Indian J. pure appl. Math., 11(2) : 165-172, February 1980

VISCOUS FLOW DOWN AN OPEN INCLINED CHANNEL WITH NATURALLY PERMEABLE BED

P. D. VERMA AND H. K. VYAS

Department of Mathematics, University of Rajasthan, Jaipur, 302004

(*Received* 22 *January* 1979; *after revision* 15 *June* 1979)

The problem of steady laminar viscous flow of fluid down an open inclined channel whose bottom is a fluid saturated permeable bed has been studied. The free surface is exposed to atmospheric pressure. Flow of fluid both in porous medium and in free fluid region is studied with the same pressure gradient. The boundary conditions due to Beavers and Joseph (1967), at the interface of fluid in free flow region and fluid flowing in porous medium are applied. Flux across the cross-section of the channel is calculated. Particular cases like fluid flowing down (i) an infinite inclined plane and (ii) channel with impermeable bed have been derived. Velocity profile of fluid in free flow region is compared graphically with that of fluid flowing over impermeable bottom. It is inferred that at all sections, from the free surface to the naturally permeable bottom both the flux and the velocities are greater than when the bottom is impermeable.

1. INTRODUCTION

The open channel flow has an important application in the field of Hydraulic Engineering. Several empirical results have been reported by many investigators e.g. Vanoni (1941) and Johnson (1944). Satyaprakash (1971) considered analytically a viscous flow down an open inclined channel with plane bottom and vertical walls under the action of gravity. The free surface was exposed to atmospheric pressure and bottom was taken as impermeable.

In the present investigation we have made an effort to derive analytically the results for velocity profile and flux of fluid in free fluid region by considering the viscous flow down an open inclined channel with naturally permeable bed. It is inferred that presence of permeable bed increases the discharge. This may lead to a practical application in the field of irrigation. If the bed of the canal is permeable and fully saturated with water then there is every possibility of the discharge to be increased.

Here we have considered the steady viscous laminar flow of fluid both in porous medium and in free fluid region, with the same pressure gradient.

Fluid flow in porous medium is governed by Darcy's Law and fluid in free flow region is governed by Navier-Stokes' equations. Boundary conditions at the permeable surface in this coupled flow motion is taken here as suggested by Beavers and Joseph (1967). Results given by Satyaprakash (1971) have been deduced as a particular case of this problem. As a very special case we have also deduced the case considered by Yih (1963) by letting the width of the channel tend to infinity and bottom as impermeable.

Velocity profiles of fluid in free fluid region on central transversal section ($z' = 0$) of the channel are shown graphically and compared with the case of channel flow with impermeable bed.

Formation of the Problem

We have considered a steady laminar viscous uniform flow of a fluid down an inclined open channel of depth d and width b, the walls of channel being normal to the surface of bottom.

The naturally permeable bottom of the channel is taken at an angle $\beta (0 \leqslant \beta \leqslant \pi/2)$ with the horizontal.

The flow in both the regions, free fluid region and porous medium, is developed due to a constant pressure gradient at the mouth of the channel.

We take the Cartesian co-ordinate axes (X, Y, Z), X-axis along the central line in the direction of the flow at the free surface, Y-axis along the depth of the channel and Z-axis along the width of the channel (Fig. 1). For this problem as specified the velocity components are (u, 0, 0).

The Navier-Stokes' equations for the free fluid region are reduced to

$$\mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \rho g \sin\beta - \frac{\partial p}{\partial x} = 0 \qquad \ldots(1.1)$$

$$\rho g \cos\beta - \frac{\partial p}{\partial y} = 0 \qquad \ldots(1.2)$$

$$-\frac{\partial p}{\partial z} = 0 \qquad \ldots(1.3)$$

and the equation of continuity is reduced to

$$\frac{\partial u}{\partial x} = 0. \qquad \ldots(1.4)$$

The Darcy's equation for the flow in porous media is

$$Q = \frac{K}{\mu}\left(-\frac{\partial p}{\partial x} + \rho g \sin\beta\right) \qquad \ldots(1.5)$$

where Q is the velocity in the porous medium; K is the permeability of the medium.

