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# ON THE RADIUS OF UNIVALENCE AND STARLIKENESS OF A CLASS OF ANALYTIC FUNCTIONS

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Let $A$ be the class of functions $f$ holomorphic in the unit disc $E$, normalized by $f(0) = 0 = f'(0) - 1$. For $|z| < 1$, $0 \leqslant \lambda < 1$, $0 < \alpha \leqslant 1$, let $f, g \in A$ satisfy

$$\left| \frac{f(z) - \lambda f(z) - (1 - \lambda)(g(z))}{f(z) + \lambda f(z) + (1 - \lambda)(g(z))} \right| < \alpha. \qquad \text{...(A)}$$

Let $A(\alpha)$ denote the family of functions $f$ satisfying (A). The aim of this paper is to determine the radius of univalence and starlikeness of the class $A(\alpha)$ where $g$ is subjected to one of the following conditions : (i) $g(z)$ is univalent; (ii) $\operatorname{Re} g(z)/z > 0$; (iii) $g(z)$ is starlike. The paper has been motivated by that of Shah (1972).

## STATEMENT OF THEOREMS

In the sequel we take $k = (1 + \lambda)/(1 - \lambda)$.

*Theorem* 1 — Let $f \in A(\alpha)$ and $g$ be univalent in $E$.

$$\text{Let} \quad \alpha_0 = \frac{(\sqrt{2} - 1)(1 + k) - [(3 - 2\sqrt{2})(k^2 + 1) - 4k(7 - 5\sqrt{2})]^{1/2}}{k(10 - 7\sqrt{2})}.$$

If $\alpha > \alpha_0, f$ is univalent in $|z| < r_1$ where $r_1$ is the smallest positive root of the equation $1 - [2\alpha(1 + k) + 1] r + k\alpha^2 r^2 - k\alpha^2 r^3 = 0$. If $0 < \alpha \leqslant \alpha_0, f$ is univalent in $|z| < r_2$ where $r_2$ is the smallest positive root of the equation

$$2[4k\alpha - (1 + k)] r^4 - 4\alpha[2k\alpha + k + 1] r^3 + 2[2(k\alpha^2 + 1) + 3\alpha(1 + k)] r^2 - 4[\alpha(k + 1) + 2] r + [4 - \alpha(1 + k)] = 0.$$

*Theorem* 2 — Let $f \in A(\alpha)$ and let $g$ in (A) satisfy $\operatorname{Re} g(z)/z > 0$ for $z$ in $E$. The radius of starlikeness of $f$ is given by the only positive root $r_3$ in $(0, 1)$ of the equation

$$k\alpha^2 r^4 + 2\alpha(k\alpha + 1)r^3 - [k\alpha^2 + 2\alpha(k - 1) + 1] r^2 - 2(\alpha + 1)r + 1 = 0$$

where $B = k\alpha \leqslant B_1$, $B_1$ being the unique positive root in $(1, \infty)$ of the equation

$$B^3 - (3\alpha + 4)B^2 + (4\alpha - 1)B + 3\alpha = 0.$$

*Theorem* 3 — Let $f \in A(\alpha)$ and let $g$ in (A) satisfy the condition that $g(z)$ is starlike. Let $\alpha_0$ be the unique root in (0, 1) of the equation $k\alpha^4 - 2k\alpha^3 - 2\alpha + 1 = 0$. Then

(i) For $0 < \alpha < \alpha_0$, the radius of starlikeness of $f$, $r_4$ is the least positive root of the equation

$$k\alpha^2 r^3 - k\alpha(\alpha + 2)r^2 - (2\alpha + 1)r + 1 = 0$$

provided $B = k\alpha \leqslant B_2$, $B_2$ being the unique root in $(1, \infty)$ of the equation

$$B^2 - 3(1 + \alpha)B + \alpha = 0.$$

(ii) For $\alpha_0 \leqslant \alpha \leqslant 1$, the radius of starlikeness of $f$, $r_5$ is the least positive root of the equation

$$(1 - \alpha) - 2(1 - \alpha)r + (1 - 2\alpha - 2k\alpha + k\alpha^2)r^2$$
$$+ 2k\alpha(1 - \alpha)r^3 - k\alpha(1 - \alpha)r^4 = 0$$

provided $B = k\alpha \leqslant B_3$, $B_3$ being the least root in $(1, \infty)$ of the equation

$$(1 - \alpha)B^3 - (4 - 3\alpha)B^2 + (3 - 4\alpha)B - (1 - \alpha) = 0.$$

Except in Theorem 1, all the results obtained are sharp. Proof of Theorem 1 may be found in a paper of the author (Rangarajan 1978).

