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A NOTE ON PARTIAL ISOMETRIES—II

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A sufficient condition is obtained for square of an operator to be a partial isometry. Also, several conditions are discussed which imply that an operator is the direct sum of an isometry and zero.

A bounded linear operator T on a Hilbert space H is called a partial isometry if T is isometric on the orthogonal complement of its kernel $N(T)$. The operators whose first N powers and all powers are partial isometries have been characterized respectively by Guyker (1976) and Halmos and Wallen (1970). In this note we establish a condition for square of an operator to be a partial isometry and we also discuss conditions which imply that an operator is the direct sum of an isometry and zero.

An operator T is said to be n-paranormal if $\| Tx \|^n \leqslant \| T^n x \|$ for every unit vector x in H, n being a positive integer; and T is said to be binormal if T^*T commutes with TT^* or equivalently if $TT^{*2}T \geqslant 0$ (Campbell 1972, 1974).

Gupta (1979) proved the following result.

Theorem A — Let T be a contraction and let $N(T) = N(T^2)$. If T^k is a partial isometry for some $k \geqslant 2$, then T is a partial isometry.

From this theorem it was deduced that if T^k is a partial isometry then T is the direct sum of an isometry and zero if any one of the following holds:

(i) T is n-paranormal

(ii) T^*T commutes with $T^* + T$.

The following result implicitly contained in Lemma 2 of Halmos and Wallen (1970) gives a characterization of binormal partial isometries.

Theorem B — Suppose T is a partial isometry. Then T^2 is a partial isometry if and only if T is binormal.

In fact in one direction, we have a stronger result:

Theorem C (Guyker 1976) — If T is a contraction and T^2 is a partial isometry then $TT^{*2}T$ is a projection.

In the other direction we have the following interesting result.

Theorem 1 — If $TT^{*2}T$ is a projection, then T^2 is a partial isometry.

PROOF : Let $T = UP$ and $T^* = VQ$ be polar decompositions of T and T^*, where U and V are partial isometries with their initial spaces $\overline{R(T^*)}$ and $\overline{R(T)}$ respectively.

If $TT^{*2}T = S$ then S is a projection and

$$(PQ)^2 = P^2Q^2 = T^*T^2T^* = S$$

so that

$$PQ = S.$$

Therefore

$$T^2 = (UP)(VQ)^* = USV^*.$$

Since $R(S) \subset \overline{R(T)} \cap \overline{R(T^*)}$, U and V are both isometric on $R(S)$. So

$$SU^*US = S \text{ and } SV^*VS = S.$$

Hence

$$T^2T^{*2}T^2 = USV^*VSU^*USV^* = USV^* = T^2$$

and it follows that T^2 is a partial isometry.

Corollary 1 — (i) $TT^{*2}T = 0$ if and only if $T^2 = 0$.

(ii) $TT^{*2}T = I$ if and only if T^2 is unitary.

PROOF : (i) is obvious.

(ii) If T^2 is unitary then $T^{*2} = T^{-2}$ and so $TT^{*2}T = I$.

Conversely, if $TT^{*2}T = I$ then $T^2x = 0$ implies that

$$Tx = TT^{*2}T^2x = 0.$$

So $$x = TT^{*2}Tx = 0.$$

Thus $N(T^2) = \{0\}$.

Similarly $N(T^{*2}) = \{0\}$ and the result follows from Theorem 1.

Corollary 2 — Let $TT^{*2}T$ be a projection. If T is invertible and

$$\sigma(T) \cap \sigma(-T) = \phi$$

then T is unitary, $\sigma(T)$ being the spectrum of T.

PROOF : By Corollary 1(ii) T^2 is unitary and since $\sigma(T) \cap \sigma(-T) = \phi$ the result follows from Embry (1968).

Corollary 3 — If $TT^{*2}T$ is a projection and $N(T) = N(T^2)$, then T^2 is the direct sum of an isometry and zero.

PROOF : Since $TT^{*2}T = T^*T^2T^*$ and $N(T) = N(T^2)$, $Tx = 0$ implies $T^2T^*x = 0$ and so $T^*x = 0$. Therefore $N(T)$ $(= N(T^2))$ reduces T. So T^2, being a partial isometry by Theorem 1, is the direct sum of an isometry and zero.

Corollary 4 — If T is a contraction operator with $N(T) = N(T^2)$ and $TT^{*2}T$ is a projection, then T is the direct sum of an isometry and zero.

PROOF : As in Corollary 3, $N(T)$ is reducing for T. Since T^2 is a partial isometry the result follows from Theorem A.

By the remark following Theorem A, we have the following.

Corollary 5 — If T is n-paranormal and $TT^{*2}T$ is a projection, then T is the direct sum of an isometry and zero.

Example : The condition that T^2 is a partial isometry is not sufficient for $TT^{*2}T$ to become a projection. This can be seen by taking

$$T = \begin{pmatrix} \alpha^{1/2} & \alpha^{-1/2}(1 - \alpha^2)^{1/2} \\ 0 & 0 \end{pmatrix}$$

where $0 < \alpha < 1$.

It has been shown by Gupta (1979) that if T and T^2 both are partial isometries with the same kernel, then T is the direct sum of an isometry and zero.

We improve this result in the following.

Theorem 2 — If T and T^{k+1} $(k \geqslant 1)$ are partial isometries and $N(T) = N(T^2)$ then T is the direct sum of an isometry and zero.

PROOF : Since $S = T^*T^{k+1}T^{*k}$ is contraction and idempotent, S is self-adjoint. It follows from this that T^*T commutes with T^kT^{*k}. Therefore $N(T)$ reduces T^kT^{*k}.

But $N(T) = N(T^2)$ implies $N(T) = N(T^n)$ for every positive integer n. Therefore if $x \in N(T)$ then $T^{*k}x \in N(T)$ and thus $N(T)$ reduces T^k.

Since T and T^{k+1} are isometric on $N(T)^\perp$ which reduces T^k, it follows that T^k is isometric on $N(T)^\perp$. But $N(T)^\perp = N(T^k)^\perp$ and so T^k is a partial isometry. Repeating the same argument we see that T^2 is a partial isometry and $N(T)$ reduces T. Therefore, T is the direct sum of an isometry and zero.

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