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A NUMERICAL TECHNIQUE FOR SOLVING NONLINEAR ELLIPTIC PROBLEMS

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The numerical solution of a second order nonlinear elliptic partial differential equation is considered by using three different linearization procedures: Outer-approximation, Picard-approximation and Newton-approximation. A set of difference equations approximating the linearized version of the given nonlinear problem is derived by using a six-point finite difference scheme. A two-step method is proposed for solving the resulting problem. The existence and stability of the numerical scheme are established. The method is illustrated by solving four test examples. Numerical results indicate that the two-step method gives significantly better results for the numerical solutions of nonlinear elliptic problems.

INTRODUCTION

Non-linear elliptic boundary value problems occur very often in the mathematical modelling of physical problems arising in Bio-medical Engineering, Numerical Weather Forecasting, Radiative Transfer, Optimal Control and other areas of science and engineering. The method of invariant imbedding, initially introduced by Bellman and Kalaba (1956), has been used for solving boundary value problems. The method consists in solving a given boundary value problem, which may be inherently unstable, through a family of stable initial value problems. An exhaustive bibliography on the subject of invariant imbedding has been compiled by Scott (1974). Most of this literature is concerned with the application of invariant imbedding technique to linear boundary value problems in ordinary differential equations (cf. Scott 1973, Meyer 1973). However, for the past few years, this method has also been applied for solving difference equations, partial differential equations and integral equations. Angel (1968) has applied discrete invariant imbedding method to difference equations arising from the discretization of linear elliptic problems over irregular regions. A general form of imbedding, using generalised Riccati transformations, has been used for solving linear partial differential equations by Golberg (1975) and Maynard and Scott (1971). But little has been done for the applicability and effectiveness of the invariant imbedding method for solving nonlinear partial differential equations.

In this paper, a two-step invariant imbedding method is proposed for solving non-linear elliptic problems. Three types of linearization procedures, (i) Outer-approximation, (ii) Picard-approximation and (iii) Newton-approximation, are used to reduce the given nonlinear problem to a sequence of lincar problems and a set of difference equations is derived for every member of the sequence in each case; this is done by using a six-point difference analogue of $O(h^2)$, where h is the mesh size. The existence and stability of the proposed numerical scheme are established on the assumption that the matrices occurring in the invariant imbedding equations are not ill-conditioned and the mesh size h is small.

THE PROBLEM AND LINEAR APPROXIMATIONS

We consider the non-linear elliptic equation

$$u_{xx} + u_{yy} + uu_x + uu_y = f(u), \text{ on } S \qquad ...(1)$$

subject to

$$u(x, y) = g(x, y), \text{ on } \partial S \qquad ...(2)$$

where S is an arbitrary simply connected bounded region with a smooth boundary ∂S and $f(u)$ and $g(x, y)$ are continuous functions in the respective domains;

$$f(u) \equiv f(x, y, u).$$

Assuming the functions $f(u)$ to be twice differentiable and denoting the numerical approximation at the nth iteration for the unknown function u by $u^{(n)}$, we consider the following three linear approximations for eqn. (1):

(i) *Outer-approximation*

$$u_{xx}^{(n+1)} + u_{yy}^{(n+1)} + u^{(n)}u_x^{(n+1)} + u^{(n)}u_y^{(n+1)}$$
$$= f(u^{(n)}) + (u^{(n+1)} - u^{(n)})\, f_u(u^{(n)}), \text{ on } S \ (n = 0, 1, 2, \ldots). \qquad ...(3)$$

(ii) *Picard-approximation*

$$u_{xx}^{(n+1)} + u_{yy}^{(n+1)} + u^{(n)}u_x^{(n)} + u^{(n)}u_y^{(n)} = f(u^{(n)}), \text{ on } S \ (n = 0, 1, 2, \ldots). \qquad ...(4)$$

(iii) *Newton-appoximation*

$$u_{xx}^{(n+1)} + u_{yy}^{(n+1)} + u^{(n+1)}u_x^{(n)} + u^{(n)}u_x^{(n+1)} - u^{(n)}u_x^{(n)}$$
$$+ u^{(n+1)}u_y^{(n)} + u^{(n)}u_y^{(n+1)} - u^{(n)}u_y^{(n)}$$
$$= f(u^{(n)}) + (u^{(n+1)} - u^{(n)})\, f_u(u^{(n)}), \text{ on } S \ (n = 0, 1, 2, \ldots). \qquad ...(5)$$

Equation (2) is discretized as

$$u^{(n+1)}(x, y) = g(x, y), \text{ on } \partial S \qquad ...(6)$$

where $u^{(0)}$ denotes the initial guess.