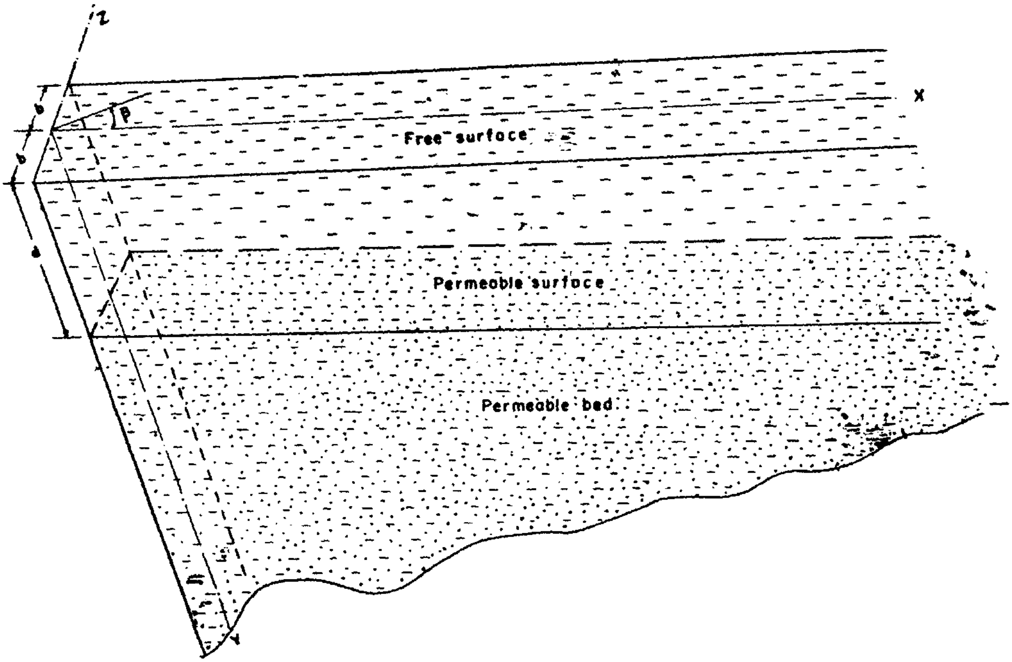

FIG. 1. Diagram of the flow.

The boundary conditions are

(i) $u = 0$ at $z = \pm b$ ...(1.6)

(ii) $\dfrac{\partial u}{\partial y} = 0$ at the free surface $y = 0$ ...(1.7)

and the condition at the interface of free flow region and porous medium as suggested by Beavers and Joseph (1967) is

(iii) $\left.\dfrac{\partial u}{\partial y}\right|_{y=d} = \dfrac{\alpha}{\sqrt{K}}(U_1 - Q)$ ...(1.8)

where $u = U_1$ at $y = d$. α is a dimensionless constant depending on the porous material.

Let the depth d of the channel be the characteristic length and the mean flow velocity i.e. rate of flow per unit cross-sectional area U be the characteristic velocity.

On introducing the following non-dimensional quantities

$$u' = \frac{u}{U}, x' = \frac{x}{d}, y' = \frac{y}{d}, z' = \frac{z}{d}, Q' = \frac{Q}{U}, K' = \frac{K}{d^2},$$

$$U_1' = \frac{U_1}{U}, \; p' = \frac{p}{\rho U^2} \qquad ...(1.9)$$

in eqns. (1.1), (1.4), (1.5) and dropping the primes, we have

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \left(R\frac{\partial p}{\partial x} - \frac{R}{F}\sin\beta\right) \qquad ...(1.10)$$

$$\frac{\partial u}{\partial x} = 0 \qquad ...(1.11)$$

$$Q = -KR\left(\frac{dp}{dx} - \frac{1}{F}\sin\beta\right) \qquad ...(1.12)$$

where R (Reynolds number) $= Ud/\nu$, and F (Froude number) $= U^2/gd$.

The boundary conditions (1.6) to (1.8) with the use of (1.9) are reduced to

$$u = 0 \quad \text{at} \quad z = \pm\sigma \qquad ...(1.13)$$

$$\frac{\partial u}{\partial y} = 0 \quad \text{at} \quad y = 0 \qquad ...(1.14)$$

and

$$\left.\frac{\partial u}{\partial y}\right|_{y=1} = \frac{\alpha}{\sqrt{K}}\left(-U_1 - KR\frac{dp}{dx} + \frac{KR}{F}\sin\beta\right) \qquad ...(1.15)$$

where $\sigma = \dfrac{b}{d}$ and $u = U_1$ at $y = 1$.

