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ON GENERALIZED BERNSTEIN POLYNOMIALS

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Lorentz (1955) and Voronowskaja (1932) proved their results for Bernstein polynomials

$$B_n^f(x) = \sum_{k=0}^{n} f(k/n)\, p_{n,k}(x)$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-1}$$

and we have proved the corresponding results of Lorentz and Voronowskaja for Lebesgue integrable function in L_1-norm by our newly defined polynomial.

$$A_n^{\alpha}(f, x) = (n+1) \sum_{k=0}^{n} \left\{ \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)\, dt \right\} p_{n,k}(x; \alpha)$$

where

$$p_{n,k}(x; \alpha) = \binom{n}{k} \frac{x(x+k\alpha)^{k-1} (1-x+(n-k)\alpha)^{n-k}}{(1+n\alpha)^n}.$$

1. INTRODUCTION AND RESULTS

If $f(x)$ is a function defined on [0, 1], the Bernstein polynomial $B_n^f(x)$ of f is defined by

$$B_n^f(x) \sum_{k=0}^{n} f(k/n)\, p_{n,k}(x) \qquad \ldots(1.1)$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$$

and let w_f be the modulus of continuity of f defined by

$$w_f(h) = \max \{ \mid f(x) - f(y) \mid x, y \in [0, 1], \mid x - y \mid \leqslant h \}. \qquad \ldots(1.2)$$

Lorentz (1955) proved that if $w_1(\delta)$ is the modulus of continuity of $f'(x)$, then

$$\mid B_n^f(x) - f(x) \mid \leqslant \tfrac{3}{4} n^{-1/2} w_1(n^{-1/2}). \qquad \ldots(1.3)$$

Later one question arises about the rapidity of convergence of $B_n^f(x)$ to $f(x)$. An answer to this question has been given in different directions. One direction is that in which $f(x)$ is supposed to be at least twice differentiable in a point x of [0, 1].

Voronowskaja (1932) proved that

$$\lim_{n\to\infty} n \mid f(x) - B_n^f(x) \mid = -\tfrac{1}{2} x(1-x) f''(x). \qquad ...(1.4)$$

A small modification of Bernstein polynomial due to Kantorovitch (1930) makes it possible to approximate Lebesgue integrable function in L_1-norm by the modified polynomials

$$P_n^f(x) = (n+1) \sum_{k=0}^{n} \left\{ \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)\, dt \right\} p_{n,k}(x) \qquad ...(1.5)$$

where $p_{n,k}(x)$ is defined by (1.2).

By Jensen's formula

$$(x+y+n\alpha)^n = \sum_{k=0}^{n} \binom{n}{k} x(x+k\alpha)^{k-1} (y+(n-k)\alpha)^{n-k}. \qquad ...(1.6)$$

If we put $y = 1 - x$, we obtain

$$(1+n\alpha)^n = \sum_{k=0}^{n} \binom{n}{k} x(x+k\alpha)^{k-1} (1-x+(n-k)\alpha)^{n-k}$$

or

$$1 = \sum_{k=0}^{n} \binom{n}{k} \frac{x(x+k\alpha)^{k-1} (1-x+(n-k)\alpha)^{n-k}}{(1+n\alpha)^n}. \qquad ...(1.7)$$

Thus defining

$$p_{n,k}(x;\alpha) = \binom{n}{k} \frac{x(x+k\alpha)^{k-1} (1-x+(n-k)\alpha)^{n-k}}{(1+n\alpha)^n} \qquad ...(1.8)$$

we have

$$\sum_{k=0}^{n} p_{n,k}(x;\alpha) = 1. \qquad \text{[by (1.7)]} \qquad ...(1.9)$$

Now we define the polynomial

$$A_n^\alpha(f, x) = (n+1) \sum_{k=0}^{n} \left\{ \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)\, dt \right\} p_{n,k}(x;\alpha) \qquad ...(1.10)$$

where $p_{n,k}(x;\alpha)$ is defined in (1.8) and moreover when $\alpha = 0$, (1.8) and (1.10) reduces to (1.2) and (1.5) respectively.