Bellman and Kalaba (1965) have established the existence and convergence of the sequence $\{u^{(n)}\}$ for the last two procedures; the proof for the convergence of $\{u^{(n)}\}$ for the first case follows on the same lines.

DISCRETIZATION

We take

$$S = \{(x, y) \in R^2 \mid 0 \leqslant x \leqslant a,\ 0 \leqslant y \leqslant b,\ a \in R \ \text{ and } \ b \in R\}$$

where R^2 denotes the two-dimensional Euclidean space. A uniform grid on S is superimposed by discretizing the domain both in the x- and y-directions and by taking $a = Nh$, $b = Mh$, where N, M are positive integers and h is the mesh size. We denote

$$u_{i,j} = u(ih, jh),\ (i = 0, 1, 2, \ldots, N),\ (j = 0, 1, 2, \ldots, M)$$

and define a six-point finite difference analogue for the Laplacian as

$$(u_{xx})_{i,j} + (u_{yy})_{i,j} \simeq \frac{(3+h)\,u_{i+1,j} + (-6-3h)\,u_{i,j} + (3+3h)\,u_{i-1,j} - hu_{i-2,j}}{3h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2}. \qquad \ldots(7)$$

The local truncation error in this scheme is $O(h^2)$.

We confine our analysis to the linearized problem defined by eqn. (5) subjet to the boundary conditions (6). Similar procedures can be applied for the analysis of the other two linear approximations.

After replacing the Laplacian in eqn. (5) by the six-point finite difference approximation and the first order derivatives by backward differences, we get the resulting system in the matrix-vector form:

$$\begin{aligned} & u_{i+1}^{(n+1)} - 2u_i^{(n+1)} + u_{i-1}^{(n+1)} + \frac{h}{3}\left(u_{i+1}^{(n+1)} - 3u_i^{(n+1)} + 3u_{i-1}^{(n+1)} - u_{i-2}^{(n+1)}\right) \\ & \quad - Qu_i^{(n+1)} + r_i + \left(U_i^{(n)} - U_{i-1}^{(n)}\right) u_i^{(n+1)} + U_i^{(n)}\left(u_i^{(n+1)} - u_{i-1}^{(n+1)}\right) \\ & \quad - U_i^{(n)}\left(u_i^{(n)} - u_{i-1}^{(n)}\right) + V_i^{(n)} u_i^{(n+1)} + U_i^{(n)}\left(Tu_i^{(n+1)} - Tu_i^{(n)}\right) \\ & = f_i^{(n)} + L_i^{(n)}\left(u_i^{(n+1)} - u_i^{(n)}\right), \qquad \ldots(8) \end{aligned}$$

$$(i = 2, 3, \ldots, N-1), \quad (n = 0, 1, 2, \ldots).$$

Here the following notations are used:

The vectors

$$u_i^{(n+1)} = \begin{bmatrix} u_{i,1}^{(n+1)} \\ u_{i,2}^{(n+1)} \\ \vdots \\ u_{i,M-1}^{(n+1)} \end{bmatrix}; \quad (i = 1, 2, \ldots, N). \qquad \ldots(9)$$

$$r_i = [r_{i,j}]; \; r_{i,j} = \begin{cases} u_{i,0} \; ; \; j = 1 \\ u_{i,M}; \; j = M-1 \\ \quad 0; \; \text{otherwise} \end{cases} \quad (i = 2, 3, \ldots, N-1). \qquad \ldots(10)$$

