Indian J. pure appl. Math., **11**(2) : 238-244, February 1980

WAVES CREATED AGAINST A VERTICAL CLIFF

A. K. PRAMANIK

Centre of Advanced Study in Applied Mathematics, University College of Science, Calcutta 700009

(*Received* 17 *April* 1979)

The initial-value problem of water waves generated by an oscillatory pressure distribution against a vertical cliff is studied and an uniformly valid asymptotic analysis of the unsteady state is given.

1. INTRODUCTION

The transient development of two-dimensional gravity waves generated by certain oscillatory pressure distribution has been considered by Miles (1962) and by Debnath (1968) in a fluid which is unbounded in both the horizontal directions. On the other hand there exists well-known solution for free harmonic oscillation in a fluid bounded on one side by a vertical cliff (Stoker 1957). But very little appears to have been done regarding the transient development of waves against a vertical cliff produced by certain pressure distribution.

In the present paper we consider the transient wave motion generated by an oscillatory pressure distribution in a fluid bounded on one side by a vertical cliff. Again the asymptotic solutions obtained by Miles (1962) and Debnath (1968) are not uniform in that these solutions fail when the stationary point of certain integrand coincides with the pole. Here we have employed a method that yields an asymptotic expression which remains uniformly valid even when the stationary point of certain integrand coincides with the pole. Finally it is shown here that when an oscillatory pressure distribution is placed in a fluid, waves propagate along both the directions and the wave on reaching the cliff is reflected back.

2. STATEMENT AND FORMAL SOLUTION OF THE PROBLEM

We take the origin on the cliff, x-axis along the undisturbed free surface and y-axis vertically upwards. The system being initially at rest waves are produced by the continued application of the pressure distribution $p(x, t)$. Let $\varphi(x, y, t)$ be the velocity potential, $\eta(x, t)$ the surface elevation, ρ the density and g the acceleration due to gravity. These functions are assumed to be defined in the generalized sense. Then we have the following intial-value problem,

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0,\ 0 \leqslant x < \infty, y \leqslant 0, t \geqslant 0 \qquad ...(1)$$

$$\frac{\partial \varphi}{\partial t} + g\eta + (p/\rho) = 0 \text{ at } y = 0 \qquad \text{...(2)}$$

$$\frac{\partial \eta}{\partial t} = \frac{\partial \varphi}{\partial y} \text{ at } y = 0 \qquad \text{...(3)}$$

$$\left.\begin{aligned} &\frac{\partial \varphi}{\partial x} = 0 \text{ at } x = 0 \\ &\varphi = 0 \text{ at } y = \infty \end{aligned}\right\} \qquad \text{...(4)}$$

and

$$\eta(x, 0) = 0, \quad \varphi(x, y, 0) = 0. \qquad \text{...(5)}$$

To solve the problem posed above we write φ and p in the following integral forms,

$$\varphi = \sqrt{2/\pi} \int_0^\infty A(k, t)\, e^{ky} \cos kx\, dk \qquad \text{...(6)}$$

$$p = \sqrt{2/\pi} \int_0^\infty \bar{p}(k, t) \cos kx\, dk. \qquad \text{...(7)}$$

Elimination of η between (2) and (3) gives the condition on φ as

$$\frac{\partial^2 \varphi}{\partial t^2} + g\frac{\partial \varphi}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial t} = 0 \text{ at } y = 0.$$

Substitution from (6) and (7) reduces this to

$$\frac{\partial^2 A}{\partial t^2} + \sigma^2 A = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial t}$$

where $\sigma^2 = gk.$

Solution of this equation after utilising the initial condition (5) gives

$$A = -\frac{1}{\rho}\int_0^t \bar{p}(k, \tau) \cos \sigma(t - \tau)\, d\tau. \qquad \text{...(8)}$$

The surface elevation η can be found from (2) with the help of eqns. (6), (7) and (8) in the following integral form

$$\eta = -\frac{1}{g\rho\sqrt{2\pi}}\int_0^\infty \cos kx\, dk \int_0^t \bar{p}(k, \tau) \sin \sigma(t - \tau)\, d\tau. \qquad \text{...(9)}$$

