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ON NÖRLUND SUMMABILITY OF FOURIER SERIES AND ITS CONJUGATE SERIES

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In this paper the authors have obtained two theorems for the Nörlund summability of Fourier series and its conjugate series respectively under very general conditions.

§1. Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series with the sequence of partial sums $\{S_n\}$. Let $\{p_n\}$ be a sequence with $p_0 > 0$ and $p_n \geqslant 0$ for $n > 0$, and let

$$P_n = p_0 + p_1 + p_2 + \dots + p_n, \ (P_{-1} = p_{-1} = 0).$$

Let

$$t_n = \sum_{\nu=0}^{n} \frac{p_{n-\nu} S_\nu}{P_n}, \ (P_n \neq 0)$$

or

$$t_n = P_n^{-1} \sum_{\nu=0}^{n} p_\nu S_{n-\nu}. \qquad \dots(1.1)$$

If $t_n \to S$ as $n \to \infty$ we write

$$\sum_{n=0}^{\infty} a_n = S(N, p_n)$$

or

$$S_n \to S(N, p_n).$$

The conditions of regularity of the method of summability (N, p_n) defined by (1.1) is

$$\lim_{n\to\infty} \frac{p_n}{P_n} = 0. \qquad \dots(1.2)$$

§2. Let $f(t)$ be 2π-periodic function and L-integrable over an interval $(-\pi, \pi)$. Let the Fourier series of $f(t)$ be given by

$$\tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \sum_{n=0}^{\infty} A_n(t) \qquad \dots(2.1)$$

and then the conjugate series of (2.1) is

$$\sum_{n=1}^{\infty} (b_n \cos nt - a_n \sin nt) \equiv \sum_{n=1}^{\infty} B_n(t). \qquad \ldots(2.2)$$

We shall use the following notations:

$$\phi(t) = \phi(x, t) = f(x+t) + f(x-t) - 2f(x)$$

$$\psi(t) = \psi(x, t) = f(x+t) - f(x-t)$$

$$\Phi(t) = \int_0^t |\phi(u)| \, du$$

$$\Psi(t) = \int_0^t |\psi(u)| \, du$$

$$N_n(t) = \frac{1}{2\pi P_n} \sum_{\nu=0}^{n} p_\nu \frac{\sin (n - \nu + \frac{1}{2}) t}{\sin \frac{1}{2} t}$$

$$\bar{N}_n(t) = \frac{1}{2\pi P_n} \sum_{\nu=0}^{n} p_\nu \frac{\cos (n - \nu + \frac{1}{2}) t}{\sin \frac{1}{2} t}$$

$$p_{(1/t)} = p_\tau \quad \text{and} \quad P_{(1/t)} = P_\tau$$

where τ denotes the integral part of $1/t$.

§3. Siddiqi (1948) proved the following theorems:

Theorem A — If

$$\Phi(t) = \int_0^t |\phi(u)| \, du = o\left[\frac{t}{\log (1/t)}\right] \qquad \ldots(3.1)$$

as $t \to +0$, then the series (2.1), at $t = x$, is summable (H) to $f(x)$.

He also proved the corresponding theorem for the conjugate series (2.2). His theorem concerning the conjugate series is

Theorem B — If

$$\Psi(t) = \int_0^t |\psi(u)| \, du = o\left[\frac{t}{\log (1/t)}\right] \qquad \ldots(3.2)$$

as $t \to +0$, then the conjugate series (2.2) is summable (H) to

$$\frac{1}{2\pi} \int_0^{\pi} \psi(t) \cot \tfrac{1}{2} t \, dt,$$

at points where this integral exists.

In this direction, Pati (1961) has proved the following theorem.

Theorem C — If (N, p_n) be a regular Nörlund method, defined by a real, non-negative, monotonic non-increasing sequence of coefficients $\{p_n\}$, such that $P_n \to \infty$, and $\log n = O(P_n)$, as $n \to \infty$, then, if

$$\Phi(t) = \int_0^t | \phi(u) | \, du = o\,[t/P_\tau] \qquad ...(3.3)$$

as $t \to +0$, the Fourier series of $f(t)$, at $t = x$, is summable (N, p_n) to $f(x)$.

