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ON COMMON FIXED POINTS OF TWO MAPPINGS

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Some results on common fixed points of two mappings have been presented.

§1. Recently, a fixed point theorem has been offered by Ćirić (1977, Theorem 3), using a quasi-contraction mapping satisfying the condition :

$$d(Tx, Ty) \leqslant \alpha \max \{2d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\},$$
$$0 \leqslant \alpha < 1$$

where T is a mapping of a metric space M into itself.

The main purpose of this note is to prove some results on common fixed points of two mappings which extend the result of Ćirić. Before we state our theorems we shall give some notations and definitions which will be used in the sequel.

The set of all non-negative integers will be denoted by I^+. Let u be a mapping of a normed linear space E into itself.

Definition 1 — A subset K of E is said to be u-convex if $u(x) + (e - u)(y) \in K$ for all $x, y \in K$, where e denotes the identity mapping of E into itself.

Definition 2 — Let d be a non-negative real valued function on $X \times X$ such that $d(x, x) = 0$ for all $x \in X$ and that $d(x, y) = d(y, x)$ for all $x, y \in X$. Then d is called a symmetric on the set X.

Definition 3 — A pair $(X, \rightarrow)$ of a set X and a subset $\rightarrow$ of the set $X^{I^+} \times X$ is called an L-space or multivalued convergence space by Novák (1964) if the following conditions are satisfied:

(1) If $x_n = x \in X$ for all $n \in I^+$, then $(\{x_n\}_{n \in I^+}, x) \in \rightarrow$.

(2) If $(s, x) \in \rightarrow$, then $(t, x) \in \rightarrow$ for every subsequence t of s.

In what follows, we shall write $s \rightarrow x$ or $x_n \rightarrow x$ instead of $(s, x) \in \rightarrow$, and read s converges to x, where $s = \{x_n\}_{n \in I^+}$. If $s = \{x_n\}_{n \in I^+}$ is a sequence in a set X, and if f is a mapping on X, then $f(s)$ denotes the sequence $\{f(x_n)\}_{n \in I^+}$.

§2. We first state the following lemma:

Lemma.— Let $(X, \rightarrow)$ be an L-space and D a family of symmetrics on X having the property that

$$d(x, y) = 0 \text{ for all } d \in D \text{ implies } x = y.$$

Let f and g be mappings of X into itself satisfying

$$d(f(x), g(y)) \leqslant \alpha(d, x, y) \,.\, \max \{cd(x, y), [d(x, f(x)) + d(y, g(y))], [d(x, g(y)) + d(y, f(x))\}$$

for all $d \in D$ and $x, y \in X$, where $c \geqslant 0$ and $\alpha : D \times X \times X \rightarrow [0, 1)$. If f has a fixed point, then it is a common fixed point of f and g.

PROOF : Let $a \in X$ be a fixed point of f. Then for each $d \in D$,

$$\begin{aligned} d(a, g(a)) &= d(f(a), g(a)) \\ &\leqslant \alpha(d, a, a) \,.\, \max \{c\, d(a, a), [d(a, f(a)) + d(a, g(a))]\} \\ &= \alpha(d, a, a) \,.\, d(a, g(a)). \end{aligned}$$

Thus, $d(a, g(a)) = 0$ and hence a is a fixed point of g.

Theorem 1 — Let u be a linear mapping of a Banach space E into itself with $\| u \| < 1$ and let S and T be mappings of a u-convex subset K of E into K satisfying

$$\| Sx - Ty \| \leqslant \alpha \,.\, \max \{c \| x - y \|, [\| x - Sx \| + \| y - Ty \|], [\| x - Ty \| + \| y - Sx \|]\} \quad \ldots(1)$$

for all $x, y \in K$, where $c > 0$ and $0 \leqslant \alpha < (1 - \| u \|)/ \| e - u \|$. If there exists an $x_0 \in K$ such that the sequence $\{x_n\}_{n \in I^+}$ defined by

$$x_{n+1} = u(x_n) + (e - u)(Sx_n) \text{ for all } n \in I^+ \quad \ldots(2)$$

or

$$x_{n+1} = u(x_n) + (e - u)(Tx_n) \text{ for all } n \in I^+ \quad \ldots(3)$$

is convergent, then the limit point is a common fixed point of S and T.

PROOF : Let a be in K such that the sequence $\{x_n\}_{n \in I^+}$ defined by (2) converges to it.

