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ABSOLUTE SUMMABILITY BY RIESZ MEANS

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In this paper the authors investigate local conditions imposed upon the generating functions of Fourier series and conjugate series to obtain the absolute Riesz summability of the general type and order one of factored Fourier series and conjugate series. In particular, they also improve the known results due to Hsiang (1970) and Yadav (1978).

1. DEFINITIONS AND NOTATIONS

Let $\sum_1^\infty a_n$ be a given infinite series. Then $\sum_1^\infty a_n \in | R, \lambda(w), r |$, if (see Obrechkoff 1928, 1929 and Mohanty 1951)

$$\int_h^\infty \frac{\lambda'(w)}{(\lambda(w))^{r+1}} \left| \sum_{n \leqslant w} \{\lambda(w) - \lambda(n)\}^{r-1} \lambda(n) a_n \right| dw < \infty$$

where h is a positive constant.

Let f be 2π-periodic and L-integrable over $(-\pi, \pi)$. Without loss of generality, let the Fourier series of f, at a point x, be given by

$$\sum_{n=1}^\infty (a_n \cos nx + b_n \sin nx) \equiv \sum_{n=1}^\infty A_n(x). \qquad \text{...(1.1)}$$

And the series, conjugate to (1.1), is given by

$$\sum_{n=1}^\infty (b_n \cos nx - a_n \sin nx) \equiv \sum_{n=1}^\infty B_n(x). \qquad \text{...(1.2)}$$

For fixed real number x, we write

$$\phi(t) = \tfrac{1}{2}\{f(x+t) + f(x-t)\} \qquad \text{...(1.3)}$$

$$\psi(t) = \tfrac{1}{2}\{f(x+t) - f(x-t)\} \qquad \text{...(1.4)}$$

$$t\phi_1(t) = \int_0^t \phi(u)\, du \qquad \text{...(1.5)}$$

$$t\psi_1(t) = \int_0^t \psi(u)\, du \qquad \text{...(1.6)}$$

$$P(t) = \phi(t) - \phi_1(t). \qquad \text{...(1.7)}$$

2. Introduction

Concerning the $|C, 1|$-summability of (1.1) and (1.2), the following theorems are due to Hsiang (1970):

Theorem A — Let $r > 0$ and let

$$\Phi(t) = \int_0^t |\phi(u)|\, du = O(t)\ (t \downarrow 0). \qquad \text{...(2.1)}$$

Then $\sum_1^\infty n^{-r} A_n(x) \in |C, 1|$.

Theorem B — Let $r > 0$ and let

$$\Psi(t) = \int_0^t |\psi(u)|\, du = O(t)\ (t \downarrow 0). \qquad \text{...(2.2)}$$

Then $\sum_1^\infty n^{-r} B_n(x) \in |C, 1|$.

Recently, Yadav (1978) proved the following results concerning absolute Riesz summability of (1.1) and (1.2).

Theorem C — Let $\alpha > 0$ and $\alpha + \beta < 0$. Then

$$\phi_1(t) \in BV(0, \pi) \Rightarrow \sum_1^\infty A_n(x)\, n^\beta \in |R, \exp(n^\alpha), 1|.$$

Theorem D — Let $\alpha > 0$ and $\alpha + \beta < 0$. Then

$$\psi_1(t) \in BV(0, \pi) \Rightarrow \sum_1^\infty B_n(x)\, n^\beta \in |R, \exp(n^\alpha), 1|.$$

We first observe that

$$\phi(t) \in BV(0, \pi) \Rightarrow \phi_1(t) \in BV(0, \pi) \qquad \text{...(2.3)}$$

but the converse is not true, in general, since if

$$\phi(t) = t \sin \frac{1}{t}\ in\ (0, \pi)$$

then $\phi(t) \notin BV(0, \pi)$ but

$$\int_0^\pi |\, d\phi_1(t)\,| = \int_0^\pi \left| d\left(\frac{1}{t}\int_0^t u \sin\frac{1}{u}\, du\right)\right|$$
$$< \infty .$$

Secondly,

$$\phi_1(t) \in BV(0, \pi) \Rightarrow \Phi(t) = O\,(t)\,(t \downarrow 0) \qquad \ldots(2.4)$$

but converse is not true, in general, since, whenever $\phi_1(t) \in BV(0, \pi)$,

$$\Phi(t) = \int_0^t |\, \phi_1(u)\, du + u d\phi_1(u)\,|$$
$$\leqslant \int_0^t |\, \phi_1(u)\,|\, du + t\int_0^t |\, d\phi_1(u)\,|$$
$$= O\,(t).$$