The object of the present paper is to obtain, for the Nörlund summability of Fourier series and its conjugate series a criterion of a different type replacing the conditions (3.1) and (3.2) by more general conditions. We prove the following theorems.

§4. *Theorem* 1 — Let $\lambda(t)$ and $K(t)$ be two positive functions. If

$$\Phi(t) = \int_0^t | \phi(u) | \, du = o\,[\lambda(1/t)\, p_\tau / K(P_\tau)], \quad \text{as} \quad t \to 0 \qquad ...(4.1)$$

and

$$\lambda(n)\, P_n = O\,[K(P_n)] \qquad ...(4.2)$$

as $n \to \infty$, then Fourier series of $f(t)$ at $t = x$ is summable (N, p_n) to $f(x)$ where $\{p_n\}$ is a real non-negative and non-increasing sequence such that $P_n \to \infty$, as $n \to \infty$.

Theorem 2 — Let the sequence $\{p_n\}$ and the functions $\lambda(t)$ and $K(t)$ be the same as in Theorem 1. Then if,

$$\Psi(t) = \int_0^t | \psi(u) | \, du = o\,[\lambda(1/t)\, p_\tau / K(P_\tau)] \qquad ...(4.3)$$

as $t \to +0$, then the conjugate series (2.2) is summable (N, p_n) to

$$\frac{1}{2\pi} \int_0^{\pi} \psi(t) \cot \tfrac{1}{2} t \, dt$$

at every point where this integral exists. For the proof of our theorems we need the following lemmas.

Lemma 1 (McFadden 1942) — If $\{p_n\}$ is non-negative and non-increasing, then for $0 \leqslant a < b \leqslant \infty$, $0 \leqslant t \leqslant \pi$, and any n

$$\left| \sum_{\nu=a}^{b} p_\nu e^{i(n-\nu)t} \right| \leqslant AP_\tau,$$

where A is an absolute constant.

Lemma 2 — If $0 \leqslant t \leqslant \frac{1}{n}$, then

$$N_n(t) = O(n),$$

we have

$$| N_n(t) | = \frac{1}{2\pi P_n} \left| \sum_{\nu=0}^{n} p_\nu \frac{\sin (n - \nu + \frac{1}{2}) t}{\sin \frac{1}{2} t} \right|$$

$$= O \left\{ P_n^{-1} \sum_{\nu=0}^{n} p_\nu \frac{(2n - 2\nu + 1) \mid \sin \frac{1}{2} t \mid}{\mid \sin \frac{1}{2} t \mid} \right\}$$

$$= O \left\{ (2n + 1) P_n^{-1} \sum_{\nu=0}^{n} p_\nu \right\}$$

$$= O(n), \quad \text{as} \quad n \to \infty.$$

Lemma 3 — For $\frac{1}{n} \leqslant t \leqslant \delta < \pi$,

$$N_n(t) = O\,[P_\tau / P_n t].$$

PROOF : We have

$$| N_n(t) | = \frac{1}{2\pi P_n} \left| \sum_{\nu=0}^{n} p_\nu \frac{\sin (n - \nu + \frac{1}{2}) t}{\sin \frac{1}{2} t} \right|$$

$$= \frac{1}{2\pi P_n \mid \sin \frac{1}{2} t \mid} \left| I_m \sum_{\nu=0}^{n} p_\nu \exp\,(i(n - \nu + \tfrac{1}{2})\,t) \right|$$

$$= \frac{1}{2\pi P_n \mid \sin \frac{1}{2} t \mid} \left| I_m \left(e^{it/2} \sum_{\nu=0}^{n} p_\nu \exp\,(i(n - \nu)\,t) \right) \right|$$

$$\leqslant \frac{1}{2\pi P_n t} \left| \sum_{\nu=0}^{n} p_\nu \exp\,(i(n - \nu)\,t) \right|$$

$$= O\,[P_\tau / P_n t], \text{ by Lemma 1.}$$

Lemma 4 — If $\frac{1}{n} \leqslant t \leqslant \delta < \pi$, then

$$\bar{N}_n(t) = \frac{1}{2\pi P_n} \sum_{\nu=0}^{n} p_\nu \frac{\cos(n-\nu+\frac{1}{2})t}{\sin\frac{1}{2}t}$$

$$= O[P_\tau/P_n t].$$

The proof is similar to that of Lemma 3.