Inversion of (7) now gives

$$\bar{p}(k, t) = \sqrt{\frac{2}{\pi}} \int_0^\infty p(x, t) \cos kx \, dx. \qquad \ldots(10)$$

In the following we shall consider the pressure distribution given by

$$p(x, t) = \bar{f}(x)\, e^{i\omega t}. \qquad \ldots(11)$$

Substituting (10) into (9) and using (11) we get the following integral representation for η

$$2\pi\rho\eta = \int_0^\infty \frac{\sigma f(k)}{\sigma - \omega} e^{i(kx+\sigma t)} dk + \int_0^\infty \frac{\sigma f(k)}{\sigma - \omega} e^{i(\sigma t - kx)} dk$$

$$- e^{i\omega t} \int_0^\infty \frac{\sigma f(k)}{\sigma - \omega} e^{ikx} dk - e^{i\omega t} \int_0^\infty \frac{\sigma f(k)}{\sigma - \omega} e^{-ikx} dk$$

$$+ \int_0^\infty \frac{\sigma f(k)}{\sigma + \omega} e^{i(kx - \sigma t)} dk + \int_0^\infty \frac{\sigma f(k)}{\sigma + \omega} e^{-i(kx+\sigma t)} dk$$

$$- e^{i\omega t} \int_0^\infty \frac{\sigma f(k)}{\sigma + \omega} e^{ikx} dk - e^{i\omega t} \int_0^\infty \frac{\sigma f(k)}{\sigma + \omega} e^{-ikx} dk \qquad \ldots(12)$$

where $$f(k) = \int_0^\infty \bar{f}(\alpha) \cos k\alpha \, d\alpha.$$

3. Asymptotic Analysis

In this section we shall study the unsteady wave motion for large time and distance. The dominant contribution to this asymptotic value comes from the poles and the stationary points of the integrands. We notice that there is one pole $\sigma = \omega$ and one stationary point $\sigma = gt/2x$ to the various integrands in (12). There is one integral in which the integrand possesses both the pole and the stationary point. Such a situation prompts us to seek an asymptotic analysis of the integral which will remain uniformly valid even when the stationary point coincides with the pole. This objective is really realized through a method which is due to Van der Waerden (1951). We shall evaluate the second integral by this method. We replace this integral by the following:

$$\frac{2}{g} \int_0^\infty \frac{\sigma^2 f(\sigma^2 g)}{\sigma - \omega} \exp\left((gt - \sigma x)\, i\sigma/g\right) d\sigma =$$

(*equation continued on p.* 241)

$$= \frac{2}{g} \int_0^\infty (\sigma + \omega) f(\sigma^2/g) \exp\left(i\left(\sigma t - \frac{\sigma^2 x}{g}\right)\right) d\sigma$$

$$+ \frac{2\omega^2}{g} \int_0^\infty \frac{f(\sigma^2/g)}{\sigma - \omega} \exp\left(i\left(\sigma t - \frac{\sigma^2 x}{g}\right)\right) d\sigma. \quad \ldots(13)$$

The method of stationary phase gives for the first σ-integral in (13) the asymptotic value

$$2\left(\frac{\pi}{gx}\right)^{1/2} \left(\frac{gt}{2x} + \omega\right) f(gt^2/4x^2) \exp\left(i(gt^2 - \pi x)/4x\right),\ gt^2/4x \gg 1. \ \ldots(14)$$

To use Van der Waerden method (1951) to second σ-integral in (13) we choose a new variable of integration

$$u = \frac{\sigma}{\omega} - \frac{gt}{2x\omega}.$$

Let us now suppose that the pole $\sigma = \omega$ is sufficiently close to the saddle point $\sigma = gt/2x$. The σ-integral is then

$$\exp(igt^2/4x) \int_{-gt/2x\omega}^{\infty} \frac{f(\sigma^2/g)}{\sigma - \omega} \frac{d\sigma}{du} \exp(-ix\omega^2 u^2/g)\, du. \quad \ldots(15)$$

