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ABSOLUTE EULER SUMMABILITY OF ALLIED SERIES OF THE FOURIER SERIES

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In this paper, necessary and sufficient condition for the sequence of factors so as to ensure $| E, q | (q > 0)$ summability of factored Fourier series are obtained. Two more theorems are proved which lead to interesting results that factored Fourier series and conjugate series are almost everywhere $| E, q | (q > 0)$ summable whenever the sequence of factors satisfies certain conditions. Lastly, a theorem which improves an earlier result due to Tripathi (1973) is proved.

1. DEFINITIONS AND NOTATIONS

Let $\sum_{n=0}^{\infty} d_n$ be a given infinite series and let q be a real or complex number such that $q \neq -1$. Then we define

$$d_n^q = (1 + q)^{-n-1} \sum_{m=0}^{n} \binom{n}{m} q^{n-m} d_m ; \; d_n^0 = d_n. \qquad ...(1.1)$$

Following Chandra (1975b), we write

$$\sum_{n=0}^{\infty} d_n \in | E, q | \Leftrightarrow \sum_{n=0}^{\infty} | d_n^q | < \infty. \qquad ...(1.2)$$

Let f be 2π-periodic and L-integrable over $(-\pi, \pi)$, and its Fourier series, at a point x, be

$$\tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} A_n(x). \qquad ...(1.3)$$

Then the series conjugate to (1.3), will be given by

$$\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) = \sum_{n=1}^{\infty} B_n(x). \qquad ...(1.4)$$

Throughout the paper we assume $a_0 = 0$. Now, we write

$$\phi(t) = \tfrac{1}{2} \{f(x + t) + f(x - t)\} \qquad ...(1.5)$$

$$\psi(t) = \tfrac{1}{2}\{f(x+t) - f(x-t)\} \qquad \text{...(1.6)}$$

$$P(t) = \phi(t) - \frac{1}{t}\int_0^t \phi(u)\,du \qquad \text{...(1.7)}$$

$$P^*_\alpha(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}P(u)\,du \quad (\alpha > 0) \qquad \text{...(1.8)}$$

$$P_\alpha(t) = \Gamma(\alpha+1)\,t^{-\alpha}P^*_\alpha(t) \quad (\alpha \geqslant 0) \qquad \text{...(1.9)}$$

$$K(n,t) = \cos nt - \frac{\sin nt}{nt} \qquad \text{...(1.10)}$$

$$K^{(1)}(n,t) = \frac{\partial}{\partial t}K(n,t) \qquad \text{...(1.11)}$$

$$\Gamma(1-\alpha)\,g(n,u) = \int_u^\pi (t-u)^{-\alpha}K(n,t)\,dt \qquad \text{...(1.12)}$$

$$v^q_m(n) = (1+q)^{-n-1}\binom{n}{m}q^{n-m} \quad (n \geqslant m \geqslant 0) \qquad \text{...(1.13)}$$

$$\log_2 = \log\log,\ \log_1 = \log. \qquad \text{...(1.14)}$$

Throughout, T denotes the integral part of $(k/t)^2$, where $0 < t \leqslant \pi$ and k is suitable positive constant and not necessarily the same at each occurrence.

2. Introduction

It is known (Chandra 1972b, c) that $P(t)\,(\log(k/t))^{1+\epsilon} \in BV(0,\pi)$, where $\epsilon > 0$, is a sufficient condition for $\sum_1^\infty A_n(x) \in |E, q|\ (q > 0)$ while neither (see Theorem 1 of this paper)

$$P(t)\log\frac{k}{t} \in BV(0,\pi) \quad (k > \pi e^2) \qquad \text{...(2.1)}$$

nor (see Tripathi 1969)

$$\phi(t) \in BV(0,\pi) \qquad \text{...(2.2)}$$

is a sufficient condition for $\sum_1^\infty A_n(x) \in |E, q|\ (q > 0)$. Also (2.1) and (2.2) are non-comparable since the later holds if and only if (see Chandra 1978b)