In this paper, we shall prove the corresponding results of approximation due to Lorentz and Voronowskaja for Lebesgue integrable function in L_1-norm by our polynomial (1.10). In fact we state our results as follows:

Theorem 1.1 — Let f be a continuous Lebesgue integrable function on [0, 1] such that its first derivative is bounded. If $w_1(\delta)$ is the modulus of continuity of $f(x)$, then for $\alpha = \alpha_n = o(1/n)$, we have

$$| A_n^{\alpha}(f, x) - f(x) | \leqslant 7n^{-1/2} w_f(n^{-1/2})/4.$$

Theorem 1.2 — Let $f(x)$ be bounded Lebesgue integrable function with its first derivative in [0, 1] and suppose that the second derivative $f''(x)$ exists at a certain point x of [0, 1], then for $\alpha = \alpha_n = o(1/n)$

$$\lim_{n\to\infty} n\,[A_n^{\alpha}(f, x) - f(x)] = \tfrac{1}{2}\,[(1-2x)\,f'(x) - x(1-x)\,f''(x)].$$

2. Proof of Theorems

We, first, prove some lemmas which would be useful for the proof of our theorems.

Lemma 2.1 — For all values of x

$$\sum_{k=0}^{n} k p_{n,k}(x;\alpha) \leqslant nx/(1+\alpha).$$

Lemma 2.2 — For all values of x

$$\sum_{k=0}^{n} k(k-1)\,p_{n,k}(x,\alpha) \leqslant n(n-1)\,x\left\{\frac{x+2\alpha}{(1+2\alpha)^2} + \frac{(n-2)\,\alpha^2}{(1+3\alpha)^3}\right\}.$$

Lemma 2.3 — For all values of $x \in |\,0, 1\,|$ and for $\alpha = \alpha_n = o(1/n)$, we have

$$(n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)} (t-x)^2\,dt\right\} p_{n,k}(x;\alpha) \leqslant x(1-x)/n.$$

Before giving the proofs of the lemmas we would like to illustrate some functions (Cheney and Sharma 1964), which are helpful for the proof of our lemmas.

The functions

$$S(\nu, n, x, y) = \sum_{k=0}^{n}\binom{n}{k}(x+k\alpha)^{k+\nu-1}\,[y+(n-k)\,\alpha]^{n-k}$$

satisfy the reduction formula

$$S(\nu, n, x, y) = xS(\nu - 1, n, x, y) + n\alpha S(\nu, n - 1, x + \alpha, y).$$

By repeated use of the reduction formula, we can show that

$$S(1, n, x, y) = \sum_{k=0}^{n} \binom{n}{k} k!\, \alpha^k (x + y + n\alpha)^{n-k} \qquad \text{[by (1.6)]}$$

as $xS(0, n, x, y) = (x + y + n\alpha)^n$.

Since $k! = \int_0^\infty t^k e^{-t} dt$ (Euler's equation of Gamma function) and therefore, we get

$$S(1, n, x, y) = \sum_{k=0}^{n} \binom{n}{k} \int_0^\infty e^{-t} t^k dt\, \alpha^k (x + y + n\alpha)^{n-k}$$

$$= \int_0^\infty e^{-t}\, dt \sum_{k=0}^{n} \binom{n}{k} t^k \alpha^k (x + y + n\alpha)^{n-k}$$

$$= \int_0^\infty e^{-t}\, dt \left[\sum_{k=0}^{n} {}^n c_k (t\alpha)^k (x + y + n\alpha)^{n-k} \right]$$

$$= \int_0^\infty e^{-t}\, dt\, [{}^n c_0 (x + y + n\alpha)^n + {}^n c_1 (t\alpha)(x + y + n\alpha)^{n-1} + \ldots + {}^n c_n (t\alpha)^n].$$