2. Solution

Let $$z = \frac{2\sigma\xi}{\pi} - \sigma. \qquad ...(2.1)$$

From eqns. (1.10) and (2.1) we have

$$\frac{\partial^2 u}{\partial y^2} + \frac{\pi^2}{4\sigma^2}\frac{\partial^2 u}{\partial \xi^2} = R\frac{dp}{dx} - \frac{R}{F}\sin\beta \qquad ...(2.2)$$

with the following boundary conditions

$$u = 0 \quad \text{at} \quad \xi = 0 \quad \text{and} \quad \xi = \pi \qquad ...(2.3)$$

$$u = U_1 \quad \text{at} \quad y = 1 \qquad ...(2.4)$$

and

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0. \qquad ...(2.5)$$

Using finite sine transform eqn. (2.2) is reduced to

$$\frac{d^2\bar{u}}{dy^2} - \frac{\pi^2 n^2}{4\sigma^2}\bar{u} = \frac{1}{n}\left(R\frac{dp}{dx} - \frac{R}{F}\sin\beta\right)(1 - \cos n\pi) \qquad ...(2.6)$$

where $\bar{u} = \int_0^\pi u \sin n\xi \, d\xi$, n is a positive integer and the boundary conditions (2.3) to (2.5) are reduced to

$$\bar{u}(y, n) = \bar{U}_1 \quad \text{at} \quad y = 1 \qquad \ldots(2.7)$$

$$\frac{d\bar{u}}{dy} = 0 \quad \text{at} \quad y = 0. \qquad \ldots(2.8)$$

The solution of eqn. (2.6) with the boundary conditions (2.7) and (2.8) is

$$\bar{u} = \frac{\bar{U}_1 \cosh(\pi n y/2\sigma)}{\cosh(\pi n/2\sigma)} + \frac{4\sigma^2}{\pi^2 n^3}\left(R\frac{dp}{dx} - \frac{R}{F}\sin\beta\right)(1 - \cos n\pi) \times \left[\frac{\cosh(\pi n y/2\sigma)}{\cosh(\pi n/2\sigma)} - 1\right]. \qquad \ldots(2.9)$$

Using inversion formula for finite sine transform (2.9) is reduced to

$$u = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\bar{U}_1 \sin n\xi}{\cosh(\pi n/2\sigma)}\cosh\left(\frac{\pi n}{2\sigma}y\right) + \frac{8\sigma^2}{\pi^3}\left(-R\frac{dp}{dx} + \frac{R}{F}\sin\beta\right) \times \sum_{n=1}^{\infty}\left(\frac{1 - \cos\pi n}{n^3}\right)\left(1 - \frac{\cosh(\pi n y/2\sigma)}{\cosh(\pi n/2\sigma)}\right)\sin n\xi. \qquad \ldots(2.10)$$

On applying finite sine transform to eqn. (1.15) we get

$$\left.\frac{\partial\bar{u}}{\partial y}\right|_{y=1} = \frac{\alpha}{\sqrt{K}}\left[-\bar{U}_1 + \frac{KR}{F}\sin\beta\left(\frac{1 - \cos n\pi}{n}\right) - KR\frac{dp}{dx}\left(\frac{1 - \cos n\pi}{n}\right)\right]. \qquad \ldots(2.11)$$

From eqns. (2.9) and (2.11) we have

$$u = \frac{A}{B} \times \frac{\sin n\xi \cosh(\pi n/2\sigma)\, y}{\cosh(\pi n/2\sigma)} + \frac{8\sigma^2}{\pi^3}\left(-R\frac{dp}{dx} + \frac{R}{F}\sin\beta\right) \times \sum_{n=1}^{\infty}\left(\frac{1 - \cos\pi n}{n^3}\right)\left\{1 - \frac{\cosh(\pi n y/2\sigma)}{\cosh(\pi n/2\sigma)}\right\}\sinh\xi \qquad \ldots(2.12)$$

where

$$A = \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{-2\sigma}{\pi n^2}\left(R\frac{dp}{dx} - \frac{R}{F}\sin\beta\right)(1 - \cos\pi n)\tanh\frac{\pi n}{2\sigma} + \left(\frac{R}{F}\sin\beta - R\frac{dp}{dx}\right)\alpha\sqrt{K}\left(\frac{1 - \cos\pi n}{n}\right)\right]$$

$$B = \frac{\alpha}{\sqrt{K}} + \frac{\pi n}{2\sigma}\tanh\frac{\pi n}{2\sigma}.$$

3. Particular Cases

Case I

For impermeable bed, we take $K \to 0$ in (2.12). Then velocity profile is given by

$$u = \frac{8\sigma^2}{\pi^3}\left(-R\frac{dp}{dx} + \frac{R}{F}\sin\beta\right)\sum_{n=1}^{\infty}\left(\frac{1-\cos\pi n}{n^3}\right)$$

$$\times\left(1 - \frac{\cosh(\pi n y/2\sigma)}{\cosh(\pi n/2\sigma)}\right)\sinh\xi$$

which is in agreement with Satyaprakash (1971).