The matrices

$$Q = (q_{i,j}); \; q_{i,j} = \begin{cases} \;\; 2; \;\; i = j \\ -1; \; |i - j| = 1 \\ \;\; 0; \;\; \text{otherwise} \end{cases} \qquad \ldots(11)$$

$$U_i^{(n)} = h \text{ Diag } (u_{i,1}^{(n)}, u_{i,2}^{(n)}, \ldots, u_{i,M-1}^{(n)}), \quad (i = 1, 2, \ldots, N-1) \qquad \ldots(12)$$

$$V_i^{(n)} = h \text{ Diag } ((u_{i,1}^{(n)} - u_{i,0}^{(n)}), (u_{i,2}^{(n)} - u_{i,1}^{(n)}), \ldots, (u_{i,M-1}^{(n)} - u_{i,M-2}^{(n)})), \qquad \ldots(13)$$
$$(i = 2, 3, \ldots, N-1)$$

$$T = (t_{i,j}); \; t_{i,j} = \begin{cases} \;\; 1; \;\; i = j \\ -1; \; |i - j| = 1, \; j < i \\ \;\; 0; \;\; \text{otherwise} \end{cases} \qquad \ldots(14)$$

$$f_i^{(n)} = h^2 \begin{bmatrix} f(u_{i,1}^{(n)}) \\ f(u_{i,2}^{(n)}) \\ \vdots \\ f(u_{i,M-1}^{(n)}) \end{bmatrix}; \quad (i = 2, 3, \ldots, N-1) \qquad \ldots(15)$$

$$L_i^{(n)} = h^2 \text{ Diag } (f_u(u_{i,1}^{(n)}), f_u(u_{i,2}^{(n)}), \ldots, f_u(u_{i,M-1}^{(n)}), \quad (i = 2, 3, \ldots, N-1). \qquad \ldots(16)$$

Equation (8) is to be solved subject to the known boundary values u_0 and u_N and the conditions (6) which can be written in the matrix-vector form as

$$P_i u_i^{(n+1)} = g_i, \quad (i = 1, 2, ..., N-1) \qquad ...(17)$$

where P_i are rectangular matrices containing 0 and 1.

Two-Step Invariant Imbedding

We seek the solution of eqn. (8) in the form

$$u_{i+1}^{(n+1)} = A_i u_i^{(n+1)} + B_i u_{i-1}^{(n+1)} + C_i, \quad (i = 1, 2, ..., N-1) \qquad ...(18)$$

where the matrices A_i, B_i and the vectors C_i are to be determined.

The recurrence relations for the matrices A_i and B_i and the vectors C_i are derived by substituting eqn. (18) in eqn. (8):

$$A_{i-1} = E_i^{-1} [-B_i - I - \tfrac{1}{3} h (B_i + 3I) + U_i^{(n)}] \qquad ...(19)$$

$$B_{i-1} = E_i^{-1} [\tfrac{1}{3} h . I] \qquad ...(20)$$

and

$$C_{i-1} = E_i^{-1} [- C_i - \tfrac{1}{3} h C_i - r_i + U_i^{(n)} (u_i^{(n)} - u_{i-1}^{(n)}) + U_i^{(n)} T u_i^{(n)} + f_i^{(n)} - L_i^{(n)} u_i^{(n)}], \quad (i = 2, 3, ..., N-1) \qquad ...(21)$$

where

$$E_i = [A_i - 2I + \tfrac{1}{3} h (A_i - 3I) - Q + 2U_i^{(n)} - U_{i-1}^{(n)} + V_i^{(n)} + U_i^{(n)} T - L_i^{(n)}] \qquad ...(22)$$

and I is the identity matrix.

These recurrence relations are subject to the initial conditions

$$A_{N-1} = \mathbf{0} = B_{N-1} \quad \text{and} \quad C_{N-1} = u_N \quad (\mathbf{0} \text{ is the zero matrix}). \qquad ...(23)$$

Computation of $u_1^{(n+1)}$

In order to find the solution $u_1^{(n+1)}$, which has been assumed to be known in our analysis, the following predictor-corrector scheme may be used:

P : Predict the solution $u_i^{(0)}$ $(i = 1, 2, ..., N-1)$ and compute the matrices A_i, B_i and vectors C_i $(i = N-2, ..., 2, 1)$ from the recurrence relations.