(i) $P(t) \in BV(0,\pi)$; (ii) $t^{-1}P(t) \in L(0,\pi)$. ...(2.3)

Thus the natural question arises as to what (y_n) should be to ensure

$$\sum_{1}^{\infty} A_n(x)\, y_n \in |\, E, q \,| \quad (q > 0) \qquad \ldots(2.4)$$

whenever (2.1) and (2.2) hold. Recently, Chandra (1978a) and Tripathi (1974) independently have shown that (2.4) with $y_n = (\log (n + 1))^{-1}$ holds whenever (2.2) holds. To supplement an answer to the above question, we prove the following:

Theorem 1 — Let $y_n = (\log (n + 1))^{-c}$. Then, in order that (2.1) $\Rightarrow$ (2.4), it is necessary and sufficient that $c > 0$.

Inspired by the above theorem, we also prove

Theorem 2 — Let $y_n = (\log (n + 1))^{-1} (\log_2 (n + 2))^{-c}$. Then in order that

$$P(t) \log_2 (k/t) \in BV(0, \pi) \qquad \ldots(2.5)$$

implies (2.4), it is necessary and sufficient that $c > 0$.

Remark 1 : It may be observed that (2.5) is not a stronger condition than (2.2). However the class of the functions satisfying (2.5) contains even those functions which are not of bounded variation in $(0, \pi)$. For example, let f be 2π-periodic and even function so that, for $x = 0$, $\phi(t) = f(t)$ and let

$$f(t) = \left(\log \frac{2\pi}{t}\right)^{1/2} \text{ in } (0, \pi).$$

Then f satisfies (2.5) but $f \notin BV(0, \pi)$.

We further prove the following theorems concerning absolute summability factors for Fourier series and conjugate series:

Theorem 3 — Let

$$\left.\begin{array}{ll} \text{(i)} & 0 < n^{\delta} y_n \uparrow \text{ with } n \geqslant 1, \text{ for } \delta > 0; \\ \text{(ii)} & \sum_{n=1}^{\infty} n^{-1/2} \log (n + 1)\, y_n < \infty. \end{array}\right\} \qquad \ldots(2.6)$$

Then (2.4) holds for every value of x for which

$$\int_0^t |\, f(x + u) - f(x) \,|\, du = O(t) \quad (t \to 0). \qquad \ldots(2.7)$$

Theorem 4 — Let (2.6) hold. Then $\sum_{1}^{\infty} B_n(x)\, y_n \in |\, E, q \,|\ (q > 0)$ for every value of x for which (2.7) holds.

Remark 2 : Since, for every L-integrable function f,

$$\int_0^t | f(x + u) - f(x) | \, du = o(t) \; (t \to 0) \qquad \ldots(2.8)$$

for almost all values of x, therefore it follows from Theorems 3 and 4 that $\sum_{n=1}^{\infty} A_n(x) y_n$ and $\sum_{n=1}^{\infty} B_n(x) y_n$ are $| E, q |$, where $q > 0$, summable almost everywhere whenever (2.6) holds.

Tripathi (1973) established the following:

Theorem A — Let $\theta < \alpha < \frac{3}{2}$ and let $\lambda > \max(\alpha - \frac{1}{2}, \frac{1}{2})$. Then

$$\phi_\alpha(t) \in BV(0, \pi) \Rightarrow \sum_{n=1}^{\infty} n^{-\lambda} A_n(x) \in | E, q | \; (q > 0).$$

It follows from Theorem 3 that $\sum_{n=1}^{\infty} n^{-\lambda} A_n(x) \in | E, q | \; (q > 0, \lambda > \frac{1}{2})$ almost everywhere. Thus the case $\alpha > 1$ of Theorem A is not interesting one. Further, it may be noted that $\lambda > \frac{1}{2}$ for $0 < \alpha \leqslant 1$, therefore Theorem A loses its significance when $0 < \alpha < 1$. In view of these observations we prove the following:

Theorem 5 — Let $0 \leqslant \alpha \leqslant 1$ and let bounded sequence (y_n) satisfy (2.6) (i) and

$$\sum_{n=1}^{\infty} n^{-1} y_n < \infty. \qquad \ldots(2.9)$$

Then $$P_\alpha(t) \in BV(0, \pi) \Rightarrow \sum_{1}^{\infty} n^{-\alpha/2} A_n(x) y_n \in | E, \alpha p | \; (p > 0).$$

Remark 3 : It may be observed that, for $\alpha \geqslant 0$, $P_\alpha(t) \in BV(0, \pi)$ is a weaker condition than $\phi_\alpha(t) \in BV(0, \pi)$. Thus Theorem 5 yields the following interesting result:

Corollary — Let $0 \leqslant \alpha \leqslant 1$. Then $P_\alpha(t) \in BV(0, \pi)$ implies

$$\sum_{n=1}^{\infty} \frac{n^{-\alpha/2} A_n(x)}{(\log (n + 1))^{1+\epsilon}} \in | E, \alpha p | \; (p > 0, \epsilon > 0).$$

3. Lemmas

We shall use the following lemmas in the proofs of the theorems:

Lemma 1 (Chandra 1975b) — Let $q > p > -1$. Then $\sum_{0}^{\infty} a_n \in | E, p |$ implies that $\sum_{0}^{\infty} a_n \in | E, q |$.

Lemma 2 — Let $\Delta > -1$ and let (y_n) be any non-negative sequence such that $(n^{1+\Delta} y_n) \uparrow$ with $n \geqslant 1$. Then uniformly in $0 < t \leqslant \pi$

$$\sum_{m=1}^{n+1} v^q_m (n+1)\, m^\Delta y_m \exp(imt) = O\{t^{-1}(n+1)^{\Delta-(1/2)} y_{n+1}\}.$$

PROOF : We have

$$\sum_{m=1}^{n+1} v^q_m (n+1)\, m^\Delta y_m \exp(imt)$$

$$= \frac{1+q}{n+2} \sum_{m=1}^{n+1} v^q_{m+1} (n+2)\, m^{1+\Delta} y_m \exp(imt)$$

$$+ \frac{1+q}{n+2} \sum_{m=1}^{n+1} v^q_{m+1} (n+2)\, m^\Delta y_m \exp(imt)$$

$$= \frac{1+q}{n+2} \sum_{m=0}^{n} v^q_{m+2} (n+2)\, (m+1)^{1+\Delta} y_{m+1} \exp\{i(m+1)\, t\}$$

$$+ \frac{(1+q)^2}{(n+2)(n+3)} \sum_{m=1}^{n+1} v^q_{m+2} (n+3) \left(1 + \frac{2}{m}\right) m^{1+\Delta} y_m \exp(imt)$$

$$= \Sigma_1 + \Sigma_2, \text{ say}$$

where

$$\sum\nolimits_2 = O\left\{\frac{1}{(n+1)^2} \sum_{m=1}^{n+1} v^q_{m+2} (n+3)\, m^{1+\Delta} y_m\right\}$$

$$= O\{(n+1)^{\Delta-1} y_{n+1}\}.$$

Let s denote the integral part of $(n-3q)/(1+q)$. Then it may be observed that $v^q_{m+3}(n+2)$ is maximum for $m = s$ and, whenever $(n-3q)/(1+q)$ is an integer,

$$v^q_{s+2}(n+2) = v^q_{s+3}(n+2).$$

Thus, by using the fact that $0 < m^{1+\Delta} y_m \uparrow$ for $m \geqslant 1$, it follows by Abel's lemma that

$$\Sigma_1 = O\{t^{-1}(n+1)^\Delta y_{n+1} v^q_{s+3}(n+2)\}$$

uniformly in $0 < t \leqslant \pi$. Further

$$v^{q}_{s+3}(n+2) = \frac{q^{n-s-1}\Gamma(n+3)}{(1+q)^{n+3}\,\Gamma(s+4)\,\Gamma(n-s)}.$$