§5. *Proof of the Theorem* 1 — Let

$$S_n(x) = \sum_{\nu=1}^{n} A_\nu(x)$$

then, we have

$$S_n(x) - f(x) = \frac{1}{2\pi}\int_0^\pi \phi(t)\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\,dt.$$

Using (1.1), we get

$$t_n - f(x) = P_n^{-1} \sum_{\nu=0}^{n} p_\nu [S_{n-\nu}(x) - f(x)]$$

$$= P_n^{-1} \sum_{\nu=0}^{n} p_\nu \frac{1}{2\pi}\int_0^\pi \phi(t)\frac{\sin(n-\nu+\frac{1}{2})t}{\sin\frac{1}{2}t}\,dt$$

$$= \int_0^\pi \phi(t)\left\{\frac{1}{2\pi P_n}\sum_{\nu=0}^{n} p_\nu \frac{\sin(n-\nu+\frac{1}{2})t}{\sin\frac{1}{2}t}\right\}dt$$

$$= \int_0^\pi \phi(t)\,N_n(t)\,dt \ (= M \text{ say}).$$

In order to prove the theorem, we have to show that, under our assumptions,

$$\int_0^\pi \phi(t)\,N_n(t)\,dt = o(1), \quad \text{as} \quad n \to \infty.$$

We write, for $0 < \delta < \pi$

$$\int_0^\pi \phi(t)\,N_n(t)\,dt = \left[\int_0^{1/n} + \int_{1/n}^{\delta} + \int_\delta^\pi\right]\phi(t)\,N_n(t)\,dt.$$

$$= M_1 + M_2 + M_3, \text{ say.} \qquad \ldots(5.1)$$

Now, by Lemma 2,

$$M_1 = O\,[n \int_0^{1/n} |\,\phi(t)\,|\, dt]$$

$$= o\,[n\lambda(n)\, p_n/K(P_n)].$$

By assumption that $\{p_n\}$ is non-negative, monotonic non-increasing, we have obviously $(n+1)\, p_n \leqslant P_n$, therefore

$$M_1 = o\,[\lambda(n)\, P_n/K(P_n)]$$

$$M_1 = o(1), \quad \text{as} \quad n \to \infty. \qquad \ldots(5.2)$$

Again by Lemma 3,

$$M_2 = \int_{1/n}^{\delta} \phi(t)\, N_n(t)\, dt$$

$$= O\Big[P_n^{-1} \int_{1/n}^{\delta} |\,\phi(t)\,|\, \frac{P_\tau}{t}\, dt\Big]$$

$$= O\Big[P_n^{-1} \Big(\Phi(t)\, \frac{P_\tau}{t}\Big)_{1/n}^{\delta}\Big]$$

$$+ O\Big[\int_{1/n}^{\delta} P_n^{-1}\, \Phi(t)\, \frac{P_\tau}{t^2}\, dt\Big] + O\Big[\int_{1/n}^{\delta} P_n^{-1}\, \Phi(t)\, \frac{1}{t}\, dP_\tau\Big]$$

$$= M_{2\cdot 1} + M_{2\cdot 2} + M_{2\cdot 3}, \text{ say.}$$

Now

$$M_{2\cdot 1} = O\Big[P_n^{-1} \Big(\Phi(t)\, \frac{P_\tau}{t}\Big)_{1/n}^{\delta}\Big]$$