## Proofs of Theorems

Let $f \in A(\alpha)$ and put $w(z) = \dfrac{g(z) - f(z)}{\alpha k f(z) + \alpha g(z)}$, $w(z)$ is analytic in $E$, $w(0) = 0$ and $| w(z) | < 1$ in $E$. Hence by Schwarz Lemma, $| w(z) | < r$ for $| z | \leqslant r < 1$. We have

$$f(z) = g(z) \frac{1 - \alpha w(z)}{1 + \alpha k w(z)} \quad \text{...(1)}$$

whence

$$\text{Re} \frac{zf'(z)}{f(z)} = \text{Re} \frac{zg'(z)}{g(z)} - \alpha(1 + k) \text{ Re} \frac{zw'(z)}{[1 - \alpha w(z)][1 + \alpha k w(z)]}. \quad \text{...(2)}$$

We now show that, for $| z | \leqslant r$,

$$\text{Re} \frac{zw'(z)}{[1 - \alpha w(z)][1 + \alpha w k(z)]} \leqslant \frac{r}{(1 - \alpha r)(1 + \alpha k r)} \text{ if } R_1 \leqslant R_2 \quad \text{...(3)}$$

$$\leqslant \frac{1}{(k + 1)^2 \alpha^2} \left[ (k - 1)\alpha - \frac{2}{1 - r^2}\{(S_1 T_1)^{1/2} - (1 + k\alpha^2 r^2)\} \right]$$
$$\text{if } R_1 \geqslant R_1 \quad \text{...(4)}$$

where $R_1 = (S_1/T_1)^{1/2},\ R_2 = (1 - \alpha r)/(1 + k\alpha r)$

$S_1 = (1 - \alpha)(1 + \alpha r^2),\ T_1 = (1 + k\alpha)(1 - k\alpha r^2).$

Since $w(z)$ is regular, $w(0) = 0$ and $|w(z)| < 1$ in $|z| < 1$, we have

$$|zw'(z) - w(z)| \leqslant \frac{|z|^2 - |w(z)|^2}{1 - |z|^2}.$$

Hence

$$\operatorname{Re}\frac{zw'(z) - w(z)}{[1 - \alpha w(z)][1 + \alpha k w(z)]} \leqslant \frac{|zw'(z) - w(z)|}{|1 - \alpha w(z)|\ |1 + \alpha k w(z)|}$$

$$\leqslant \frac{r^2 - |w(z)|^2}{(1 - r^2)\,|1 - \alpha w(z)|\ |1 + \alpha w k(z)|}$$

Therefore

$$\operatorname{Re}\frac{zw'(z)}{|1 - \alpha w(z)|\ |1 + \alpha k w(z)|} \leqslant \operatorname{Re}\frac{w(z)}{|1 - \alpha w(z)|\ |1 + \alpha k w(z)|} + \frac{r^2 - |w(z)|^2}{(1 - r^2)\,|1 - \alpha w(z)|\ |1 + \alpha k w(z)|}. \quad \ldots(5)$$

Put $$w_1(z) = \frac{1 - \alpha w(z)}{1 + \alpha k w(z)}.$$

This maps the disk $|w(z)| \leqslant r$ onto the disk $|w_1(z) - a| \leqslant d$ where

$$a = \frac{1 + \alpha^2 k r^2}{1 - k^2\alpha^2 r^2},\quad d = \frac{\alpha r(1 + k)}{(1 - k^2\alpha^2 r^2)}.$$

If $r < \min(1, 1/k\alpha)$, $a - d = (1 - \alpha r)/(1 + k\alpha r)$ and $a + d = (1 + \alpha r)/(1 - k\alpha r)$ are both positive. With this transformation, the right-hand side of (5) becomes

$$\frac{1}{\alpha^2(k + 1)^2}\left[\alpha \operatorname{Re}\left\{\frac{1}{w_1(z)} + (k - 1) - k w_1(z)\right\} + \frac{r^2\alpha^2\,|k w_1(z) + 1|^2 - |1 - w_1(z)|^2}{(1 - r^2)\,|w_1(z)|}\right]. \quad \ldots(6)$$

Put $w_1(z) = a + u + iv$, $R = |a + u + iv|$. The above expression reduces to

$$\frac{1}{\alpha^2(k + 1)^2} S(u, v)$$

where $$S(u, v) = \alpha\left[\frac{a + u}{R^2} + (k - 1) - k(a + u)\right] + \frac{(1 - k^2\alpha^2 r^2)}{(1 - r^2)}\,\frac{(d^2 - u^2 - v^2)}{R}$$

$$\frac{\partial S}{\partial v} = -\frac{v}{R^4}\left[2\alpha(a + u) + \frac{(1 - k^2\alpha^2 r^2)}{(1 - r^2)}\{2R^3 + R(d^2 - u^2 - v^2)\}\right]$$

$$= -\frac{v}{R^4}\,(+ve).$$

The maximum of $S(u, v)$ is obtained on the diameter $v = 0$ i.e., putting $R = a + u$ in it we have

$$S(u, 0) = S(R)$$
$$= \alpha\left[\frac{1}{R} + (k-1) - kR\right] + \frac{(1 - k^2\alpha^2r^2)(d^2 - u^2)}{(1 - r^2)R}.$$