Using binomial theorem, we get

$$= \int_0^\infty e^{-t}\, dt\, [\{(x + y + n\alpha) + t\alpha\}^n]$$

$$= \int_0^\infty e^{-t} (x + y + n\alpha + t\alpha)^n\, dt. \qquad \ldots(1.11)$$

Similarly

$$S(2, n, x, y) = \sum_{k=0}^{\infty} (x + k\alpha) \binom{n}{k} k!\, \alpha^k S(1, n - k, x + k\alpha, y)$$

becomes

$$S(2, n, x, y) = \int_0^\infty e^{-t}\, dt \int_0^\infty e^{-s}\, ds\, \{x(x + y + n\alpha + t\alpha + s\alpha)^n + n\alpha^2 s (x + y + n\alpha + t\alpha + s\alpha)^{n-1}\}. \qquad (1.12)$$

Proof of Lemma 2.1

$$\sum_{k=0}^{n} kp_{n,k}(x;\alpha) = \sum_{k=0}^{n} k\binom{n}{k}\frac{x(x+k\alpha)^{k-1}(1-x+(n-k)\alpha)^{n-k}}{(1+n\alpha)^n}$$

$$= nx\sum_{k=1}^{n}\binom{n-1}{k-1}\frac{x(x+k\alpha)^{k-1}(1-x+(n-k)\alpha)^{n-k}}{(1+n\alpha)^n}$$

$$= \frac{nx}{(1+n\alpha)^n}\,S(1, n-1, x+\alpha, 1-x)$$

$$= \frac{nx}{(1+n\alpha)^n}\int_0^\infty e^{-t}(1+n\alpha+t\alpha)^{n-1}\,dt \qquad \text{(by eqn. 1.11)}$$

$$= \frac{nx}{(1+n\alpha)^n}\int_0^\infty e^{-t}\left(1+\frac{t\alpha}{1+n\alpha}\right)^{n-1}dt(1+n\alpha)^{n-1}$$

$$= \frac{nx}{\alpha}\int_0^\infty e^{-((1+n\alpha)/\alpha)u}(1+u)^{n-1}\,du$$

$$\leqslant \frac{nx}{\alpha}\int_0^\infty e^{-((1/\alpha)+n)u}\,e^{(n-1)u}\,du$$

$$= \frac{nx}{\alpha}\int_0^\infty e^{-((1/\alpha)+1)u}\,du$$

$$= \frac{nx}{\alpha}\int_0^\infty e^{-u'}\,\frac{\alpha}{1+\alpha}\,du'$$

$$= \frac{nx}{1+\alpha}\int_0^\infty e^{-u'}\,du'$$

$$= \frac{nx}{1+\alpha}$$

and therefore

$$\sum_{k=0}^{n} kp_{n,k}(x;\alpha) \leqslant nx/(1+\alpha)$$

which completes the proof.

Proof of Lemma 2.2

$$\sum_{k=0}^{n} k(k-1)\, p_{n,k}(x;\alpha)$$

$$= \frac{n(n-1)\,x}{(1+n\alpha)^n} \sum_{k=2}^{n} \binom{n-2}{k-2} (x+k\alpha)^{k-1} (1-x+(n-k)\,\alpha)^{n-k}$$

$$= \frac{n(n-1)\,x}{(1+n\alpha)^n} S(2, n-2, x+2\alpha, 1-x)$$

$$= \frac{n(n-1)\,x}{(1+n\alpha)^n} \Big[\int_0^\infty e^{-t}\,dt \int_0^\infty e^{-s}\,ds\,(x+2\alpha)\,(1+n\alpha+t\alpha+s\alpha)^{n-2}$$

$$+ (n-2)\,\alpha^2 s (1+n\alpha+t\alpha+s\alpha)^{n-3} \Big] \qquad \text{(by eqn. 1.12)}$$