E : Evaluate $u_2^{(1)} = A_1 u_1^{(0)} + B_1 u_0 + C_1$.

C : Correct $u_1^{(0)}$ to $u_1^{(1)}$ by using

$$u_{1,j}^{(1)} = \tfrac{1}{2}(u_{0,j} + u_{2,j}^{(1)});\ j = 1, 2, \ldots, M-1.$$

E : Evaluate $u_2^{(2)} = A_1 u_1^{(1)} + B_1 u_0 + C_1$.

C : Correct $u_1^{(1)}$ to $u_1^{(2)}$ by using

$$u_{1,j}^{(2)} = \tfrac{1}{2}(u_{0,j} + u_{2,j}^{(2)});\ j = 1, 2, \ldots, M-1.$$

This procedure is continued till $\max\limits_{1 \leqslant j \leqslant M-1} | u_{1,j}^{(n+1)} - u_{1,j}^{(n)} | \leqslant \epsilon$, where ϵ is the prescribed tolerance limit. The vector $u_1^{(n+1)}$ thus obtained may then be taken as the solution $u_1^{(n+1)}$.

In certain situations, one may like to use one-step method (Angel 1968) for computing the solution of a given problem. In that case, the value of $u_1^{(n+1)}$ becomes known for application in the two-step method. But the use of the one-step method for evaluating $u_1^{(n+1)}$ only is not economical in view of the heavy demand on the storage and the computer time.

The computational procedure is to compute matrices A_i, B_i and vectors C_i from eqns. (19) – (21) subject to eqn. (23). They are used to obtain the solution $u_i^{(n+1)}$ from eqn. (18) in a forward sweep with the help of known values u_0 and $u_1^{(n+1)}$. This procedure is repeated till the prescribed convergence criterion is satisfied.

Non-singularity and Stability·

We prove the existence of the matrices A_i and B_i and the vectors C_i appearing in the invariant imbedding eqns. (19) – (21) by showing that the inverse of the matrix E_i exists. We restrict our analysis to the case when $f(u) = \exp(u)$.

Since Q is a real and symmetric matrix, it can be reduced to the diagonal form

$$RQR' = \begin{bmatrix} \mu_1 & & 0 \\ & \mu_2 & \\ 0 & & \mu_{M-1} \end{bmatrix}$$

where R is an orthogonal matrix. Moreover, as Q is positive definite, we have

$$\mu_j > 0,\ j = 1, 2, \ldots, M-1.$$

Let the eigenvalues of the matrices $U_i^{(n)}$, $V_i^{(n)}$ and $L_i^{(n)}$ be denoted by $\alpha_j^{(i)}$, $\gamma_j^{(i)}$ and $\delta_j^{(i)}$, respectively. Then from eqns. (12), (13) and (14), we have

$$\alpha_j^{(i)} = hu_{i,j}^{(n)}$$

$$\gamma_j^{(i)} = h(u_{i,j}^{(n)} - u_{i,j-1}^{(n)})$$

and

$$\delta_j^{(i)} = h^2 f_u(u_{i,j}^{(n)}) = h^2 \exp (u_{i,j}^{(n)}), \ (j = 1, 2, \ldots, M-1; i = 1, 2, \ldots, N-1). \quad \ldots(24)$$

Again, let $\lambda_j^{(i)}$, $\beta_j^{(i)}$ and $e_j^{(i)}$ denote the eigenvalues of the matrices A_i, B_i and E_i respectively. Using Schur's Theorem (Bellman 1970),

$$e_j^{(i)} = (\lambda_j^{(i)} - 2 + \tfrac{1}{3} h \lambda_j^{(i)} - h - \mu_i + 3\alpha_j^{(i)} - \alpha_j^{(i-1)} + \gamma_j^{(i)} - \delta_j^{(i)}). \quad \ldots(25)$$