Since $\sigma = \omega$ is a simple pole of $f(\sigma^2/g)\,(\sigma - \omega)^{-1}$, we may write

$$\frac{f(\sigma^2/g)}{\sigma - \omega} \frac{d\sigma}{du} = \frac{A(\omega)}{u - u_0} + \sum_{n=0}^{\infty} \alpha_n u^n$$

where $u = u_0$ is the position of the pole in the new variable and is given as

$$u_0 = 1 - (gt/2x\omega).$$

It follows immediately that

$$A(\omega) = f(\omega^2/g), \quad \alpha_0 = \frac{\omega\,[f(gt^2/4x^2) - f(\omega^2/g)]}{(gt/2x) - \omega}. \quad \ldots(16)$$

For $(\omega^2 x/g) \to \infty$, Watson's lemma gives

$$\int_{-gt/2x\omega}^{\infty} \frac{f(\sigma^2/g)}{\sigma - \omega} \frac{d\sigma}{du} \exp(-ix\omega^2 u^2/g)\, du \simeq f(\omega^2/g) \int_{-\infty}^{\infty} \frac{\exp(-ix\omega^2 u^2/g)}{u - u_0}\, du$$

$$+ \alpha_0 \int_{-\infty}^{\infty} \exp(-ix\omega^2 u^2/g)\, du. \quad \ldots(17)$$

The extension of the range of integration is possible as it contributes a term $O(x^{-1})$. A little manipulation shows that the first integral on the right of (17) is equal to

$$-\pi(1 + i)\,\mathrm{sgn}\,u_0\,\mathrm{cis}\,(\omega^2 x u_0^2/g)\exp(-i\omega^2 x u_0^2/g) \qquad ...(18)$$

where $\mathrm{cis}\,(x) = C(x) + iS(x)$,

$$C(x) = \frac{1}{\sqrt{2\pi}}\int_0^x \frac{\cos x}{x^{1/2}}\,dx,\quad S(x) = \frac{1}{\sqrt{2\pi}}\int_0^x \frac{\sin x}{x^{1/2}}\,dx.$$

The second integral in (17) is easily found to be

$$(1 - i)(\pi g/2\omega^2 x)^{1/2}. \qquad ...(19)$$

Combining the results (14) to (19) the asymptotic value for the second integral in (12) is obtained as follows:

$$\begin{aligned}&2(\pi/gx)^{1/2}\left(\frac{gt}{2x} + \omega\right) f(gt^2/4x^2)\exp(i(gt^2 - \pi x)/4x)\\ &-\frac{2\pi\omega^2(1 + i)}{g} f(\omega^2/g)\,\mathrm{sgn}\left(1 - \frac{gt}{2x\omega}\right)\mathrm{cis}\,(\omega^2 x u_0^2/g)\\ &\times\exp(i\omega(gt - \omega x)/g) + 2(1 - i)\,\alpha_0\omega\,(\pi/2gx)^{1/2}\exp(igt^2/4x).\end{aligned} \qquad ...(20)$$

The fifth integral contains no pole. The contribution to the asymptotic value comes from the stationary point. The following asymptotic value of this integral is obtained by the method of stationary phase,

$$2(\pi/gx)^{1/2}\frac{f(gt^2/4x^2)\,(gt/2x)^2}{(gt/2x) + \omega}\exp(i(\pi x - gt^2)/4x). \qquad ...(21)$$

The asymptotic value of the first, third and fourth integral of (12) which come from the only pole $\sigma = \omega$ may be evaluated by the previous method (Debnath 1968). The followings are respectively the asymptotic values of the first, third and fourth integrals

$$\left.\begin{aligned}&\frac{2\pi i\omega^2}{g} f(\omega^2/g)\exp\left(i\left(\omega t + \frac{\omega^2}{g}x\right)\right),\\ &-\frac{2\pi i\omega^2}{g} f(\omega^2/g)\exp\left(i\left(\omega t + \frac{\omega^2}{g}x\right)\right),\\ &\frac{2\pi i\omega^2}{g} f(\omega^2/g)\exp\left(i\left(\omega t - \frac{\omega^2}{g}x\right)\right).\end{aligned}\right\} \qquad ...(22)$$