Therefore, by using Stirling's asymptotic values (see Hobson 1957, p. 70) for $\Gamma(n+3)$, $\Gamma(s+4)$ and $\Gamma(n-s)$, it follows that

$$v^{q}_{s+3}(n+2) = O\left(\frac{1}{\sqrt{n+1}}\right).$$

Collecting the results, proof of the lemma follows.

Lemma 3 — Let $\beta \geqslant 0$ and $s = 1, 2$. Then, uniformly in $0 < t \leqslant \pi$,

$$\int_0^t \frac{\sin nu}{nu\,(\log_s k/u)^{\beta}}\,du = O\left\{\frac{1}{n\,(\log_s n)^{\beta}}\right\}.$$

For $s = 1$ see Chandra (1972a, d), and for $s = 2$ see Chandra (1975a); Lemma 4.

Lemma 4 — Let $s = 1, 2$. Then for all real β

$$\int_0^{\pi} \left(\log_s \frac{k}{t}\right)^{\beta} \frac{\sin nt}{t}\,dt \sim \frac{\pi}{2}\,(\log_s n)^{\beta}$$

where $k \geqslant \pi e^2$.

Proof: We sketch a proof for $s = 2$. The proof for $s = 1$ is similar. For arbitrarily large $\Delta > 0$, we write

$$\int_0^{\pi} (\log_2 (k/t))^{\beta} \frac{\sin nt}{t}\,dt = \int_0^{n\pi} \left(\log_2 \frac{kn}{\theta}\right)^{\beta} \frac{\sin\theta}{\theta}\,d\theta$$

$$= \left(\int_0^{\Delta} + \int_{\Delta}^{\sqrt{n}} + \int_{\sqrt{n}}^{n\pi}\right)\left(\left(\log_2 \frac{kn}{\theta}\right)^{\beta} \frac{\sin\theta}{\theta}\,d\theta\right)$$

$$= I_1 + I_2 + I_3, \text{ say.}$$

Now

$$I_1 = (\log_2 n)^{\beta} \int_0^{\Delta} \left[1 + (\log_2 n)^{-1} \log\left\{1 + \frac{\log (k/\theta)}{\log n}\right\}\right]^{\beta} \frac{\sin\theta}{\theta}\,d\theta$$

(*equation continued on p.* 221)

$$\to (\log_2 n)^\beta \int_0^\Delta \frac{\sin\theta}{\theta}\, d\theta \quad (n \to \infty)$$

$$= (\log_2 n)^\beta \frac{\pi}{2} \left\{1 + O\left(\frac{1}{\Delta}\right)\right\}. \qquad \text{...(3.1)}$$

By the second mean value theorem for η and η' in $(\Delta, \sqrt{n})$, we have

$$I_2 = \begin{cases} (\log_2 kn^{1/2})^\beta \int_\eta^{\sqrt{n}} \frac{\sin\theta}{\theta}\, d\theta & (\beta \leqslant 0) \\ \left(\log_2 \frac{kn}{\Delta}\right)^\beta \int_\Delta^{\eta'} \frac{\sin\theta}{\theta}\, d\theta & (\beta > 0) \end{cases}$$

$$= O\{(\log_2 n)^\beta/\Delta\}. \qquad \text{...(3.2)}$$

Again, by using the second mean value theorem for ξ and ξ' in $(\sqrt{n}, n\pi)$, we get

