Indian J. pure appl. Math., **11**(2) : 268–270, February 1980

SOME RESULTS ON FIXED POINTS IN COMPACT METRIC SPACES

PHULLENDU DAS

Department of Mathematics, Visva-Bharati University, Santiniketan, West Bengal

(*Received* 15 *May* 1979)

A sufficient condition for the existence of a unique fixed point of a self-map of a compact metric space has been obtained.

Edelstein (1962) established the existence of a unique fixed point of a self-map T of a compact metric space satisfying the inequality $\rho(Tx, Ty) < \rho(x, y)$. Fisher (1977) obtained a generalization of this result. We shall now prove the following generalization of Fisher's Theorem.

Theorem 1 — Let T be a self-map of a compact metric space (X, ρ) such that for some positive integer m, T^m is continuous and for every $x, y \in X$ with $x \neq y$, $T^m x \neq T^m y$

$$\rho(T^m x, T^m y) < \frac{\alpha_1 \rho(x, T^m x)\, \rho(y, T^m y)}{\rho(x, y)} + \frac{\alpha_2 \rho(x, T^m x)\, \rho(y, T^m x)}{\rho(T^m x, T^m y)}$$
$$+ \frac{\alpha_3 \rho(x, T^m y)\, \rho(y, T^m y)}{\rho(T^m x, T^m y)} + \beta_1 \rho(x, y) + \beta_2 \rho(x, T^m x)$$
$$+ \beta_3 \rho(y, T^m y) + \beta_4 \rho(x, T^m y) + \beta_5 \rho(y, T^m x) \qquad \ldots(A)$$

where $\alpha_1 + \alpha_3 + \beta_3 + \beta_4 < 1$, $\alpha_3 \geqslant 0$, $\beta_4 \geqslant 0$, $\alpha_1 + 2\alpha_3 + \beta_1 + \beta_2 + \beta_3 + 2\beta_4 = 1$ and $\beta_1 + \beta_4 + \beta_5 < 1$.

Then T has a unique fixed point.

PROOF : Define $f : X \to [0, \infty)$ by $f(x) = \rho(x, T^m x)$ for every $x \in X$.

Continuity of ρ and T^m ensure the continuity of f. Then compactness of X yields a point $z \in X$ such that $f(z) = \inf \{f(x) \,;\, x \in X\}$.

Then $f(z) \neq 0$ gives $z \neq T^m z$. If $T^m z = T^{2m} z$, then $T^m z$ is a fixed point of T^m. So we assume that $T^m z \neq T^{2m} z$.

Then $f(T^m z) = \rho(T^m z, T^{2m} z)$

$$< \frac{\alpha_1 \rho(z, T^m z)\, \rho(T^m z, T^{2m} z)}{\rho(z, T^m z)} + \frac{\alpha_2 \rho(z, T^m z)\, \rho(T^m z, T^m z)}{\rho(T^m z, T^{2m} z)}$$
$$+ \frac{\alpha_3 \rho(z, T^{2m} z)\, \rho(T^m z, T^{2m} z)}{\rho(T^m z, T^{2m} z)} + \beta_1 \rho(z, T^m z) + \beta_2 \rho(z, T^m z)$$
$$+ \beta_3 \rho(T^m z, T^{2m} z) + \beta_4 \rho(z, T^{2m} z) + \beta_5 \rho(T^m z, T^m z) \qquad \ldots\{\text{by (A)}\}$$
$$\leqslant (\alpha_1 + \alpha_3 + \beta_3 + \beta_4) f(T^m z) + (\alpha_3 + \beta_1 + \beta_2 + \beta_4) f(z) \; (\because \alpha_3, \beta_4 \geqslant 0).$$

Therefore $f(T^m z) < \dfrac{\alpha_3 + \beta_1 + \beta_2 + \beta_4}{1 - \alpha_1 - \alpha_3 - \beta_3 - \beta_4} f(z)$ ($\because \alpha_1 + \alpha_3 + \beta_3 + \beta_4 < 1$)

$$= f(z) \; (\because \alpha_1 + 2\alpha_3 + \beta_1 + \beta_2 + \beta_3 + 2\beta_4 = 1)$$

But this contradicts the definition of $f(z)$; thus $f(z) = 0$ and this gives $z = T^m z$.

If possible, let T^m possess another fixed point z'.

$$\text{Then } \rho(z, z') = \rho(T^m z, T^m z')$$
$$< \beta_1 \rho(z, z') + \beta_4 \rho(z, z') + \beta_5 \rho(z', z) \quad \{\text{by (A)}\}$$
$$\leqslant \rho(z, z'), \; (\text{as } \beta_1 + \beta_4 + \beta_5 \leqslant 1)$$

which is a contradiction.

Thus z is the unique fixed point of T^m. Then $Tz = T(T^m z) = T^m(Tz)$ and the unicity of the fixed point of T^m yield $Tz = z$. The unicity of the fixed point of T follows from that of T^m and the fact that any fixed point of T is a fixed point of T^m.