Each of the integrals sixth, seventh and eighth in (12) having neither any pole nor a stationary point has no significant contribution to the asymptotic value of η. Collecting all the results so obtained we get finally the following uniformly valid asymptotic expression for η

$$\rho\eta = \frac{i\omega^2}{g} f(\omega^2/g)\left[1-(1-i)\,\mathrm{sgn}\left(1-\frac{gt}{2x\omega}\right)\mathrm{cis}\left(\frac{\omega^2 x u_0^2}{g}\right)\right]$$

$$\times \exp\left(i\left(\omega t - \frac{\omega^2}{g}x\right)\right) + \frac{f(gt^2/4x^2)}{(\pi g x)^{1/2}}\left[\left(\frac{gt}{2x}+\omega\right)\exp\left(i\left(\frac{gt^2}{2x}-\frac{\pi}{4}\right)\right)\right.$$

$$\left. + \frac{(gt/2x)^2}{(gt/2x)+\omega}\exp\left(i\left(\frac{\pi}{4}-\frac{gt^2}{4x}\right)\right)\right] + \frac{\alpha_0\omega}{(\pi g x)^{1/2}}\exp\left(i\left(\frac{gt^2}{4x}-\frac{\pi}{4}\right)\right). \quad \ldots(23)$$

Evidently the wave motion depicted to (23) consists of a system of decaying dispersive waves superposed on a travelling wave represented by the first term on the right-hand side of (23). The travelling wave has a slowly varying amplitude and moves with a velocity g/ω away from the cliff.

We now consider the important special case $t \gg 1$. It can be easily verified that for all pressure distributions of physical interest excepting the one concentrated at a single point the transient terms in (23) disappear and the motion reduces to a steady wave motion. To obtain this steady state wave we note

$$\mathrm{cis}\left(\frac{\omega^2 x u_0^2}{g}\right) \simeq \tfrac{1}{2}(1+i) + O(x^{-1/2}).$$

The steady state progressive wave η_s is then given by

$$\eta_s = \frac{2i\omega^2}{g\rho} f(\omega^2/g)\exp\left(i(\omega t - \omega^2 x/g)\right) \text{ for } x < gt/2\omega,$$

$$= 0 \text{ for } x > gt/2\omega.$$

Comparison of this result with that in a fluid unbounded in both the horizontal directions reveals a very interesting fact about the wave created against a vertical cliff. It is known that in the previous case the steady state wave system has two components

$$\frac{\pi i\omega^2}{g} f_1(\omega^2/g)\exp\left(i\left(\omega t - \frac{\omega^2 x}{g}\right)\right)$$

and

$$\frac{\pi i\omega^2}{g} f_1(-\,\omega^2/g)\exp\left(i\left(\omega t + \frac{\omega^2 x}{g}\right)\right)$$

where $f_1(k) = \int_{-\infty}^{\infty} \bar{f}(x)\, e^{ikx}dx$, and the components propagate respectively along the positive and the negative x-axis direction. Now if the latter is reflected from the cliff and is then added up with the former we get the result of the present case. This clearly indicates that when an oscillatory pressure distribution is placed in a fluid bounded on one side by a vertical cliff, waves propagate along both the directions and the wave propagating towards the cliff is reflected back on reaching it. That reflection occurs is however physically reasonable. For the wave that moves towards the cliff carries certain energy with it. Now when the wave reaches the cliff, in ideal fluid there being no mechanism to absorb the incoming energy it must be reflected back.

ACKNOWLEDGEMENT

The author is grateful to Prof. A. R. Sen of Viswabharati University for his help during the preparation of this article.

REFERENCES

Debnath, L. (1968). On transient development of surface waves. *Z. angew. Math. Phys.*, **19**, 948.

Miles, J. W. (1962). Transient gravity response to an oscillatory present. *J. Fluid Mech.*, **13**, 145.

Stoker, J. J. (1957). Water Waves. Inter Swice Publications, Inc., New York.

Van der Waerden, B. L. (1951). *Appl. Sci. Res.*, 2B, 353.