However, if

$$\phi(t) = 2t \sin\frac{1}{t} - \cos\frac{1}{t},$$

then

$$\Phi(t) = \int_0^t \left| 2u \sin\frac{1}{u} - \cos\frac{1}{u}\right| du$$
$$= O(t)\,(t \downarrow 0)$$

but on the other hand,

$$\phi_1(t) = \frac{1}{t}\int_0^t \left(2u \sin\frac{1}{u} - \cos\frac{1}{u}\right) du$$
$$= t \sin\frac{1}{t}$$
$$\notin BV(0, \pi).$$

Now, finally, we remark that the relation

$$\Phi(t) = O\,(t) \Rightarrow \int_0^t |\, P(u)\,|\, du = O\,(t) \qquad \ldots(2.5)$$

is also strict, since, for $\phi(t) = \log(1/t)$,

$$\Phi(t) = t\left(1 + \log\frac{1}{t}\right)$$

while

$$P(t) = 1.$$

Therefore, in view of the inclusion relations (2.3), (2.4) and (2.5), we propose to study $| R, \lambda(w), 1 |$ -summability factors for Fourier series under the condition :

$$\int_0^t | P(u) | \, du = O(t) \quad (t \downarrow 0) \qquad \ldots(2.6)$$

and deduce its corollaries improving Theorems A and C. Precisely, we prove the following :

Theorem 1 — Let the type of Riesz mean $\lambda(w)$ and the function $y(w)$ satisfy the following conditions :

$$0 < \lambda(w)\, y(w) \downarrow \text{ for } w \geqslant w_0 \qquad \ldots(2.7)$$

$$\int_h^\infty w^{-1} y(w)\, dw < \infty \qquad \ldots(2.8)$$

$$\int_h^\infty \frac{\lambda'(w)}{\lambda(w)}\, y(w) \log w \, dw < \infty, \qquad \ldots(2.9)$$

where h is some suitable positive constant. Then

$$(2.6) \Rightarrow \sum_{n=1}^{\infty} A_n(x)\, y(n) \in | R, \lambda(w), 1 | . \qquad \ldots(2.10)$$

We also state the following theorem for the conjugate series:

Theorem 2 — Let the type of Riesz mean $\lambda(w)$ and the function $y(w)$ satisfy the conditions (2.7), (2.8) and (2.9). Then

$$\int_0^t | \psi(u) | \, du = O(t) \quad (t \downarrow 0) \qquad \ldots(2.11)$$

implies that $\sum_{n=1}^{\infty} B_n(x)\, y(n) \in | R, \lambda(w), 1 |$.

3. Proof of the Theorems

Proof of Theorem 1 — We have (see Chandra 1974, Theorem 1)

$$A_n(x) = \frac{2}{\pi} \int_0^{\pi} P(t)\left(\cos nt - \frac{\sin nt}{nt}\right) dt.$$

Then, $\sum_{n=1}^{\infty} A_n(x)\, y(n) \in |R, \lambda(w), 1|$ if

$$I = \frac{2}{\pi} \int_h^{\infty} \frac{\lambda'(w)}{(\lambda(w))^2} \left| \int_0^{\pi} P(t)\left(\sum_{n \leqslant w} \lambda(n)\, y(n)\left(\cos nt - \frac{\sin nt}{nt}\right)\right) dt \right| dw$$

is convergent. Now, splitting up the inner integral into $\int_0^{w^{-1}}$ and $\int_{w^{-1}}^{\pi}$, we write the corresponding I by I_1 and I_2. Then

$$I_1 \leqslant \frac{4}{\pi} \int_h^{\infty} \frac{\lambda'(w)}{(\lambda(w))^2} \int_0^{w^{-1}} |P(t)| \left(\sum_{n \leqslant w} \lambda(n)\, y(n)\right) dt\, dw.$$

However,

$$\sum_{n \leqslant w} \lambda(n)\, y(n) = \left(\sum_{n=1}^{[w_0]} + \sum_{1+[w_0]}^{[w]-1}\right)(\lambda(n)\, y(n)) + \lambda([w])\, y([w])$$

$$\leqslant O(1) + \sum_{1+[w_0]}^{[w]-1} \lambda(n)\, y(n) + \lambda(w)\, y(w)$$

and, by (2.7),

$$\sum_{1+[w_0]}^{[w]-1} \lambda(n)\, y(n) = \sum_{1+[w_0]}^{[w]-1} \lambda(n)\, y(n) \int_n^{n+1} dp$$

$$\leqslant \sum_{1+[w_0]}^{[w[-1} \int_n^{n+1} \lambda(p)\, y(p)\, dp$$

$$= \int_{1+[w_0]}^{[w]} \lambda(p)\, y(p)\, dp.$$

Therefore

$$\sum_{n \leqslant w} \lambda(n)\, y(n) = O\{\lambda(w)\, y(w)\} + O\left\{\int_h^{w} \lambda(p)\, y(p)\, dp\right\}$$

which yields that

$$I_1 = O(1) \int_h^{\infty} \frac{\lambda'(w)}{w\lambda(w)}\, y(w)\, dw + O(1) \int_h^{\infty} \frac{\lambda'(w)}{w(\lambda(w))^2} \left\{\int_h^{w} \lambda(p)\, y(p)\, dp\right\} dw$$