Case II

If fluid is flowing down an infinite inclined plane with bottom as saturated permeable bed then by taking $\sigma \to \infty$ (2.12) gives

$$u = \frac{\sqrt{K}}{\alpha}\left(R\frac{dp}{dx} - \frac{R}{F}\sin\beta\right)(1 - \alpha\sqrt{K}) + \left(-R\frac{dp}{dx} + \frac{R}{F}\sin\beta\right)\left(\frac{1-y^2}{2}\right) \qquad \ldots(3.1)$$

where $\frac{dp}{dx} = 0$ and $K \to 0$, (3.1) is reduced to

$$u = \frac{R}{F}\sin\beta\left(\frac{1-y^2}{2}\right) \qquad \ldots(3.2)$$

which is in agreement with Yih (1963).

4. Flux

Volume rate of flow across the cross-section of the channel perpendicular to the X-axis is given by

$$V = \int_{z=-\sigma}^{\sigma}\int_{y=0}^{1} U\,dy\,dz$$

$$= \sum_{m=0}^{\infty}\frac{C}{D}\times\tanh\frac{\pi(2m+1)}{2\sigma} + \frac{64\sigma^3}{\pi^4}\left(-R\frac{dp}{dx} + \frac{R}{F}\sin\beta\right)$$

$$\times\sum_{m=0}^{\infty}\frac{1}{(2m+1)^4}\left[1 - \frac{2\sigma}{(2m+1)\pi}\tanh\left(\frac{2m+1}{2\sigma}\right)\pi\right]$$

$$C = \left[\frac{32\sigma^2\alpha\sqrt{K}}{\pi^3(2m+1)^3}\left(\frac{R}{F}\sin\beta - R\frac{dp}{dx}\right) - \frac{32\sigma^3}{\pi^4(2m+1)^4} \times \left(R\frac{dp}{dx} - \frac{R}{F}\sin\beta\right)\tanh\frac{\pi(2m+1)}{2\sigma}\right]$$

$$D = \frac{\alpha}{\sqrt{K}} + \frac{\pi(2m+1)}{2\sigma}\tanh\frac{\pi(2m+1)}{2\sigma}$$

where m is an integer. For $K \to 0$, the flux V is in agreement with Satyaprakash (1971).

5. Numerical Discussion

The velocity profiles FU/R, at the central transverse section ($\xi = \frac{1}{2}\pi$) of the channel flow are shown graphically (Fig. 2) for $\beta = \frac{1}{6}\pi$, $\sigma = \frac{1}{2}\pi$, $K = 10^{-4}$, $\alpha = 0.01$ with different values of pressure gradients viz. $-F\frac{dp}{dx} = 0, 0.5, 1.0, 1.5, 2$. These velocity profiles are also compared with that of, when bed is impermeable. Normally on the first thinking one can imagine that the flux on the naturally permeable bottom should be less than when the bottom is impermeable due to seepage of the fluid in the porous material but in the present investigation we find that at all sections from the free surface to the naturally permeable bottom the velocities are greater than when the bottom is impermeable. This phenomena is due to flow in the porous material which is coming up in the analysis where in the case of naturally permeable bed, a slip-velocity boundary condition has been assumed.