$$= O\,[P_n^{-1}] + o\Big[\frac{1}{P_n}\, \frac{\lambda(n)\, p_n}{K(P_n)}\, nP_n\Big]$$

$$M_{2\cdot 1} = o(1), \quad \text{as} \quad n \to \infty. \qquad \ldots(5.3)$$

$$M_{2\cdot 2} = O\Big[P_n^{-1} \int_{1/n}^{\delta} \Phi(t)\, \frac{P_\tau}{t^2}\, dt\Big]$$

$$= o(1) + P_n^{-1} \sum_{m=1}^{n-1} \int_m^{m+1} \Phi(1/v)\, P_{[v]}\, dv$$

but

$$\int_m^{m+1} \Phi(1/v)\, P_{[v]}\, dv \leqslant \Phi(1/m)\, P_m$$

$$= o\left[P_m \frac{\lambda(m)\, p_m}{K(P_m)}\right]$$

$$= o\,[p_m], \quad \text{as} \quad m \to \infty$$

so

$$M_{2\cdot 2} = o(1) + o\left[P_n^{-1} \sum_{m=1}^{n-1} p_m\right]$$

$$M_{2\cdot 2} = o(1) \qquad \ldots(5.4)$$

and finally by the hypothesis of the theorem, we have

$$M_{2\cdot 3} = P_n^{-1} \int_{1/n}^{\delta} \Phi(t) \frac{1}{t}\, dP_\tau$$

$$= P_n^{-1} \int_{1/\delta}^{n} \Phi(1/v)\, v\, dP_{[v]}$$

$$= o(1) + O(P_n^{-1} \sum_{m=1}^{n-1} m p_m \Phi(1/m))$$

$$= o(1) + O\left\{P_n^{-1} \sum_{m=1}^{n-1} P_m \Phi(1/m)\right\}$$

$$= o(1) + o\left[P_n^{-1} \sum_{m=1}^{n-1} P_m \frac{\lambda(m)\, p_m}{K(P_m)}\right]$$

$$M_{2\cdot 3} = o(1), \quad \text{as} \quad n \to \infty. \qquad \ldots(5.5)$$

Collecting (5.3), (5.4) and (5.5) we get

$$M_2 = o(1). \qquad \ldots(5.6)$$

Lastly, by virtue of Riemann-Lebesgue theorem and regularity of the method of summation, we have

$$M_3 = \int_\delta^\pi \phi(t)\, N_n(t)\, dt$$

$$= o\left[P_n^{-1} \sum_{v=0}^{n} p_v\right]$$

$$M_3 = o(1), \quad n \to \infty. \qquad \ldots(5.7)$$

Hence on collecting (5.2), (5.6) and (5.7), we get

$$M = o(1)$$

which completes the proof of Theorem 1.

§6. *Proof of the Theorem* 2 — Let $\bar{S}_n(x)$ denote the nth partial sum of the series $\Sigma B_n(x)$. Then we have

$$\bar{S}_n(x) = \frac{1}{2\pi} \int_0^\pi \psi(t) \frac{\cos \frac{1}{2} t - \cos (n + \frac{1}{2}) t}{\sin \frac{1}{2} t} dt.$$

For $\Sigma B_n(x)$, making use of the formula (1.1), we get

$$\bar{t}_n - \frac{1}{2\pi} \int_0^\pi \psi(t) \cot \tfrac{1}{2} t \, dt = P_n^{-1} \sum_{\nu=0}^{n} p_\nu \bar{S}_{n-\nu}(x) - \frac{1}{2\pi} \int_0^\pi \psi(t) \cot \tfrac{1}{2} t \, dt$$

$$= P_n^{-1} \sum_{\nu=0}^{n} p_\nu \frac{1}{2\pi} \int_0^\pi \psi(t) \frac{\cos \frac{1}{2} t - \cos (n - \nu + \frac{1}{2}) t}{\sin \frac{1}{2} t} dt - \frac{1}{2\pi} \int_0^\pi \psi(t) \cot \tfrac{1}{2} t \, dt$$

$$= - \int_0^\pi \psi(t) \left\{ \frac{1}{2\pi P_n} \sum_{\nu=0}^{n} p_\nu \frac{\cos (n - \nu + \frac{1}{2}) t}{\sin \frac{1}{2} t} \right\} dt$$

$$= - \int_0^\pi \psi(t) \bar{N}_n(t) \, dt \; (= R \text{ say}).$$

In order to prove the theorem, we have to show that, under our assumptions

$$\int_0^\pi \psi(t) \bar{N}_n(t) \, dt = o(1),$$

as $n \to \infty$.