The absolute maximum of $S(R)$ is obtained at

$$R = R_1 = \sqrt{\frac{(1-\alpha)(1+\alpha r^2)}{(1+k\alpha)(1-k\alpha r^2)}}, \text{ provided } a - d \leqslant R_1 \leqslant a + d.$$

It is easy to show that $R_1 \leqslant a + d$ but $R_1$ need not always be greater than $R_2 = a - d$. If $R_2 \leqslant R_1$, the maximum of $S(R)$ is obtained at $R = R_2 = a - d$ i.e., $u = -d$ and this maximum value is

$$S(-d, 0) = \alpha\left[\frac{1}{a-d} + (k-1) - k(a-d)\right]$$
$$= \frac{\alpha^2(k+1)^2 r}{(1-\alpha r)(1+\alpha kr)}.$$

Therefore

$$S(u, v) \leqslant \frac{\alpha^2(k+1)^2 r}{(1-\alpha r)(1+\alpha kr)}. \quad \ldots(7)$$

(5), (6) and (7) together prove (3) and (4) and (2) becomes

$$\text{Re}\,\frac{zf'(z)}{f(z)} > \text{Re}\,\frac{zg'(z)}{g(z)} - \frac{\alpha(1+k)r}{(1-\alpha r)(1+\alpha kr)}, \text{ if } R_1 \leqslant R_2 \quad \ldots(8)$$

$$> \text{Re}\,\frac{zg'(z)}{g(z)} - \frac{(k-1)}{(k+1)} + \frac{2}{(k+1)\alpha(1-r^2)}[(S_1T_1)^{1/2} - (1 + k\alpha^2r^2)],$$
$$\text{if } R_1 \geqslant R_2 \quad \ldots(9)$$

We note that $R_1 \lesseqgtr R_2$ according as

$$\frac{(1-\alpha)(1+\alpha r^2)}{(1+k\alpha r)(1-k\alpha r^2)} \lesseqgtr \frac{(1-\alpha r)^2}{(1+k\alpha r)^2}$$

which reduces to $P(r) \gtreqless 0$ where

$$P(r) = 1 - 2r - (1 - k\alpha^2 - 2\alpha + 2k\alpha)r^2 + 2k\alpha^2r^3 + k\alpha^2r^4.$$

It is easy to show that $P(r) = 0$ has a unique root $r^*$ in (0, 1).

Proof of Theorem 2: When $\text{Re}\,(g(z)/z) > 0$ for $|z| < 1$, $f(z) \in A(\alpha)$ it is well known that [Anh and Tuan (1977), eqn. (3.2)]

$$\text{Re}\,\frac{zf'(z)}{f(z)} > \frac{(1 - 2r - r^2)}{1 - r^2} - \frac{\alpha(1+k)r}{(1-\alpha r)(1+\alpha kr)}, \text{ if } R_1 \leqslant R_2, \text{ using (8).}$$

For condition of starlikeness, $r$ must satisfy

$$(1 - 2r - r^2)(1 - \alpha r)(1 + \alpha kr) - \alpha r(1 + k)(1 - r^2) > 0$$

i.e. $\quad Q(r) > 0$

where $\quad Q(r) = k\alpha^2 r^4 + 2\alpha(k\alpha + 1)r^3 - [k\alpha^2 + 2\alpha(k - 1) + 1]\, r^2 - 2(\alpha + 1)r + 1.$

An analysis of the equation $Q(r) = 0$ shows that

$$Q(0) = 1 = +\text{ve},\ Q(1) = -2k\alpha(1 - \alpha) = -\text{ve}.$$

There cannot be more than two positive real roots of the equation and only one of them lies in (0, 1) and we take it as $r_3$ in Theorem 2. We observe that

$$P(r_3) = P(r_3) - Q(r_3) = 2\alpha r_3(1 + k\alpha r_3)(1 - r_3^2)$$
$$= +\text{ve}$$

Hence $r_3 \leqslant r^*$. Also $r_3$ lies in the range $(0, 1/k\alpha)$, if and only if $Q(1/k\alpha) \leqslant 0$. Simplifying this and putting $k\alpha = B$, we obtain the condition

$$B^3 - (3\alpha + 4)B^2 + (4\alpha - 1)B + 3\alpha \leqslant 0.$$

Writing the left-hand side as $R(B)$, $R(1) = 4(\alpha - 1) = -\text{ve}$

$$R(\infty) = +\text{ve}.$$

Since a root of $R(B) = 0$ exists in $(-\infty, 0)$, it is easy to see that a unique root $B_1$ in $(1, \infty)$ exists and the root $r_3$ lies in the range $(0, 1/B)$ for $1 \leqslant B \leqslant B_1$.