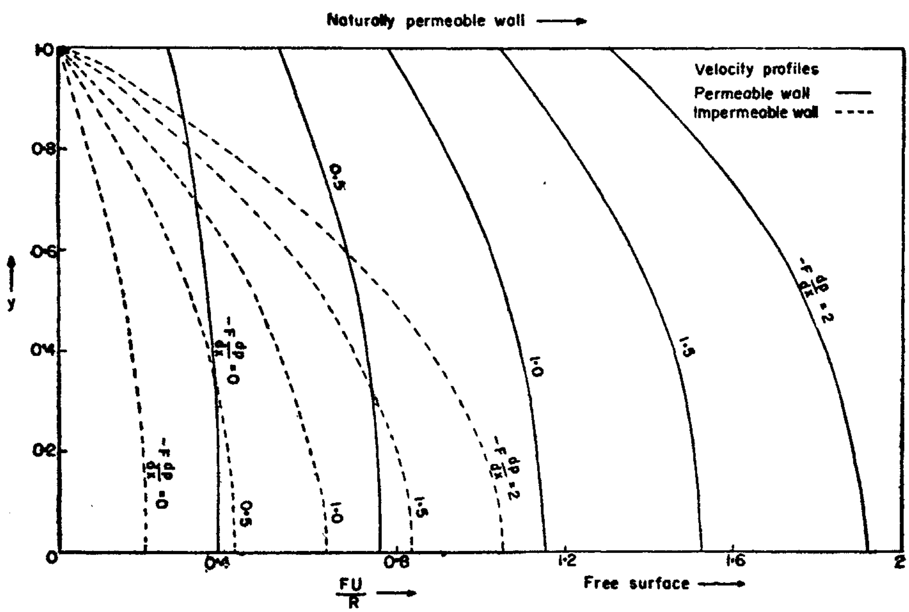

Fig. 2. Velocity profiles at the central section for $\beta = \pi/6$, $\sigma = \pi/2$, $K = 10^{-4}$ and $\alpha = 0.01$ with different pressure gradients.

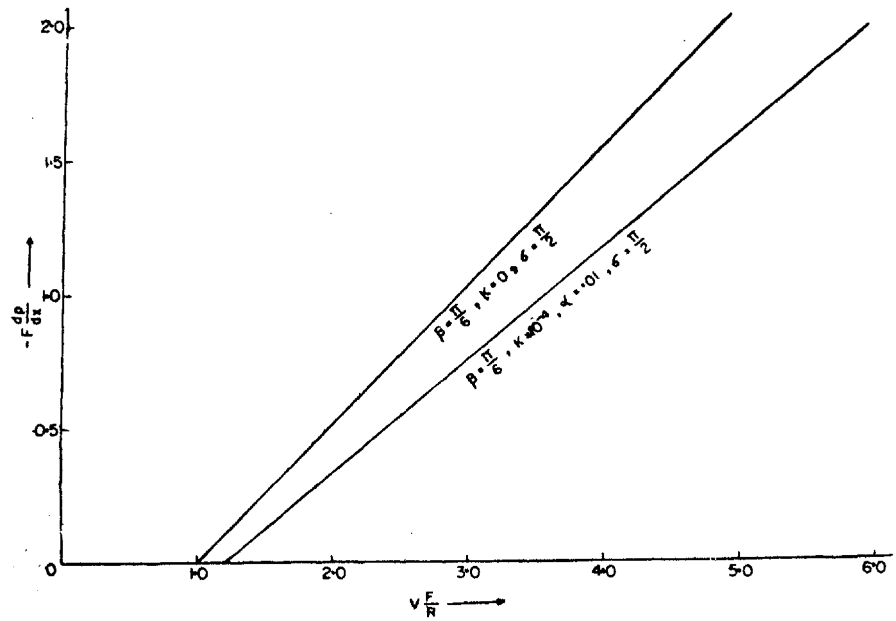

FIG. 3. Flux across the transverse cross-section of the channel.

Flux across the cross-section of the channel perpendicular to the X-axis is also shown graphically (Fig. 3) for $\beta = \frac{1}{6}\pi$, $\sigma = \frac{1}{2}\pi$, $K = 10^{-4}$, $\alpha = 0.01$ with different values of pressure gradients. The flux is also compared with that when bed is impermeable. Here, again the flux across every cross-section is greater when the bed is naturally permeable than impermeable bed. This may be due to the flow in porous media which is governed by Darcy's law and to the slip condition taken on the porous boundary.

ACKNOWLEDGEMENT

One of the authors (H.K.V.) is grateful to the University Grants Commission for awarding him a Teacher Fellowship. The authors are thankful to the referee for his valuable comments.

REFERENCES

Beavers, G. S., and Joseph, D. D. (1967). Boundary conditions at a naturally permeable wall. *J. Fluid. Mech.*, **30**, 1, 197.

Daugherty, R. L., and Franzini, J. B. (1965). Fluid Mechanics with Engineering Applications, sixth edition. McGraw-Hill Book Co., Inc., New York., pp. 286 and 287.

Johnson, J. W. (1944). Rectangular roughness in open channels. *Trans. Am. Geophysics Un.*, **25**, 906–14.

Satyaprakash (1971). Liquid flowing down an open inclined channel. *Indian J. pure appl. Math.*, **2**, 103–109.

Vanoni, V. A. (1941). Velocity distribution in open channels. *Civil Engineering*, **11**, 356–57.

Yih, C. S. (1963). Stability of liquid flowing down an inclined plane. *Phys. Fluids*, **6**, 321.