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ON FURTHER EXTENSION OF A THEOREM OF EDELSTEIN

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The present note generalizes the results of Edelstein and Sehgal in a topological space which is not necessarily a metric space.

Let (X, d) denote a metric space. A mapping $T : X \to X$ is said to be contractive if for every pair of distinct $x, y \in X$, $d(Tx, Ty) < d(x, y)$. Edelstein (1962) proved the following theorem :

Theorem E 1 — Let T be a contractive mapping of a metric space X into itself. If there exists a point $x_0 \in X$ such that the sequence of iterates $\{T^n x_0\}$ has a subsequence $\{T^{n_k} x_0\}$ which converges to a point $\xi \in X$, then is a unique fixed point of T.

Sehgal (1972) extended the above result and proved the following.

Theorem S 2 — If T be a self-mapping of a metric space such that for each pair of distinct $x, y \in X$, $d(Tx, Ty) < \max \{d(x,y), d(x, Tx), d(y, Ty)\}$. If $\xi \in X$ is a cluster point of $\{T^n x_0\}$, then ξ is the unique fixed point of T.

The purpose of the present note is to establish the generalizations of the above two results in a topological space which is not necessarily a metric space. First we present the following theorem.

Theorem 1 — Let T_1 and T_2 be two self-mappings of a Hausdorff space X and let $F : X \times X \to [0, \infty)$ be a continuous symmetric mapping such that $F(x, y) = 0$ for $x = y$ and for each pair of distinct $x, y \in X$ one has

(i) $F(T_1 x, T_2 y) < \max \{[F(x, y), F(x, T_1 x), F(y, T_2 y)]$

$$\cup \min [F(x, T_2 y), F(y, T_1 x)]\}$$

(ii) for some $x_0 \in X$ the sequence $\{x_n\}$ where $x_{2n+1} = T_1 x_{2n}$, $x_{2(n+1)} = T_2 x_{2n+1}$ has a subsequence converging to a point $\xi \in X$. If T_1 and $T_2 T_1$ or T_2 and $T_1 T_2$ are continuous at ξ, then is a fixed point of T_1 or T_2.

PROOF : Let $x_0 \in X$, $x_{2n+1} = T_1 x_{2n}$, $x_{2(n+1)} = T_2 x_{2n+1}$, $n = 0, 1, \ldots$. We may assume that $x_n \neq x_{n+1}$ for each n. From (i), $F(x_0, x_1) > F(x_1, x_2) > \ldots$ and so the sequence $C_n = F(x_n, x_{n+1})$ tends to a real number r as $n \to \infty$. Since $\{x_n\}$ has a subsequence in X which converges to a point ξ in X we may put $\xi = \lim x_{2n_k}$. Also

$x_{2n_k+1} = T_1 x_{2n_k} \to T_1 \xi$ and $x_{2(n_k+1)} = T_2 T_1 x_{2n_k} \to T_2 T_1 \xi$ as $k \to \infty$ since T_1 and $T_2\, T_1$ are continuous at ξ. We have,

$$r = \lim_{k \to \infty} F(x_{2n_k}, x_{2n_k+1}) = \lim_{k \to \infty} F(x_{2n_k}, T_1 x_{2n_k}) = F(\xi, T_1 \xi)$$

$$r = \lim_{k \to \infty} F(x_{2n_k+1}, x_{2(n_k+1)}) = \lim_{k \to \infty} F(T_1 x_{2n_k}, T_2 T_1 x_{2n_k}) = F(T_1 \xi, T_2 T_1 \xi)$$

Suppose $\xi \neq T_1 \xi$. Then from (1) we have

$$F(T_1\xi, T_2T_1\xi) < \max \{[F(\xi, T_1\xi), F(T_1\xi, \xi), F(T_1\xi, T_2T_1\xi)] \cup \min [F(\xi, T_2T_1\xi), F(T_1\xi, T_1\xi)]\}$$

i.e. $F(T_1\xi, T_2T_1\xi) < F(\xi, T_1\xi)$ which gives a contradiction. Hence $\xi = T_1\xi$. Similarly if we take $\xi = \lim_{k \to \infty} x_{2n_k+1}$ then one can show that $\xi = T_2\xi$. As a consequence of Theorem 1 we have the following corollary.

Corollary — Let T_1 and T_2 be two mappings of a metric space into itself such that for each pair of distinct $x, y \in X$ one has

$$d(T_1x, T_2y) < \max \{[d(x, y), d(x, T_1x), d(y, T_2y)]\} \cup \min \{d(x, T_2y), d(y, T_1x)\}.$$

If for some $x_0 \in X$, the sequence $\{x_n\}$, where $x_{2n+1} = T_1 x_{2n}$ $x_{2(n+1)} = T_2 x_{2n+1}$, has a subsequence converging to a point $\xi \in X$ such that T_1, T_2T_1, T_1T_2 are continuous at $\xi \in X$, then ξ is a fixed point of T_1 or T_2.

PROOF : The proof follows easily from Theorem 1 with d playing the role of F.

If $T_1 = T_2 = T$, then Theorem E 1 and Theorem S 2 follow from the above Corollary.

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