}\frac{zf'(z)}{f(z)}\right)^{1-\gamma}\left(1+\frac{zf''(z)}{f'(z)} + i\tan\alpha\,\frac{zf'(z)}{f(z)}\right)^{\gamma}\right]\right| < \pi/2 \ (z \in E). \qquad \ldots(8)$$

Let $0 \leqslant \delta \leqslant \gamma$. Then

$$\left(e^{i\alpha}\frac{zf'(z)}{f(z)}\right)^{1-\delta}\left(1+\frac{zf''(z)}{f'(z)}+i\tan\alpha\,\frac{zf'(z)}{f(z)}\right)^{\delta}$$

$$=\left[\left(e^{i\alpha}\frac{zf'(z)}{f(z)}\right)^{1-\gamma}\left(1+\frac{zf''(z)}{f'(z)}+i\tan\alpha\,\frac{zf'(z)}{f(z)}\right)^{\gamma}\right]^{\delta/\gamma} \times\left[e^{i\alpha}\frac{zf'(z)}{f(z)}\right]^{1-(\delta/\gamma)}$$

and hence

$$\left|\arg\left[\left(e^{i\alpha}\frac{zf'(z)}{f(z)}\right)^{1-\delta}\left(1+\frac{zf''(z)}{f'(z)}+i\tan\alpha\,\frac{zf'(z)}{f(z)}\right)^{\delta}\right]\right| < \frac{\delta}{\gamma}\,\frac{\pi}{2}+\left(1-\frac{\delta}{\gamma}\right)\frac{\pi}{2}=\frac{\pi}{2} \quad ...(9)$$

where we have made use of (8) and the fact that $f \in S_p(\alpha, \gamma)$ implies that

$$\operatorname{Re}\,(e^{i\alpha}zf'(z)/f(z)) > 0,\ z \in E.$$

From (9) we conclude that $f \in S_p(\alpha, \delta)$ and hence $S_p(\alpha, \gamma) \subset S_p(\alpha, \delta)$ for $0 \leqslant \delta \leqslant \gamma$. We can similarly prove this assertion for the case $\gamma \leqslant \delta \leqslant 0$. This completes the proof of (i).

Now if $f \in S_p(\alpha, \gamma)$ and $\gamma \geqslant 1$, then from (8) we have

$$\gamma\left|\arg\left(1+\frac{zf''(z)}{f'(z)}+i\tan\alpha\,\frac{zf'(z)}{f(z)}\right)\right| \leqslant \frac{\pi}{2}+(\gamma-1)\left|\arg\left(e^{i\alpha}\frac{zf'(z)}{f(z)}\right)\right|$$

$$\leqslant \frac{\pi}{2}+(\gamma-1)\frac{\pi}{2}$$

$$=\frac{\gamma\pi}{2}$$

and hence

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}+i\tan\alpha\,\frac{zf'(z)}{f(z)}\right) > 0 \ (z \in E).$$

This completes the proof of (ii).

Taking $\alpha = 0$ and $\alpha = 0$, $\gamma \geqslant 1$ in (i) and (ii) of Theorem 2, respectively, we obtain the following results due to Lewandowski *et al.* (1974).

Corollary 1 — For $0 \leqslant \delta \leqslant \gamma$ or $\gamma \leqslant \delta \leqslant 0$,

$$S_\gamma \equiv S_p(0, \gamma) \subset S_p(0, \delta) \equiv S_\delta.$$

Corollary 2 — $S_p(0, \gamma) = S_\gamma$, $\gamma \geqslant 1$ is a subclass of K.

Proceeding as in Theorem 1, we can easily establish the following result.

Theorem 3 — If $f \in A$, and $zf'(z)/f(z)$ and $\{1 + (zf''(z)/f'(z)) + i \tan \alpha \times (zf'(z)/f(z))\}$ are non-vanishing in E, then for every integer $n \geqslant 0$, the condition

$$\operatorname{Re}\left[\left(e^{i\alpha}\frac{zf'(z)}{f(z)}\right)^{2n+1}\left(1+\frac{zf''(z)}{f'(z)}+i\tan\alpha\,\frac{zf'(z)}{f(z)}\right)^{2n+1}\right] > 0 \quad (z \in E)$$

implies that f is an α-spiral-like function and hence univalent in E.

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