Also, from eqns. (19), (20) and (23),

$$\lambda_j^{(i-1)} = (-\beta_j^{(i)} - 1 - \tfrac{1}{3} h\beta_j^{(i)} - h + \alpha_j^{(i)})/e_j^{(i)} \quad \ldots(26)$$

and

$$\beta_j^{(i-1)} = \tfrac{1}{3} h/e_j^{(i)} \quad \ldots(27)$$

$$\lambda_j^{(N-1)} = 0 = \beta_j^{(N-1)}. \quad \ldots(28)$$

Proceeding inductively with eqns. (25) – (28), one finds that the matrices E_i are non-singular provided

$$e_j^{(i)} < 0, \ 0 < \lambda_j^{(i-1)} < 1, \ -\tfrac{1}{3} h < \beta_j^{(i-1)} < 0 \quad \ldots(29)$$

for $j = 1, 2, \ldots, M-1$ and $i = 2, \ldots, N-1$.

This completes the proof for the existence of the matrices A_i, B_i and the vector C_i.

The stability of the numerical scheme can be proved on the assumption that the initial errors in computing the matrices are small.

Test Examples

To illustrate the method, we consider the following four examples.

Example 1

$$u_{xx} + u_{yy} + uu_x + uu_y = e^u, \quad \text{on } S$$

$$u(x, y) = x^2 + y, \quad \text{on } \partial S$$

where S is the squared region

$$S : \{(x, y); 0 \leqslant x \leqslant \tfrac{1}{2}, 0 \leqslant y \leqslant \tfrac{1}{2}\} \qquad \ldots(30)$$

and ∂S is its boundary.

Example 2

$$u_{xx} + u_{yy} - uu_x - uu_y = e^{-x} \sin \pi y(1 - \pi^2 + e^{-x} \sin \pi y - \pi e^{-x} \cos \pi y), \text{ on } S$$

$$u(x, y) = e^{-x} \sin \pi y, \text{ on } \partial S \qquad \ldots(31)$$

where S is the octant with vertices (3/20, 0), (7/20, 0), (1/2, 3/20), (1/2, 7/20), (7/20, 1/2), (3/20, 1/2), (0, 7/20) and (0, 3/20).

Example 3

$$u_{xx} + u_{yy} - uu_y = e^{-x} \sin \pi y(1 - \pi^2 - \pi e^{-x} \cos \pi y), \text{ on } S$$

$$u(x, y) = e^{-x} \sin \pi y, \text{ on } \partial S \qquad \ldots(32)$$

where S is the same octant as defined in Example 2.

Example 4

$$u_{xx} + u_{yy} + uu_y = e^{-x} \sin \pi y(1 - \pi^2 + \pi e^{-x} \cos xy), \text{ on } S$$

$$u(x, y) = e^{-x} \sin \pi y, \text{ on } \partial S \qquad \ldots(33)$$

where S is the same octant as defined in Example 2.

Numerical Results and Discussion

Using all the three linearization procedures and by taking mesh size $h = 1/10$, 1/20 and 1/40, we have computed the solutions of the four test examples by the one-step method (Angel 1968) and the two-step method.

Figures 1 and 2 exhibit the numerical solutions for example 1 by the Outer- and Picard-approximations respectively. It is found that the computational results by the two-step method are significantly better as compared to those obtained by the one-step method. In Figs. 1(a) and 2(a), one observes a slow variation of the solution as one moves towards the right-hand boundary of the region, where one gets an abrupt change in the solution. The numerical results obtained by the two-step method [cf. Figs. 1(b) and 2(b)] are free from these defects; here, the figures show a smooth behaviour of the solution in the whole region of computation. The numerical solution obtained by the two-step method improves as the mesh size is refined. It may be observed that the solution obtained by the one-step method for all the three mesh sizes contains errors in a large part of the computational region [cf. Figs. 1(a) and 2(a)].