$$I_3 = \begin{cases} \left(\log_2 \frac{k}{\pi}\right)^\beta \int_{\sqrt{n}}^{\xi} \frac{\sin\theta}{\theta}\, d\theta & (\beta \leqslant 0) \\ (\log_2 k\sqrt{n})^\beta \int_{\sqrt{n}}^{\xi'} \frac{\sin\theta}{\theta}\, d\theta & (\beta > 0). \end{cases}$$

$$= O\{n^{-1/2}(\log_2 n)^\beta\}. \qquad \text{...(3.3)}$$

Combining (3.1), (3.2) and (3.3), we obtain that

$$\int_0^\pi \left(\log_2 \frac{k}{t}\right)^\beta \frac{\sin nt}{t}\, dt = \frac{\pi}{2} (\log_2 n)^\beta \left[1 + O\left(\frac{1}{\Delta}\right) + O(n^{-1/2})\right].$$

Since $\Delta > 0$ is arbitrarily large, the desired result follows on letting $n \to \infty$.

Lemma 5 — Let $0 < u < \pi$ and $0 < \alpha < 1$. Then

$$\Gamma(1 - \alpha)\, g(n, u) = \frac{n^{\alpha-1}}{1-\alpha} K(n, z)$$

$$+ n^{\alpha-1} \left\{\sin ny - \sin (nu + 1) - \int_{nu+1}^{ny} \frac{\sin\theta}{\theta}\, d\theta\right\}$$

where $\quad u \leqslant z \leqslant u + \frac{1}{n}$ and $u + \frac{1}{n} < y < \pi$.

PROOF : $$\Gamma(1-\alpha)\, g(n, u) = \int_u^\pi (t-u)^{-\alpha} K(n, t)\, dt$$

$$= \left(\int_u^{u+(1/n)} + \int_{u+(1/n)}^\pi\right) ((t-u)^{-\alpha} K(n, t)\, dt)$$

$$= I_1 + I_2, \text{ say.}$$

By the first mean value theorem

$$I_1 = K(n, z) \int_u^{u+(1/n)} (t-u)^{-\alpha}\, dt \quad (u \leqslant z \leqslant u + n^{-1})$$

$$= \frac{n^{\alpha-1}}{1-\alpha} K(n, z).$$

And, by the second mean value theorem

$$I_2 = n^\alpha \int_{u+(1/n)}^{y} K(n, t)\, dt \quad \left(u + \frac{1}{n} < y < \pi\right)$$

$$= n^\alpha \left\{ \int_{u+(1/n)}^{y} \cos nt\, dt - \int_{u+(1/n)}^{y} \frac{\sin nt}{nt}\, dt \right\}$$

$$= n^{\alpha-1} \left\{ \sin ny - \sin (nu+1) - \int_{nu+1}^{ny} \frac{\sin \theta}{\theta}\, d\theta \right\}.$$

This completes the proof of the lemma.

4. PROOF OF THEOREMS 1 AND 2

For $s = 1, 2$, we write

$$g(t, s) = P(t) \log_s (k/t)$$

and

$$h(n, s) = (\log (n+1))^{1-s} (\log_s (n+s))^{-\epsilon}.$$

Then (see Chandra 1974)

$$A_n(x) = \frac{2}{\pi} \int_0^\pi P(t)\, K(n, t)\, dt$$

(equation continued on p. 223)

$$= \frac{2}{\pi} g(\pi, s) \int_0^\pi \left(\log_s \frac{k}{t}\right)^{-1} K(n, t)\, dt$$

$$- \frac{2}{\pi} \int_0^\pi dg(t, s) \int_0^t \left(\log_s \frac{k}{u}\right)^{-1} K(n, u)\, du.$$

And, for $0 < t \leqslant \pi$,

$$\int_0^t \left(\log_s \frac{k}{u}\right)^{-1} K(n, u)\, du$$

$$= \frac{\sin nt}{n} \left(\log_s \frac{k}{t}\right)^{-1} + O\left\{\frac{1}{n \log_s (n + s)}\right\},$$

by Lemma 2. Therefore

$$A_n(x) = O\left\{\frac{1}{n \log_s (n + s)}\right\} - \frac{2}{\pi} \int_0^\pi dg(t, s) \frac{\sin nt}{n \log_s (k/t)}.$$

