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A REMARK ON A PAPER OF V. ISTRĂŢESCU

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Let T be a bounded linear operator on a Hilbert space H. We denote by Z_T and R_T respectively the centre and radius of the smallest circular disc containing the spectrum of T. In this note we prove that

$$\sup_{\|x\|=1} \{\|Tx\|^2 - |(Tx, x)|^2\} = R_T^2$$

if $T - Z_T I$ is normaloid and thereby extend the results of Björk and Thomee (1962) for the case of normal operators and that of Istrǎtescu (1972) for the case of transloid operators.

Let H be a complex Hilbert space. For a bounded linear operator T on H, we denote by σ_T and $r_\sigma(T)$ respectively the spectrum of T and spectral radius of T. Let d_T be the smallest circular disc containing σ_T, C_T its boundary, Z_T its centre and R_T its radius. In this note we study a property of a bounded linear operator T on H for which $T - Z_T I$ is a normaloid.

Björk and Thomee (1962) have proved the following theorem for normal operators.

Theorem 1 — If N is a normal operator,

$$\sup_{u \neq 0} \left\{ \frac{\|Nu\|^2}{\|u\|^2} - \frac{|(Nu, u)|^2}{\|u\|^4} \right\} = R_N^2 \qquad \text{...(I)}$$

Istrǎţescu (1972) has extended this result to the case of a transloid operator T (i.e. $T - \lambda I$ is normaloid for all complex λ) (Istrǎţescu 1972). In both the cases the proof is mainly based on proving that $Z_T \in$ convex null $(\sigma_T \cap C_T)$ and finding distinct points $\lambda_1, \lambda_2, \ldots, \lambda_r$ ($r = 2$ or 3) such that $|\lambda_r - Z_T| = R_T$ and expressing Z_T as a linear combination of the λ_r's. Naturally this leads one to question whether the condition $T - \lambda I$ is normaloid for all complex λ is necessary for the equality (*I*) to hold. In fact our investigation shows that even under a weaker condition namely $T - \lambda I$ is normaloid only for $\lambda = Z_T$, the equality holds.

Theorem — Let $T \in \beta(H)$ and Z_T, R_T as defined above. If $T - Z_T I$ is normaloid, then

$$\sup_{x \neq 0} \left\{ \frac{\| Tx \|^2}{\| x \|^2} - \frac{| (Tx, x) |^2}{\| x \|^4} \right\} = R_T^2 .$$

Lemma 1 — Let $T \in \beta(H)$. $\sigma_T \cap C_T \neq \phi$.

PROOF : Suppose $\sigma_T \cap C_T = \phi$; σ_T and C_T being compact subsets of the complex plane, $d = \text{dist}\,(\sigma_T, C_T) > 0$. Then the circular disc with centre Z_T and radius $R_T - \frac{1}{2} d$ contains σ_T contradicting the fact that d_T is the smallest circular dics containing σ_T.

Lemma 2 — $r_\sigma (T - Z_T I) = R_T$.

PROOF : By Lemma 1, $| Z_T - \lambda_0 | = R_T$ for some $\lambda_0 \in \sigma_T$. Since

$$| Z_T - \lambda | \leqslant R_T \;\forall\; \lambda \in \sigma_T$$

the result follows from the definition of the spectral radius.

Lemma 3 — $Z_T \in$ convex hull $\{\sigma_T \cap C_T\}$.

PROOF : Suppose not. Then we can find a closed half circle $h \subset C_T$ such that $\sigma_T \cap h = \phi$. Since h and σ_T are compact, $d(h, \sigma_T) > 0$. Hence it follows that σ_T is contained in a smaller circular disc than d_T.

Proof of Theorem:

$$\| Tx \|^2 - | (Tx, x) |^2 = \| Tx - \lambda x \|^2 - | (Tx, x) - \lambda |^2$$

for any complex λ.

In particular taking $\lambda = Z_T$ and taking supremum,

$$\sup_{\|x\|=1} \left\{ \frac{\| Tx \|^2}{\| x \|^2} - \frac{| (Tx, x) |^2}{\| x \|^4} \right\}$$

$$= \sup_{\|x\|=1} \{ \| Tx - Z_T x \|^2 - | (Tx, x) - Z_T |^2 \}$$

$$\leqslant \sup_{\|x\|=1} \| (T - Z_T I)\, x \|^2 = \| T - Z_T I \|^2$$

$$= [\, r_\sigma(T - Z_T I)]^2 \text{ (since } T - Z_T I \text{ is normaloid).}$$

$$= R_T^2 \text{ by Lemma 2.}$$

Now we will prove that the supremum is in fact attained.

Since $Z_T \in$ convex null $(\sigma_T \cap C_T)$, there exist two or three distinct complex numbers $\lambda_1 \ldots \lambda_r$ $(r = 2 \text{ or } 3) \in \sigma_T \cap C_T$ such that

$$| \lambda_i - Z_T |^2 = R_T^2$$

and $$Z_T = \sum_{i=1}^{r} m_i \lambda_i \quad \text{with} \quad \sum_{i=1}^{r} m_i = 1 \quad \text{and } m_i > 0.$$

Since

$$\lambda_i - Z_T \in \sigma(T - Z_T I)$$

and $$| \lambda_i - Z_T | = R_T = r_\sigma(T - Z_T I) = \| T - Z_T I \|,$$

there exist sequences $(\{x_n^{(i)}\})$ of unit vectors such that

$$\| (T - Z_T I)\, x_n^{(i)} - (\lambda_i - Z_T)\, x_n^{(i)} \| = \| (T - \lambda_i)\, x_n^{(i)} \| \to 0$$

and

$$\| (T^* - \bar{Z}_T I)\, x_n^{(i)} - (\bar{\lambda}_i - \bar{Z}_T)\, x_n^{(i)} \| = \| (T^* - \bar{\lambda}_i)\, x_n^{(i)} \| \to 0.$$

Now

$$\lim_{n\to\infty} (\lambda_i - \lambda_j)(x_n^{(i)}, x_n^{(j)}) = \lim_{n\to\infty} \{(\lambda_i x_n^{(i)}, x_n^{(j)}) - (x_n^{(i)}, \bar{\lambda}_j x_n^{(j)})\}$$

$$= (T x_n^{(i)}, x_n^{(j)}) - (x_n^{(i)}, T^* x_n^{(j)})$$

$$= 0.$$

This implies $\lim_{n\to\infty} (x_n^{(i)}, x_n^{(j)}) = 0$ for $i \neq j$. Now we define

$$x_n = \sum_{i=1}^{r} \sqrt{m_i}\, x_n^{(i)}.$$

Then as $n \to \infty$,

$$\| x_n \| \to 1, \quad \| T x_n - Z_T x_n \|^2 \to R_T^2$$

and

$$(T x_n, x_n) \to Z_T.$$

Hence the supremum is attained and

$$\sup_{\|x\|=1} \left\{ \frac{\| Tx \|^2}{\| x \|^2} - \frac{| (Tx, x) |^2}{\| x \|^4} \right\} = R_T^2.$$

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