Corollary — Let T_1 and T_2 be two commuting self-maps of a compact metric space (X, ρ) satisfying the conditions of Theorem 1 with T replaced by $T_1 . T_2$. Then T_1 and T_2 have a unique common fixed point.

PROOF : By Theorem 1, $T_1 . T_2 (= T_2 . T_1)$ has a unique fixed point u (say). Then $T_1 . T_2(T_1 u) = T_1 u$ gives $T_1 u = u$. Similarly $T_2 u = u$. Since any common fixed point of T_1 and T_2 is a fixed point of $T_1 . T_2$, the unicity of the common fixed point of T_1 and T_2 follows from that of $T_1 . T_2$.

Theorem 2 — Let T be as in Theorem 1. Also, in addition, let $\alpha_2 \geqslant 0$, $\beta_2 \geqslant 0$, $\alpha_2 + \beta_2 + \beta_5 < 1$, $2\alpha_2 + \beta_1 + 2\beta_2 + \beta_4 + \beta_5 = 1$. If $\rho(T^m x, u) < \rho(x, u)$ for every $x \in X$ with $x \neq u$ where u is the unique fixed point of T (which exists by Theorem 1), then for every $x \in X$, $\operatorname{Lt}_{n\to\infty} T^{mn} x = u$.

PROOF : Let $x \in X$. Since X is compact, there exists a subsequence $\{T^{mn_i} x\}$ of $\{T^{mn} x\}$ which converges to some point z (say). If $T^{mn} x = u$ for any n, then $\operatorname{Lt}_{n\to\infty} T^{mn} x = u$. So we assume that $T^{mn} x \neq u$ for any n. Therefore

$$\rho(T^{mn} x, u) = \rho(T^m T^{m(n-1)} x, T^m u)$$
$$< \frac{\alpha_1 \rho(T^{m(n-1)} x, T^{mn} x)\, \rho(u, T^m u)}{\rho(T^{m(n-1)} x, u)} + \frac{\alpha_2 \rho(T^{m(n-1)} x, T^{mn} x)\, \rho(u, T^{mn} x)}{\rho(T^{mn} x, u)}$$
$$+ \frac{\alpha_3 \rho(T^{m(n-1)} x, u)\, \rho(u, T^m u)}{\rho(T^{mn} x, u)} + \beta_1 \rho(T^{m(n-1)} x, u) + \beta_2 \rho(T^{m(n-1)} x, T^{mn} x)$$
$$+ \beta_3 \rho(u, T^m u) + \beta_4 \rho(T^{m(n-1)} x, T^m u) + \beta_5 \rho(u, T^{mn} x) \qquad \{\text{by (A)}\}$$

$$\leqslant (\alpha_2 + \beta_2 + \beta_5)\,\rho(T^{mn}x, u) + (\alpha_2 + \beta_1 + \beta_2 + \beta_4)\,\rho(T^{m(n-1)}x, u),$$

(as $\alpha_2, \beta_2 \geqslant 0$)

Thus $\rho(T^{mn}x, u) \leqslant \dfrac{\alpha_2 + \beta_1 + \beta_2 + \beta_4}{1 - \alpha_2 - \beta_2 - \beta_5}\rho(T^{m(n-1)}x, u)$ (as $\alpha_2 + \beta_2 + \beta_5 < 1$)

$$= \rho(T^{m(n-1)}x, u) \quad (\text{as } 2\alpha_2 + \beta_1 + 2\beta_2 + \beta_4 + \beta_5 = 1)$$

Therefore $\{\rho(T^{mn}x, u)\}$ is convergent.

Since $\underset{i\to\infty}{\text{Lt}}\, T^{mn_i}x = z$, $\underset{i\to\infty}{\text{Lt}}\, \rho(T^{mn_i}x, u) = \rho(z, u)$, $\underset{n\to\infty}{\text{Lt}}\, \rho(T^{mn}x, u) = \rho(z, u)$.

Now $\rho(T^{m(n_i+1)}x, u) \leqslant \rho(T^{m(n_i+1)}x, T^m z) + \rho(T^m z, u)$

Therefore $\underset{i\to\infty}{\text{Lt}}\, \rho(T^{m(n_i+1)}x, u) \leqslant \rho(T^m z, u)$. This gives $\rho(z, u) \leqslant \rho(T^m z, u)$ which contradicts the inequality $\rho(T^m x, u) < \rho(x, u)$ unless $z = u$. Thus

$$z = u, \Rightarrow \underset{n\to\infty}{\text{Lt}}\, \rho(T^{mn}x, u) = 0. \Rightarrow \underset{n\to\infty}{\text{Lt}}\, T^{mn}x = u.$$

References

Edelstein, M. (1962). On fixed and periodic points under contractive mappings. *J. Lond. Math. Soc.*, **37**, 74–79.

Fisher, B. (1977). On three fixed point mappings for compact metric spaces. *Indian J. pure appl. math.*, **8**, 479–81.