4. Sufficient and Necessary Conditions

(i) *The condition is sufficient* — By Lemma 1,

$$\sum_{n=1}^{\infty} \frac{h(n, s)}{n \log_s (n + s)} \in |\, E, q \,| \; (q \geqslant 0).$$

Therefore it is sufficient to show that

$$\sum = \sum_{n=0}^{\infty} \frac{1}{n + 1} \left| \sum_{m=1}^{n+1} v_m^q (n + 1) \sin mt\, h(m, s) \right| = O\{\log_s (k/t)\}$$

uniformly in $0 < t < \pi$, since

$$v_m^q (n) = (1 + q) \frac{m + 1}{n + 1} v_{m+1}^q (n + 1) \quad (m \geqslant 0).$$

Now, we write

$$\Sigma = \sum_{n \leqslant T} + \sum_{n > T}.$$

The inner sum in $\sum_{n \leqslant T}$ does not, in modulus, exceed $O\{h(n, s)\}$, therefore uniformly in $0 < t < \pi$

$$\sum_{n \leqslant T} = O\{\log_s (k/t)\}.$$

And, by Lemma 2, it follows that

$$\sum_{n>T} = O(t^{-1}) \sum_{n>T} n^{-3/2}$$
$$= O(1)$$

uniformly in $0 < t < \pi$. Thus condition is sufficient.

(ii) *The condition is necessary* — Let

$$\sum_{n=1}^{\infty} A_n(x)\, h(n, s) \in |\, E, q\,|\ (q > 0),$$

whenever $g(t, s) \in BV(0, \pi)$. Then, for $g(t, s) = 1$ in $(0, \pi)$,

$$A_n(x) = \frac{2}{\pi} \int_0^{\pi} \frac{\cos nt}{\log_s (k/t)}\, dt - \frac{2}{\pi} \int_0^{\pi} \frac{\sin nt}{nt \log_s (k/t)}\, dt$$

$$= -\frac{2}{\pi} \int_0^{\pi} \frac{\sin nt}{tn} \left[\frac{1}{\log_s (k/t)} + t \frac{d}{dt} \left(\frac{1}{\log_s (k/t)}\right)\right] dt$$

$$\sim -\frac{1}{n \log_s n},$$

by Lemma 4, and in order that $\Sigma\, A_n(x)\, h(n, s) \in |\, E, q\,|\ (q > 0)$, it is necessary that

$$\sum_{n=1}^{\infty} \frac{h(n, s)}{n \log_s (n + s)} < \infty.$$

Hence the condition is necessary.

This proves Theorems 1 and 2 completely.

5. Proof of Theorems 3 and 4

Proof of Theorem 3 — We have

$$A_n(x) = \frac{2}{\pi} \int_0^{\pi} \phi^*(t) \cos nt\, dt$$

where $\phi^*(t) = \phi(t) - f(x)$. Then $\sum_{n=0}^{\infty} A_{n+1}(x)\, y_{n+1} \in |\, E, q\,|\ (q > 0)$ if

$$\Sigma = \sum_{n=0}^{\infty} \left|\, \sum_{m=0}^{n} v_m^q (n)\, y_{m+1} A_{m+1}(x) \,\right| < \infty. \qquad \ldots(5.1)$$

However

$$\sum = \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{1+q}{1+n} \left| \int_0^{\pi} \phi^*(t) \left(\sum_{m=1}^{n+1} v_m^q \,(n+1)\, m y_m \cos mt \right) dt \right| . \qquad \ldots(5.2)$$