$$= \frac{n(n-1)\,x}{(1+n\alpha)^n} (x+2\alpha) \int_0^\infty e^{-t}\,dt \int_0^\infty e^{-s}(1+n\alpha+t\alpha+s\alpha)^{n-2}\,ds$$

$$+ \frac{n(n-1)\,(n-2)\,\alpha^2 x}{(1+n\alpha)^n} \int_0^\infty e^{-t}\,dt \int_0^\infty s e^{-s}(1+n\alpha+t\alpha+s\alpha)^{n-3}\,ds$$

$$= I_1 + I_2 \text{ (say)}. \qquad \ldots(2.1)$$

Now we evaluate I_1 :

$$I_1 = \frac{n(n-1)}{(1+n\alpha)^n} x(x+2\alpha) \int_0^\infty e^{-t}\,dt \int_0^\infty e^{-s} \left(1 + \frac{t\alpha + s\alpha}{1+n\alpha}\right)^{n-2} ds\,(1+n\alpha)^{n-2}$$

$$\leqslant \frac{n(n-1)}{(1+n\alpha)^2} x(x+2\alpha) \int_0^\infty e^{-t}\,dt \int_0^\infty e^{-s} \exp \Big\{(n-2)$$

$$\times \left(\frac{t\alpha}{1+n\alpha} + \frac{s\alpha}{1+n\alpha}\right)\Big\}\,ds$$

$$= \frac{n(n-1)}{(1+n\alpha)^2} x(x+2\alpha) \int_0^\infty \exp\left(-t\left(\frac{1+2\alpha}{1+n\alpha}\right)\right) dt$$

$$\times \int_0^\infty \exp\left(-s\left(\frac{1+2\alpha}{1+n\alpha}\right)\right) ds$$

(*equation continued on p.* 183)

$$= \frac{n(n-1)\,x(x+2\alpha)}{(1+n\alpha)^2} \int_0^\infty e^{-t'} \frac{1+n\alpha}{1+2\alpha}\,dt' \int_0^\infty e^{-s'} \frac{1+n\alpha}{1+2\alpha}\,ds'$$

$$= \frac{n(n-1)\,x(x+2\alpha)}{(1+2\alpha)^2} \int_0^\infty e^{-t'}\,dt' \int_0^\infty e^{-s'}\,ds'$$

$$I_1 \leqslant \frac{n(n-1)\,x(x+2\alpha)}{(1+2\alpha)^2}. \qquad \ldots(2.2)$$

We evaluate I_2 :

$$I_2 = \frac{n(n-1)\,(n-2)\,\alpha^2 x}{(1+n\alpha)^n} \int_0^\infty e^{-t}\,dt \int_0^\infty se^{-s}(1+n\alpha+t\alpha+s\alpha)^{n-3}\,ds$$

$$= \frac{n(n-1)\,(n-2)\,\alpha^2 x}{(1+n\alpha)^3} \int_0^\infty e^{-t}\,dt \int_0^\infty se^{-s}\left(1+\frac{t\alpha+s\alpha}{1+n\alpha}\right)^{n-3} ds$$

$$\leqslant \frac{n(n-1)\,(n-2)\,\alpha^2 x}{(1+n\alpha)^3} \int_0^\infty e^{-t}\,dt$$

$$\times \int_0^\infty se^{-s} \exp\left((n-3)\left(\frac{t\alpha}{1+n\alpha}+\frac{s\alpha}{1+n\alpha}\right)\right) ds$$

$$= \frac{n(n-1)(n-2)\,\alpha^2 x}{(1+n\alpha)^3} \int_0^\infty \exp\left(-t(1+3\alpha)/(1+n\alpha)\right) dt$$

$$\times \int_0^\infty s \exp\left(-s(1+3\alpha)/(1+n\alpha)\right) ds$$

$$= \frac{n(n-1)\,(n-2)\,\alpha^2 x}{(1+3\alpha)^3} \int_0^\infty e^{-t'}\,dt' \int_0^\infty s' e^{-s'}\,ds'$$