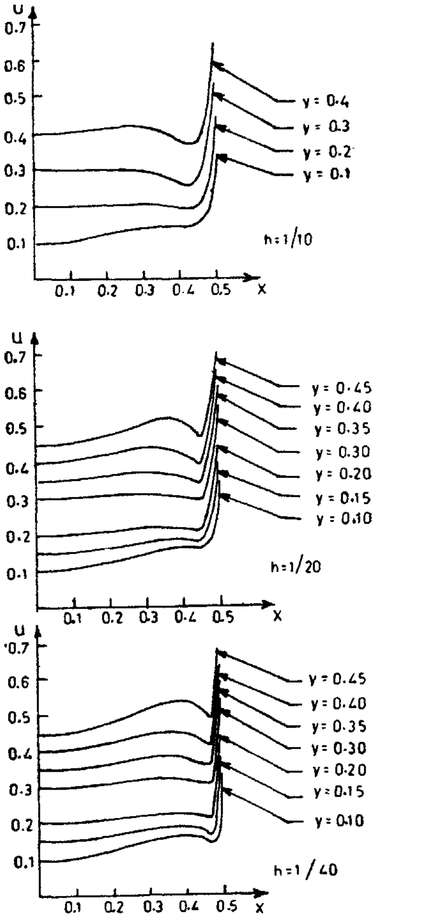

FIG. 1(a). Numerical results for Example 1 using Outer-approximation and one-step method.

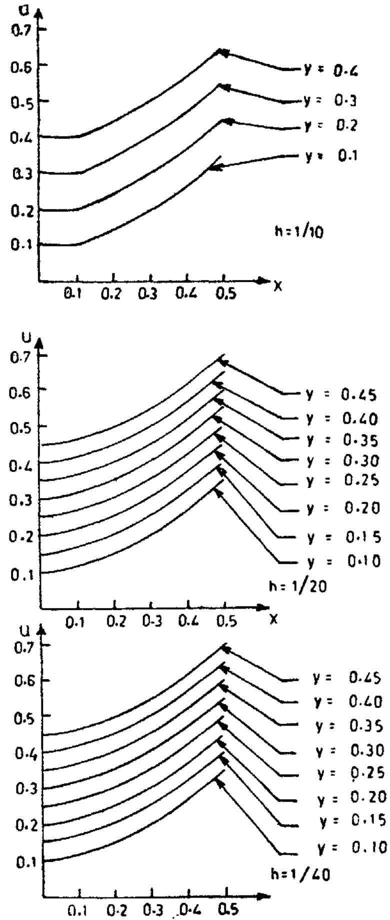

FIG. 1(b). Numerical results for Example 1 using Outer-approximation and two-step method.

In Figs. 1(a) and 2(a), we have noticed the building up of the errors by the Outer- and Picard-approximations and the one-step method, while the corresponding two-step method yields substantially better results. In the case of the Newton-approximation, the improvement in the numerical results by the use of the two-step method is marginal compared to the numerical results by the one-step method [cf. Figs. 3(a) and 3(b)].

The numerical results for Example 2 are tabulated at some typical mesh points in Table I. On examining the results, one finds an improvement in the solutions as the mesh is refined. The solution obtained by the two-step method combined with the Newton-approximation is the best out of all the three linearization procedures and

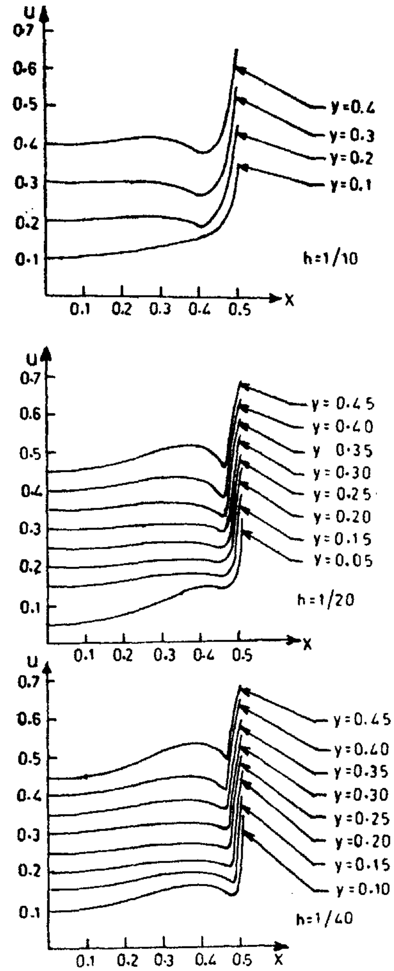

FIG. 2(a). Numerical results for Example 1 using Picard-approximation and one-step method.