Splitting up the integral $\int_0^{\pi}$ into $\int_0^{\pi/n}$ and $\int_{\pi/n}^{\pi}$ and denoting them by I_1 and I_2, respectively, we have

$$| I_1 | \leqslant \int_0^{\pi/n} | \phi^*(t) | \left(\sum_{m=1}^{n+1} v_m^q \,(n+1)\, m y_m \right) dt$$

$$\leqslant (n+1)\, y_{n+1} \int_0^{\pi/n} | \phi^*(t) | \, dt \qquad \text{(by (2.6i))}$$

$$= O\{y_{n+1}\}$$

by (2.7). And, by Lemma 2,

$$I_2 = \int_{\pi/n}^{\pi} | \phi^*(t) | \, O(t^{-1} \sqrt{(n+1)}\, y_{n+1})\, dt$$

$$= O\left\{(n+1)^{1/2}\, y_{n+1} \int_{\pi/n}^{\pi} \frac{| \phi^*(t) |}{t}\, dt \right\}$$

$$= O\{(n+1)^{1/2} \log (n+1)\, y_{n+1}\},$$

integrating by parts and using (2.7).

Combining I_1 and I_2 and using (2.6ii), we observe that (5.1) holds. Consequently, this completes the proof of Theorem 3.

Proof of Theorem 4 — We have

$$B_n(x) = \frac{2}{\pi} \int_0^{\pi} \psi(t) \sin nt \, dt$$

where, by (2.7),

$$\int_0^t | \psi(u) | \, du = O(t).$$

The series $\sum_{n=1}^{\infty} B_n(x)\, y_n \in | E, q | \; (q > 0)$ if (5.1) with $B_{m+1}(x)$ for $A_{m+1}(x)$ holds. On replacing $\phi^*(t)$ and $\cos nt$ by, respectively, $\psi(t)$ and $\sin nt$ in (5.2) and proceeding as in Theorem 3, the proof of the theorem follows.

6. Proof of Theorem 5

The proof of the theorem for $\alpha = 0$ follows from Chandra (1975c). Thus we prove the theorem for $0 < \alpha \leqslant 1$.

Writing $q = \alpha p$ $(0 < \alpha \leqslant 1)$; we first consider the following:

Case I: *When* $\alpha = 1$ — Integrating by parts and using $P_1(t) \in BV(0, \pi)$, we obtain that

$$A_n(x) = 2P_1(\pi) \cos n\pi - \frac{2}{\pi} \int_0^\pi P_1(t)\, tK^{(1)}(n, t)\, dt$$

$$= 2P_1(\pi) \cos n\pi - \frac{2}{\pi} P_1(\pi) \int_0^\pi tK^{(1)}(n, t)\, dt$$

$$+ \frac{2}{\pi} \int_0^\pi dP_1(t) \int_0^t uK^{(1)}(n, u)\, du$$

$$= O\left(\frac{1}{n}\right) + \frac{2}{\pi} \int_0^\pi t \cos nt\, dP_1(t).$$

Following (2.9) and Lemma 1, it is sufficient to show, uniformly in $0 < t < \pi$, that

$$\sum_{n=0}^{\infty} (1 + n)^{-1} \left| \sum_{m=1}^{n+1} v^q_m (n + 1) \sqrt{m}\, y_m \cos mt \right| = O(t^{-1})$$

which follows from Lemma 2 ($\Delta = \frac{1}{2}$).