$$I_2 \leqslant \frac{n(n-1)\,(n-2)\,\alpha^2 x}{(1+3\alpha)^3} \qquad \ldots(2.3)$$

and hence from (2.1), (2.2) and (2.3), we have

$$\sum_{k=0}^{n} k(k-1)\,p_{n,k}(x;\alpha) \leqslant n(n-1)\,x\left[\frac{x+2\alpha}{(1+2\alpha)^2}+\frac{(n-2)\,\alpha^2}{(1+3\alpha)^3}\right]$$

which completes the proof.

Proof of Lemma 2.3 :

$$(n+1)\sum_{k=0}^{n}\left(\int_{k/(n+1)}^{(k+1)/(n+1)}(t-x)^2\,dt\right)p_{n,k}(x;\alpha)$$

$$=\sum_{k=0}^{n}\left[x^2-\frac{2kx+x}{(n+1)}+\frac{k^2+k}{(n+1)^2}+\frac{1}{3(n+1)^2}\right]p_{n,k}(x;\alpha)$$

$$\leqslant x^2-\frac{1}{(n+1)}\left[\frac{2nx^2}{1+\alpha}+x\right]+\frac{1}{(n+1)^2}\left[n(n-1)\,x\,\frac{x+2\alpha}{(1+2\alpha)^2}+\frac{(n-2)\,\alpha^2}{(1+3\alpha)^3}+\frac{2nx}{1+\alpha}\right]+\frac{1}{3(n+1)^2}\text{ (by Lemmas 2.1 and 2.2)}$$

$$\leqslant\frac{1}{n(1-\alpha)(1+2\alpha)^2(1+3\alpha)^3}\,[x(1-x)+\alpha x(1-x)(2n+9)+x$$
$$+\alpha^2x(1-x)(17n+23)+9x$$
$$+\alpha^3x(1-x)(57n-13)+7nx^2+x(5n^2+35)$$
$$+\alpha^4x(1-x)(96n-144)+86nx^2+(65-12n)$$
$$+\alpha^5x(1-x)(54n-216)+x(4n^2-12n+46)+162nx^2$$
$$+\alpha^6-108x(1-x)+108nx^2]+1/3n^2$$

$$\leqslant\frac{x(1-x)}{n}\text{ for }\alpha=\alpha_n=o\left(\frac{1}{n}\right)\text{ and for large }n$$

which completes the proof of Lemma 2.3.

Proof of Theorem 1.1

For arbitrary x', x'' in $[0, 1]$ and $\delta > 0$ we denote $\lambda = \lambda(x', x''; \delta)$ the integer $[\,|\,x' - x''\,|\,\delta^{-1}]$, the difference $f(x') - f(x'')$ is then a sum of $(\lambda + 1)$ difference of $f'(x)$ on intervals of length less than δ, therefore

$$|\,f'(x') - f'(x'')\,| \leqslant (\lambda + 1)\, w_1(\delta).$$

We have

$$f(x_1) - f(x_2) = (x_1 - x_2)\, f'(u) + (x_1 - x_2)\,[f'(u) - f'(x_1)]$$

where $x_1 < u < x_2$. The absolute value of the last term does not exceed

$$|\,x_1 - x_2\,|\,(\lambda + 1)\, w_1(\delta). \qquad \text{(by hypothesis)}$$

From the above condition, we have

$$\left|\,A_n^{\alpha}(f, x) - f(x)\right| = \left|(n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)} f(x) - f(t)\,dt\right\}p_{n,k}(x;\alpha)\right|$$