Since,

$$\begin{aligned} x_{n+1} - Ta &= u(x_n) + (e - u)(Sx_n) - Ta \\ &= (e - u)(Sx_n - Ta) - u(Ta - x_n) \end{aligned}$$

we have,

$$\begin{aligned} \| x_{n+1} - Ta \| &\leqslant \| e - u \| \| Sx_n - Ta \| + \| u \| \| Ta - x_n \| \\ &\leqslant \| u \| \| Ta - x_n \| + \| e - u \| \alpha \\ &\quad \times \max \{c \| x_n - a \|, [\| x_n - Sx_n \| + \| a - Ta \|], [\| x_n - Ta \| + \| a - Sx_n \|]\} \quad \ldots(4) \end{aligned}$$

for all $n \in I^+$. Now since $\| u \| < 1$, the mapping $e - u$ has a continuous inverse. Hence it follows from

$$(e - u)(Sx_n - x_n) = u(x_n) - x_n + (e - u)(Sx_n)$$
$$= x_{n+1} - x_n$$

that $Sx_n - x_n \to 0$ and so $Sx_n \to a$ as $n \to \infty$. Consequently passing to limit as $n \to \infty$ in (4), we have

$$\| a - Ta \| \leqslant \| u \| \, \| Ta - a \| + \| e - u \| \, . \alpha \, . \, \| a - Ta \|.$$

This implies $\| a - Ta \| = 0$ or $Ta = a$, since $\| u \| + \| e - u \| \alpha < 1$. Therefore, the theorem immediately follows from the above lemma. Now we shall state a theorem which partially extends Theorem 1.

Theorem 2 — Let $(X, \to)$ be an L-space, and D a family of continuous symmetrics on X having the property that $d(x, y) = 0$ for all $d \in D$ implies $x = y$. Let f and g be mappings of X into itself satisfying

$$d(f(x), g(y)) \leqslant \alpha(d) \, . \max \{2d(x, y), d(x, f(x)), d(y, g(y)),$$
$$d(x, g(y)), d(y, f(x))\}$$

for all $d \in D$ and $x, y \in X$, where $\alpha : D \to [0, 1)$. If the sequence $\{f^n(x)\}_{n \in I^+}$ or $\{g^n(x)\}_{n \in I^+}$ is convergent for some $x \in X$, then the limit point is a fixed point of both f and g.

PROOF : Let $f^n(x) \to a$ for some $x, a \in X$. In view of the lemma, we need only show that a is a fixed point of g. Suppose that $g(a) \neq a$. Then we can find a $d \in D$ such that $d(a, g(a)) \neq 0$. Since $(f^n(x), f^{n+1}(x)) \to (a, a)$ in the product space of $(X, \to)$ with itself, we have

$$d(f^{n_i}(x), f^{n_i+1}(x)) \to d(a, a) = 0$$

for some subsequence $\{f^{n_i}(x)\}_{n \in I^+}$ of $\{f^n(x)\}_{n \in I^+}$.

Selecting subsequences subsequently, but denoting them by the same symbol, one can assume that

$$d(f^{n_i+1}(x), g(a)) \to d(a, g(a)), \; d(f^{n_i}(x), g(a)) \to d(a, g(a))$$

and

$$d(f^{n_i}(x), a) \to d(a, a) = 0$$

as $i \to \infty$. Select η such that

$$0 < \eta < \tfrac{1}{2}[1 - \alpha(d)] \, . \, d(a, g(a)).$$

Hence there is an $m \in I^+$ such that

$$d(f^{n_i}(x), a) < d(a, g(a)), d(a, f^{n_i+1}(x))$$
$$< d(a, g(a)), d(f^{n_i}(x), f^{n_i+1}(x)) < d(a, g(a))$$

and $$d(f^{n_i}(x), g(a)) < d(a, g(a)) + \frac{\eta}{\alpha(d)}$$

for every $i \geqslant m$. Therefore we have

$$\begin{aligned} d(f^{n_i+1}(x), g(a)) &\leqslant \alpha(d) . \max \{2d(f^{n_i}(x), a), \\ &\quad d(f^{n_i}(x), f^{n_i+1}(x)), d(a, g(a)), d(f^{n_i}(x), g(a)), \\ &\quad d(a, f^{n_i+1}(x))\} \\ &\leqslant \alpha(d)\left[d(a, g(a)) + \frac{\eta}{\alpha(d)}\right] \\ &< d(a, g(a)) - \eta \end{aligned}$$

for every $i \geqslant m$, which gives

$$d(a, g(a)) \leqslant d(a, g(a)) - \eta, \text{ a contradiction.}$$

This completes the proof.

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