For $0 < \delta < \pi$, we have

$$\int_0^\pi \psi(t) \bar{N}_n(t) \, dt = \left[\int_0^{1/n} + \int_{1/n}^{\delta} + \int_0^{\pi} \right] \psi(t) \bar{N}_n(t) \, dt$$

$$R = R_1 + R_2 + R_3, \text{ say.} \qquad \dots(6.1)$$

Since the conjugate function exists, therefore

$$\frac{1}{2\pi}\int_0^{1/n} \psi(t) \cot \tfrac{1}{2}t \, dt = o(1).$$

Also

$$\frac{1}{2\pi P_n}\sum_{\nu=0}^{n} p_\nu \frac{\cos \frac{1}{2}t - \cos (n-\nu+\frac{1}{2})t}{\sin \frac{1}{2}t}$$

$$= \frac{1}{2\pi P_n}\sum_{\nu=0}^{n} p_\nu \sum_{k=0}^{n-\nu} 2 \sin kt$$

$$= O\left[P_n^{-1} \sum_{\nu=0}^{n} p_\nu \sum_{k=0}^{n-\nu} |\sin kt|\right]$$

$$= O\left[P_n^{-1} \sum_{\nu=0}^{n} p_\nu (n-\nu)\right]$$

$$= O(n), \quad \text{for} \quad 0 \leqslant t \leqslant \pi.$$

Therefore

$$R_1 = \int_0^{1/n} \psi(t)\, \bar{N}_n(t)\, dt$$

$$= \int_0^{1/n} \frac{\psi(t)}{2\pi P_n}\sum_{\nu=0}^{n} p_\nu \frac{\cos (n-\nu+\frac{1}{2})t}{\sin \frac{1}{2}t}\, dt$$

$$= -\int_0^{1/n} \frac{\psi(t)}{2\pi P_n}\sum_{\nu=0}^{n} p_\nu \frac{\cos \frac{1}{2}t - \cos (n-\nu+\frac{1}{2})t}{\sin \frac{1}{2}t}\, dt$$

$$+ \frac{1}{2\pi P_n}\sum_{\nu=0}^{n} p_\nu \int_0^{1/n} \psi(t) \cot \tfrac{1}{2}t \, dt$$

$$= O\left(n \int_0^{1/n} |\psi(t)|\, dt\right) + o(1)$$

$$= O\,[n\Psi(1/n)] + o(1)$$

$$= o\,[n\lambda(n)\, p_n/K(P_n)]$$

$$= o\,[\lambda(n) P_n/K(P_n)]$$

$$R_1 = o(1), \ \text{since} \ np_n \leqslant P_n. \qquad \ldots(6.2)$$

Now, for $1/n \leqslant t \leqslant \delta$

$$R_2 = O\left[\int_{1/n}^{\delta} | \psi(t) | \; | \bar{N}_n(t) | \, dt \right]$$

$$= O\left[\int_{1/n}^{\delta} | \psi(t) | \frac{P_\tau}{P_n t} \, dt \right]$$

$$R_2 = o(1), \text{ as in } M_2. \qquad \ldots(6.3)$$

Also

$$R_3 = o(1) \qquad \ldots(6.4)$$

by virtue of Riemann-Lebesgue theorem and the regularity of the method of summation.

Hence on collecting (6.2), (6.3) and (6.4), we get

$$R = o(1)$$

which completes the proof of Theorem 2.

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