(*equation continued on p.* 185)

$$\leqslant \left|(n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)}(x-t)f'(x)\,dt\right\}p_{n,k}(x;\alpha)\right|$$

$$+\left|(n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)}(x-t)(f'(u)-f'(x))\,dt\right\}p_{n,k}(x;\alpha)\right|$$

$$\leqslant \left|(n+1)\sum_{k=0}^{n}\left[\frac{x}{n+1}-\frac{2k+1}{2(n+1)}\right]f'(x)\,p_{n,k}(x;\alpha)\right|$$

$$+(n+1)\,w_1(\delta)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)}|\,t-x\,|\,(1+\lambda)\,dt\right\}p_{n,k}(x;\alpha)$$

$$=\left|\sum_{k=0}^{n}\left[x-\frac{2k+1}{2(n+1)}\right]f'(x)\,p_{n,k}(x;\alpha)\right|$$

$$+w_1(\delta)\,(n+1)\left[\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)}|\,t-x\,|\,dt\right\}p_{n,k}(x;\alpha)\right.$$

$$\left.+\sum_{\lambda\geqslant 1}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)}|\,t-x\,|\,\lambda(t,x;\delta)\,dt\right\}p_{n,k}(x;\alpha)\right]$$

$$\leqslant\left|\left[x-\frac{nx}{(1+\alpha)(n+1)}-\frac{1}{2(n+1)}\right]f'(x)\right|$$

$$+w_1(\delta)\,(n+1)\left[\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)}|\,t-x\,|\,dt\right\}p_{n,k}(x;\alpha)\right]$$

$$+w_1(\delta)\,(n+1)\left[\delta^{-1}\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)}(t-x)^2\,dt\right\}p_{n,k}(x;\alpha)\right]$$

$$=I_3+I_4+I_5,\ \text{(say)} \qquad \ldots(2.4)$$

where

$$I_3=\left|\left[\frac{x(1+\alpha)(n+1)-nx}{(1+\alpha)(n+1)}-\frac{1}{2(n+1)}\right]f'(x)\right|$$

(*equation continued on p.* 186)

$$= \left| \frac{(2x-1)(1+\alpha)+2nx_\alpha}{2(1+\alpha)(n+1)} f'(x) \right|$$

$$\leqslant \frac{M}{n} \qquad \text{...(2.5)}$$

where $|f'(x)| \leqslant M$ and $\alpha = \alpha_n = o\left(\frac{1}{n}\right)$.

We evaluate I_4:

$$I_4 = w_1(\delta)(n+1)\left[\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)} |t-x|\,dt\right\} p_{n,k}(x;\alpha)\right]$$

Now applying Cauchy's inequality in the above equation, we have

$$I_4 = w_1(\delta)(n+1)\left[\left\{\sum_{k=0}^{n}\int_{k/(n+1)}^{(k+1)/(n+1)} (|t-x|\,p_{n,k}^{1/2}(x;\alpha))^2\,dt\right\}^{1/2}\right.$$

$$\left.\times\left\{\sum_{k=0}^{n}\int_{k/(n+1)}^{(k+1)/(n+1)} (p_{n,k}^{1/2}(x;\alpha))^2\,dt\right\}^{1/2}\right]$$

$$= w_1(\delta)\left[\left\{(n+1)\sum_{k=0}^{n}\int_{k/(n+1)}^{(k+1)/(n+1)} (t-x)^2\,p_{n,k}(x;\alpha)\right\}^{1/2}\right.$$

$$\left.\times\left\{(n+1)\sum_{k=0}^{n}\left(\int_{k/(n+1)}^{(k+1)/(n+1)} dt\right) p_{n,k}(x;\alpha)\right\}^{1/2}\right]$$

$$= w_1(\delta)\left[(n+1)\sum_{k=0}^{n}\left(\int_{k/(n+1)}^{(k+1)/(n+1)} (t-x)^2\,dt\right) p_{n,k}(x;\alpha)\right]^{1/2}$$

since $x(1-x) \leqslant \frac{1}{4}$ on $x \in [0, 1]$, then by Lemma 2.3, we have