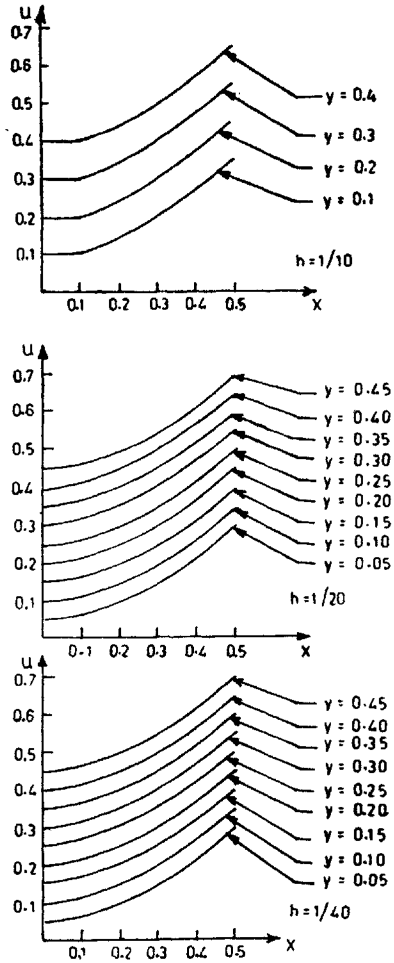

FIG. 2(b). Numerical results for Example 1 using Picard-approximation and two-step method.

is found to be in good agreement with the exact solution. It is also found that the solution obtained in the Newton-approximation case for $h = 1/20$ is significantly better than that obtained by using the Outer- or Picard-approximation with $h = 1/40$. The numerical results obtained by the one-step method are also listed in Table I for the Outer-approximation case. On comparing the solutions obtained by the two-step method with those obtained by the one-step method for the Newton-approximation, it is seen that there is only a marginal improvement in the solution. For the other two-linearization procedures, the two-step method yields significantly better results than the one-step method.

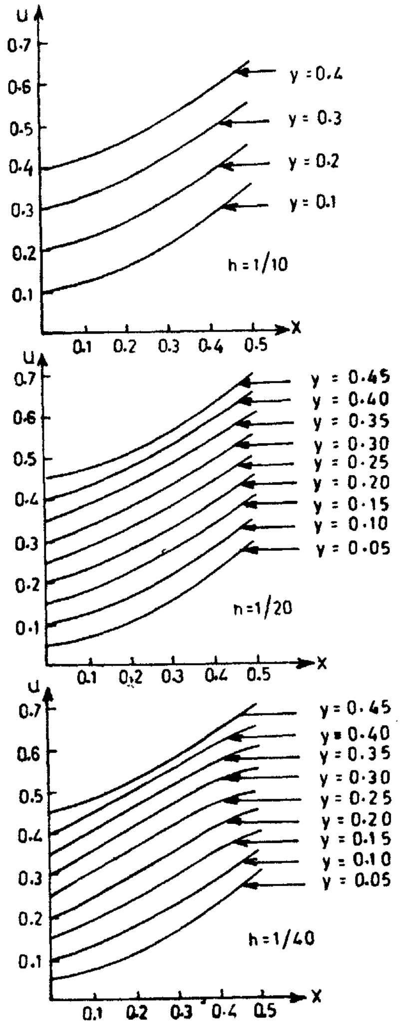

FIG. 3(a). Numerical results for Example 1 using Newton-approximation and one-step method.