Finally, we consider the following:

Case II : *When* $0 < \alpha < 1$ — We have

$$A_n(x) = \frac{2}{\pi(1 - \alpha)} \int_0^\pi K(n, t)\, dt \int_0^t (t - u)^{-\alpha}\, dP^*_\alpha(u)$$

$$= \frac{2}{\pi(1 - \alpha)} \int_0^\pi dP^*_\alpha(u) \int_u^\pi (\theta - u)^{-\alpha} K(n, \theta)\, d\theta$$

(*equation continued on p.* 227)

$$= \frac{2\alpha P_\alpha(\pi)}{\pi\Gamma(\alpha+1)} \int_0^\pi u^{\alpha-1} g(n, u)\, du$$

$$+ \frac{2}{\pi\Gamma(\alpha+1)} \int_0^\pi dP_\alpha(t) \int_0^t u^\alpha \frac{d}{du} g(n, u)\, du.$$

Writing

$$A_n^{(1)}(x) = \int_0^\pi u^{\alpha-1} g(n, u)\, du \qquad \ldots(6.1)$$

it follows that

$$A_n^{(1)}(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^\pi K(n, t)\, dt \int_0^t u^{\alpha-1}(t-u)^{-\alpha}\, du$$

$$= \Gamma(\alpha) \int_0^\pi K(n, t)\, dt$$

$$= O\left(\frac{1}{n}\right). \qquad \ldots(6.2)$$

Thus following Lemma 1, it is enough to show the uniform boundedness in $0 < t < \pi$ of the following:

$$\sum_{n\geqslant 0} = \sum_{n=0}^{\infty} (n+1)^{-1} \left| \sum_{m=1}^{n+1} v_m^q\, (n+1)\, m^{1-(\alpha/2)} y_m \int_0^t u^\alpha \frac{d}{du} g(m, u)\, du \right|.$$

Now, we first consider those values of n for which $n \leqslant T$. Integrating by parts

$$\int_0^t u^\alpha \frac{d}{du} g(m, u)\, du = t^\alpha g(m, t) - \alpha \int_0^t u^{\alpha-1} g(m, u)\, du \qquad \ldots(6.3)$$

$$= O(t^\alpha m^{\alpha-1})$$

since, by Lemma 5, $g(n, u) = O(n^{\alpha-1})$. Therefore

$$\sum_{n\leqslant T} = O(t^\alpha) \sum_{n\leqslant T} \frac{1}{n+1} \sum_{m=1}^{n+1} v_m^q\, (n+1)\, m^{\alpha/2} y_m$$

$$= O(t^\alpha) \sum_{n\leqslant T} (n+1)^{(\alpha/2)-1} y_{n+1}$$

$$= O(1)$$

uniformly in $0 < t < \pi$. And, following (6.1) and (6.3),

$$\int_0^t u^\alpha \frac{d}{du} g(m, u)\, du = t^\alpha g(m, t) - \alpha A_m^{(1)}(x) + \alpha \int_t^\pi u^{\alpha-1} g(n, u)\, du$$

Therefore, following (6.2), (2.9) and Lemma 1, it is enough, for $n > T$, to show that

$$\sum_{n>T} = \sum_{n>T} (n+1)^{-1} \left| \sum_{m=1}^{n+1} v_m^q (n+1)\, m^{1-(\alpha/2)} y_m \{t^\alpha g(m, t) + \alpha \int_t^\pi u^{\alpha-1} g(m, u)\, du\} \right|$$

$$= O(1), \text{ uniformly in } 0 < t < \pi.$$

However by Lemmas 5 and 2,

$$\sum_{m=1}^{n+1} v_m^q (n+1)\, m^{1-(\alpha/2)} y_m \{t^\alpha g(m, t) + \alpha\} \int_t^\pi u^{\alpha-1} g(m, u)\, du$$

$$= O(t^{\alpha-1}) \sum_{m=1}^{n+1} v_m^q (n+1)\, m^{(\alpha/2)-1} y_m + O(t^{\alpha-1})\, y_{n+1}(n+1)^{(\alpha-1)/2}$$

$$= O\{t^{\alpha-1}(n+1)^{(\alpha-1)/2}\}.$$

Therefore

$$\sum_{n>T} = O(t^{\alpha-1}) \sum_{n>T} (n+1)^{((\alpha-1)/2)-1}$$

$$= O(1)$$

uniformly in $0 < t < \pi$.

This completes the proof of the theorem.

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