$$\leqslant w_1(\delta)(1/4n)^{1/2}$$

and therefore

$$I_4 \leqslant w_1(\delta)\, 1/2\sqrt{n} \qquad \text{...(2.6)}$$

and

$$I_5 = w_1(\delta)(n+1)\left[\delta^{-1}\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)} (t-x)^2\,dt\right\} p_{n,k}(x;\alpha)\right]$$

$$\leqslant w_1(\delta)\,\delta^{-1}(1/4n). \qquad \text{...(2.7)}$$

{by Lemma 2.3 and also since $x(1-x) \leqslant \frac{1}{4}$ on $x \in [0, 1]$}

Hence from (2.4), (2.5), (2.6) and (2.7), we have

$$| A_n^\alpha (f, x) - f(x) | \leqslant \frac{M}{n} + (w_1(\delta)/2\sqrt{n}) + w_1(\delta)\, \delta^{-1}(1/4n)$$
$$= \frac{M}{n} + w_1(\delta)\, [1/2\, \sqrt{n} + \delta^{-1}(1/4n)].$$

For $\delta = n^{-1/2}$, we obtain

$$= w_1(n^{-1/2})\, [\tfrac{3}{4}\, n^{-1/2}] + \frac{M}{n}$$
$$= w_1(n^{-1/2})\, (\tfrac{3}{4} n^{-1/2}) + n^{-1/2}\, w_1(n^{-1/2})$$
$$= 7 n^{-1/2}\, w_1(n^{-1/2})/4$$

which completes the proof.

Proof of Theorem 1.2

We write (in view of Taylor's theorem)

$$f(t) = f(x) + (t - x) f'(x) + (t - x)^2 [\tfrac{1}{2} f''(x) + \eta(t - x)] \qquad \text{...(2.8)}$$

where $\eta(h)$ is bounded $| \eta(h) | \leqslant H$ for all h and converges to zero with h.

Multiplying eqn. (2.8) by $(n + 1)\, p_{n,k}(x; \alpha)$ and integrating it from $k/(n + 1)$ to $(k + 1)/(n + 1)$, then on summing, we get

$$(n + 1) \sum_{k=0}^{n} \left\{ \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)\, dt \right\} p_{n,k}(x; \alpha)$$
$$= (n + 1) \sum_{k=0}^{n} \left\{ \int_{k/(n+1)}^{(k+1)/(n+1)} f(x)\, dt \right\} p_{n,k}(x; \alpha)$$
$$+ (n + 1) \sum_{k=0}^{n} \left\{ \int_{k/(n+1)}^{(k+1)/(n+1)} (t - x) f'(x)\, dt \right\} p_{n,k}(x; \alpha)$$
$$+ \tfrac{1}{2}(n + 1) \sum_{k=0}^{n} \left\{ \int_{k/(n+1)}^{(k+1)/(n+1)} (t - x)^2 f''(x)\, dt \right\} p_{n,k}(x; \alpha)$$
$$+ (n + 1) \sum_{k=0}^{n} \left\{ \int_{k=(n+1)}^{(k+1)/(n+1)} (t - x)^2\, \eta(t - x)\, dt \right\} p_{n,k}(x; \alpha)$$
$$= I_6 + I_7 + I_8 + I_9 \text{ (say).} \qquad \text{...(2.9)}$$

Now first we evaluate I_6:

$$I_6 = (n+1) \sum_{k=0}^{n} \left\{ \int_{k/(n+1)}^{(k+1)/(n+1)} f(x)\, dt \right\} p_{n,k}(x;\alpha)$$

$$= f(x) \qquad \ldots(2.10)$$

and then

$$I_7 = (n+1) \sum_{k=0}^{n} \left\{ \int_{k/(n+1)}^{(k+1)/(n+1)} (t-x)\, f'(x)\, dt \right\} p_{n,k}(x;\alpha)$$

$$= \sum_{k=0}^{n} \left(\frac{2k+1}{2(n+1)} - x \right) f'(x)\, p_{n,k}(x;\alpha)$$

$$\leqslant \frac{(1-2x)}{2n} f'(x), \quad \text{for } \alpha = \alpha_n = o\left(\frac{1}{n}\right). \qquad \ldots(2.11)$$