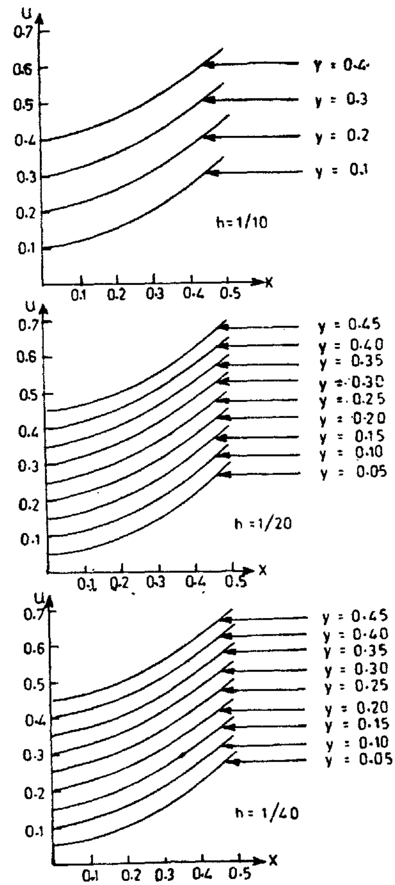

FIG. 3(b). Numerical results for Example 1 using Newton-approximation and two-step method.

The behaviour of the numerical solutions for Examples 3 and 4 was found to be of the same nature as for the second example.

Based on these numerical experiments, we make the following observations:

Numerical results obtained by the two-step method improve as one goes from the Picard-approximation to the Outer-approximation and then to the Newton-approximation.

The two-step method gives substantially better results (especially when the given problem is of highly nonlinear nature like Examples 1 and 2) than the one-step method by using the Outer- and Picard-linearization procedures.

The two-step method gives good results compared with the exact solution for a crude mesh $h = 1/20$. On the other hand, by using the Outer- and Picard-approximations and the one-step method ($h = 1/20$), the results are crude compared to the exact solution.

By using the Newton-approximation, the two-step method shows appreciable improvement (Example 1) and marginal improvement (Examples 2, 3 and 4) in the solutions over the one-step method. Hence, the two-step method combined with the Newton-approximation is likely to yield good numerical results for highly non-linear problems of elliptic type.

TABLE I

Numerical results for Example 2

Mesh points	Two-step method						One-step method	
	Outer-approximation		Picard-approximation		Newton-approximation		Outer-approximation	Exact solution
	$h = 1/20$	$h = 1/40$	$h = 1/20$	$h = 1/40$	$h = 1/20$	$h = 1/40$	$h = 1/20$	
(0.10, 0.10)	0.279951	0.279781	0.280136	0.279950	0.279465	0.279524	0.280967	0.279610
(0.10, 0.20)	0.532863	0.532422	0.533117	0.532728	0.531356	0.531566	0.536644	0.531850
(0.10, 0.30)	0.734181	0.733168	0.734157	0.733331	0.731318	0.731630	0.739967	0.732029
(0.15, 0.15)	0.391990	0.391338	0.392608	0.391895	0.390324	0.390503	0.394326	0.390753
(0.15, 0.25)	0.611503	0.609961	0.611915	0.610481	0.607824	0.608165	0.615627	0.608612
(0.15, 0.35)	0.773326	0.769663	0.772694	0.769676	0.766019	0.766411	0.777507	0.766896
(0.20, 0.10)	0.253981	0.253439	0.254541	0.253949	0.252710	0.252828	0.255432	0.253002
(0.20, 0.20)	0.483819	0.482396	0.484479	0.483102	0.480557	0.480846	0.486923	0.481238
(0.25, 0.25)	0.554989	0.552581	0.555305	0.553120	0.549846	0.550212	0.558633	0.550695
(0.25, 0.30)	0.636433	0.632866	0.636273	0.633172	0.629104	0.629524	0.640221	0.630063
(0.30, 0.15)	0.338112	0.337099	0.338567	0.337560	0.335878	0.336063	0.340638	0.336324
(0.35, 0.05)	0.110564	0.110375	0.110681	0.110477	0.110154	0.110188	0.111143	0.110564
(0.35, 0.30)	0.574782	0.572124	0.574621	0.572266	0.569373	0.569692	0.580013	0.570105
(0.40, 0.10)	0.207602	0.207336	0.207693	0.207428	0.207021	0.207069	0.209051	0.207140
(0.40, 0.40)	0.640696	0.638891	0.640403	0.638833	0.637192	0.637336	0.645810	0.637512
(0.45, 0.15)	0.289934	0.289667	0.289977	0.289723	0.289355	0.289407	0.292221	0.289477
(0.45, 0.35)	0.569366	0.568663	0.569293	0.568672	0.567897	0.568002	0.574624	0.568131

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