{by (2.6)}

(*equation continued on p.* 235)

$$= O(1)\int_h^\infty \frac{y(w)\,\lambda'(w)}{w\lambda(w)}\,dw + O(1)\int_h^\infty \frac{y(w)}{w}\,dw$$

$$= O(1)$$

by (2.8) and (2.9). And, by (2.7) and Abel's lemma, we have

$$\sum_{n\leqslant w} \lambda(n)\,y(n)\left(\cos nt - \frac{\sin nt}{nt}\right) = O(t^{-1}\lambda(w)\,y(w)) \qquad \ldots(3.1)$$

uniformly in $0 < t \leqslant \pi$. And hence

$$I_2 = O(1)\int_h^\infty \frac{\lambda'(w)\,y(w)}{\lambda(w)}\left(\int_{w^{-1}}^{\pi} t^{-1}\,|\,P(t)\,|\,dt\right)dw$$

$$= O(1)$$

uniformly in $0 < t < \pi$, since the integration by parts and the condition (2.6) yield that

$$\int_{w^{-1}}^{\pi} t^{-1}\,|\,P(t)\,|\,dt = O\,(\log w).$$

This completes the proof of Theorem 1.

Proof of Theorem 2 — We have

$$B_n(x) = \frac{2}{\pi}\int_0^\pi \psi(t)\sin nt\,dt.$$

Now, proceeding as in Theorem 1 and using

$$\sum_{n\leqslant w} \lambda(n)\,y(n)\sin nt = O\{t^{-1}\,\lambda(w)\,y(w)\} \qquad \ldots(3.2)$$

in place of (3.1), we may obtain the proof of the theorem.

4. Corollaries

Now we deduce a few corollaries from the theorems.

On taking $\lambda(w) = w$ and $y(n) = (\log(n+1))^{-2-\epsilon}$ in Theorem 1 and using the fact that

$$|\,R, w, 1\,| \sim |\,C, 1\,|$$

we obtain the following.

Corollary 1 — Let (2.6) hold. Then

$$\sum_{n=1}^{\infty} \frac{A_n(x)}{(\log (n+1))^{2+\epsilon}} \in |C, 1|, \text{ where } \epsilon > 0.$$

Next, we choose in Theorem 1

$$\lambda(w) = \exp(w^{\alpha}) \ (0 < \alpha < 1) \text{ and } y(n) = \frac{n^{-\alpha}}{(\log (n+1))^{2+\epsilon}} \ (\epsilon > 0).$$

Then we get the following.

Corollary 2 — Let $0 < \alpha < 1$ and $\epsilon > 0$. Then

$$(2.6) \Rightarrow \sum_{n=1}^{\infty} \frac{A_n(x)}{n^{\alpha} (\log (n+1))^{2+\epsilon}} \in |R, \exp(w^{\alpha}), 1|.$$

It follows from (2.4) and (2.5) that (2.6) is lighter than the hypothesis $\phi_1(t) \in BV(0, \pi)$ of Theorem C and the absolute summability factor of Corollary 2 is better than that of Theorem C.

Now, we state the following corollary which yields a new result.

Corollary 3 — Let (2.6) hold. Then

$$\sum_{n=1}^{\infty} \frac{A_n(x)}{(\log (n+1))^{1+\epsilon}} \in |R, \log w, 1| .$$

For the proof, we take $\lambda(w) = (\log w)^2$ and $y(n) = (\log (n+1))^{-1-\epsilon}$ $(1 \geqslant \epsilon > 0)$ in Theorem 1 and then we use the second theorem of consistency for the absolute Riesz summability.

Finally we state the following corollaries of Theorem 2 which improve Theorems B and D.

Corollary 4 — Let $\epsilon > 0$. Then

$$(2.11) \Rightarrow \sum_{n=1}^{\infty} \frac{B_n(x)}{(\log (n+1))^{2+\epsilon}} \in |C, 1| .$$

Corollary 5 — Let $0 < \alpha < 1$ and $\epsilon > 0$. Then

$$(2.11) \Rightarrow \sum_{n=1}^{\infty} \frac{n^{-\alpha} B_n(x)}{(\log (n+1))^{2+\epsilon}} \in |R, \exp(w^{\alpha}), 1| .$$

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