Now we evaluate I_8:

$$I_8 = \tfrac{1}{2}(n+1) \sum_{k=0}^{n} \left\{ \int_{k/(n+1)}^{(k+1)/(n+1)} (t-x)^2 f''(x)\, dt \right\} p_{n,k}(x;\alpha)$$

$$\leqslant x(1-x)\, f''(x)/2n \qquad \text{(by Lemma 2.3)} \qquad \ldots(2.12)$$

and then at last we evaluate I_9:

$$I_9 = (n+1) \sum_{k=0}^{n} \left\{ \int_{k/(n+1)}^{(k+1)/(n+1)} (t-x)^2\, (t-x)\, dt \right\} p_{n,k}(x;\alpha)$$

I_9 can be estimated easily. Let $\epsilon > 0$ be arbitrary and $\delta > 0$ such that $|\eta(h)| < \epsilon$ for $|h| < \delta$. Thus breaking up the sum I_9 into two parts corresponding to those values of t for which $|t-x| < \delta$ and those for which $|t-x| \geqslant \delta$, and since in the given range of t, $\left|\frac{k}{n} - x\right| \sim |t-x|$, we have

$$|I_9| \leqslant \epsilon \sum_{|(k/n)-x|<\delta} (n+1)\, p_{n,k}(x;\alpha) \left| \int_{k/(n+1)}^{(k+1)/(n+1)} (t-x)^2\, dt \right|$$

$$+ H \sum_{|(k/n)-x|\geqslant\delta} (n+1)\, p_{n,k}(x;\alpha) \left| \int_{k/(n+1)}^{(k+1)/(n+1)} dt \right|$$

$$= I_{10} + I_{11} \text{ (say)}$$

$$| I_{10} | \leqslant \frac{\epsilon}{n} | \{x(1-x)\} | , \text{ for } \alpha = \alpha_n = o\left(\frac{1}{n}\right)$$

$$I_{11} = (n+1) H \sum_{|(k/n)-x| \geqslant \delta} \left\{ \int_{k/(n+1)}^{(k+1)/(n+1)} dt \right\} p_{n,k}(x;\alpha)$$

$$= (n+1) \sum_{|(k/n)-x| \geqslant \delta} p_{n,k}(x,\alpha) \frac{1}{n+1}$$

But if $\delta = n^{-\beta}$, $0 < \beta < \frac{1}{2}$ (see also Kantorovitch 1930), then for $\alpha = \alpha_n = o\left(\frac{1}{n}\right)$

$$\sum_{|(k/n)-x| \geqslant n^{-\beta}} p_{n,k}(x;\alpha) \leqslant Cn^{-\nu}$$

for $\nu > 0$, the constant $C = C(\beta, \nu)$.

Whence $I_{11} < \epsilon/(n+1) < \epsilon/n$ for all n sufficiently large and therefore it follows readily that

$$I_9 < \epsilon/n, \text{ for all sufficiently large } n. \qquad \ldots(2.13)$$

Hence from (2.9), (2.10), (2.11), (2.12) and (2.13), we have

$$(n+1) \sum_{k=0}^{n} \left\{ \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)\, dt \right\} p_{n,k}(x;\alpha)$$

$$= f(x) + [\{(1-2x) f'(x) + x(1-x) f''(x)\}/2n] + (\epsilon_n/n)$$

and therefore, finally we get

$$\lim_{n\to\infty} n [A_n^{\alpha} (f, x) - f(x)] = \tfrac{1}{2} [(1-2x) f'(x) + x(1-x) f''(x)]$$

where $\epsilon_n \to 0$ as $n \to \infty$

which completes the proof.

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