*Remark*: The result is seen to be sharp if we take

$$f(z) = \frac{(1 - \alpha z)}{(1 + \alpha kz)} \frac{z(1 - z)}{(1 + z)}.$$

*Corollary* 1 — Putting $k = 1$ i.e. $\lambda = 0$ and $\alpha = 1$ in the condition on the family $A(\alpha)$, we have $\operatorname{Re} f(z)/g(z) > 0$ and the radius of starlikeness of $f$ is given by $\sqrt{5} - 2$. This is due to Theorem 1 of Ratti (1968).

PROOF OF THEOREM 3: When $g(z)$ is starlike it is well known that

$$\operatorname{Re} \frac{zg'(z)}{g(z)} \geqslant \frac{1 - r}{1 + r},\ |z| = r < 1$$

Using (8) and (9) with the above, we observe that if $R_1 \leqslant R_2$, the equation giving the radius of starlikeness of $f$ is $U(r) = 0$ where

$$U(r) = k\alpha^2 r^3 - k\alpha(\alpha + 2)r^2 - (2\alpha + 1)r + 1. \qquad ...(10)$$

If $R_1 \geqslant R_2$, the equation is $V(r) = 0$ where

$$V(r) = (1 - \alpha) - 2(1 - \alpha)r + (1 - 2\alpha - 2k\alpha + k\alpha^2)r^2 + 2k\alpha(1 - \alpha)r^3 - k\alpha(1 - \alpha)r^4 \quad \ldots(11)$$

Clearly the two maxima given by (3) and (4) become equal to one another for such values of $\alpha$, $k$ $(0 < \alpha \leqslant 1, k \geqslant 1)$ for which

$$R_1 = R_2 \text{ i.e., } P(r) = 0. \quad \ldots(12)$$

As we are interested in those real roots of (10) and (11) for which $0 < r < 1$, it is easy to see that the radii of starlikeness $r_4$ and $r_5$ are given by the least positive roots of these equations. For suitable values of $\alpha$, $k$ it may happen that $r_4 = r_5$. Such values are obtained by eliminating $r$ from (10) and (12). Now $(1 - r)\, U(r) - P(r) = 0$ gives

$$(k\alpha r + 1)(r - \alpha) = 0$$

which yields $r = \alpha$ which is permissible as $0 < \alpha \leqslant 1$. Putting this in (10) we obtain

$$k\alpha^4 - 2k\alpha^3 - 2\alpha + 1 = 0.$$

It is evident that $\alpha_0$ in the theorem is the smallest positive root of this equation.

$r_4$ lies in the range $\left(0, \dfrac{1}{k\alpha}\right)$, provided $U\left(\dfrac{1}{k\alpha}\right) \leqslant 0$. Putting $k\alpha = B$ and simplifying this condition reduces to

$$B^2 - 3(1 + \alpha)B + \alpha \leqslant 0, \text{ say } T(B) \leqslant 0.$$

$$T(1) = -2(1 + \alpha) = -\text{ve}, \; T(\infty) = +\text{ve}.$$

Thus there exists a unique root $B_2$ in $(1, \infty)$ and $r_4$ is the radius of starlikeness, provided $B \leqslant B_2$. A similar argument establishes the second part of the theorem.

*Remark* : The result is seen to be sharp if we take

$$f(z) = \frac{z}{(1 + z)^2} \frac{(1 - \alpha z)}{1 + \alpha k z} \text{ if } R_1 \leqslant R_2$$

and

$$f(z) = \frac{z}{(1 + z)^2} \frac{1 - \alpha w_1(z)}{1 + \alpha k w_1(z)} \text{ if } R_1 \geqslant R_2$$

where $w_1(z) = \dfrac{z(z - q)}{(1 - qz)}$, $q$ being determined by

$$\text{Re}\left[\frac{1 - \alpha w_1(z)}{1 + k\alpha w_1(z)}\right] = R_1 \text{ at } z = r.$$

*Corollary* 2 — For $k = 1$, $\alpha = 1$, we obtain the result of Ratti (1968, Theorem 3) putting the order of starlike of $g$ as 0 therein.

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