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SETS OF SUMMABILITY

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This paper presents some results on conditions for a set, E, on the unit circle, C, to be a set of summability, i.e., there exists a Taylor series $\Sigma\ a_m z^m$, with partial sums $s_m(z)$, of a function regular in $| z | < 1$ such that the series is summable by a regular Toeplitz matrix at each point of E and at no point of C-E. It is shown that a sufficient condition for E to be a set of summability is that E be of type F_σ. Some related results for row-finite matrices are given, and two lemmas of general usefulness are proved, one on absolute equivalence for bounded sequences, and one on the behaviour of the transform of a divergent sequence satisfying certain assumptions on the rapidity of growth.

1. INTRODUCTION

This paper presents some results on conditions for a set on the unit circle, C, to be a set of summability, i.e., there exists a Taylor series $\Sigma\ a_m z^m$ with partial sums $s_m(z)$, of a function regular in $| z | < 1$ such that the series is summable by a regular Toeplitz matrix at each point of the set and at no point of the complement of the set.

2. PRELIMINARY RESULTS

In this section two lemmas will be proved which are used in the following section, but whose applications are not restricted to the present context; a brief description is also given of a function given by Herzog and Piranian (1955-56) which is used here in Theorem 1.

The definition of absolute equivalence for regular matrices is needed. Let A and B be regular Toeplitz matrices, and let $z'_n = \Sigma\ a_{nk} z_k$, $z''_n = \Sigma\ b_{nk} z_k$ be the transforms of the divergent sequence $\{z_n\}$ by A and B, respectively. Then A and B are said to be 'absolutely equivalent' for a given class of sequences $\{z_k\}$ whenever $z'_n - z''_n \to 0$ as $n \to \infty$, for all $\{z_k\}$ in the class. Further, A and B are said to be mutually consistent for a class of sequences $\{z\}$ if, whenever z'_n tends to a limit, then z'' tends to the same limit, and vice versa. Absolute equivalence implies mutual consistency, but not conversely.

Lemma 1 — Let B be the class of all bounded sequences $x \equiv \{x_n\}_{n=1}^{n=\infty}$. If A is any regular Toeplitz matrix, there exists a row-finite regular Toeplitz matrix A' such that A and A' are absolutely equivalent for all $x \in B$.

PROOF: By hypothesis, $A = (a_{nk})$ satisfies the conditions of the regularity theorem (Hardy 1949, Th. 2); these allow the following construction. First, let $k_0 = 0 = n_0$ and let k_1 be any positive integer. Choose $n_1 > n_0$ such that

$$\sum_{k<k_1} | a_{nk} | < \frac{1}{2^2} \text{ for } n > n_1.$$

Now choose $k_2 > k_1$ such that

$$\max_n \left(\sum_{k<k_0} + \sum_{k>k_2} \right) | a_{nk} | < \frac{1}{2^3} \text{ for } n_0 \leqslant n < n_1.$$

Next choose $n_2 > n_1$ such that

$$\sum_{k<k_2} | a_{nk} | < \frac{1}{2^3} \text{ for } n > n_2$$

and $k_3 > k_2$ such that

$$\max_n \left(\sum_{k<k_1} + \sum_{k>k_3} \right) | a_{nk} | < \frac{1}{2^4} \text{ for } n_1 \leqq n < n_2.$$

Continuing in this way, two increasing sequences $\{k_r\}$ and $\{n_r\}$ are constructed such that

$$\lim_{r\to\infty} \max_{n_r \leqq n < n_{r+1}} \left(\sum_{k<k_r} + \sum_{k>k_{r+2}} \right) | a_{nk} | = 0. \qquad ...(1)$$

Now construct the matrix $A' = (a'_{nk})$ as follows. Let

$$a'_{nk} = \begin{cases} 0 \text{ if } a_{nk} \text{ occurs in (1)} \\ a_{nk} \text{ otherwise.} \end{cases}$$

Then A' is also regular by construction.

Now let $s_n = \Sigma_{k=0}^{\infty} | C_{nk} |$, where $C_{nk} = a_{nk} - a'_{nk}$. Then $s_n \to 0$ as $n \to \infty$, for (a) $C_{nk} = 0$ for all elements a_{nk} which do not occur in (9), and since (b) otherwise $C_{nk} = a_{nk}$, (1) implies $s_n \to 0$. But, by Theorem 5.4, I of Cooke (1950), $s_n \to 0$ is a sufficient condition for A and A' to be absolutely equivalent (and hence mutually consistent) for all bounded sequences. This completes the proof of Lemma 1.

Lemma 2 — Let $A = (a_{nk})$ be a regular Toeplitz matrix, and let $\{s_n\}$ be a divergent sequence which is either (1) bounded or (2) if unbounded is such that $| s_n |$ grows slowly enough (this is made precise below by a condition on the rapidity of increase). If $\{s_n\}$ has a sufficiently long block of terms with a common value b then the transform As has a corresponding element whose value is approximately b.

PROOF : Only the argument for case (2) is given; the proof for bounded sequences follows directly from the application of the construction used in the proof of Lemma 1.

Using again the fact that A is regular, from the sequences $\{n\}$ and $\{k\}$ of indices, subsequences $\{n_m\}$ and $\{k_m\}$ are selected as follows.

Choose n_1 such that

$$\left| 1 - \sum_{k=1}^{\infty} a_{nk} \right| < \tfrac{1}{2} \quad (n \geqq n_1)$$

and choose k_1 $(k_1 > 1)$ such that

$$\sum_{k=k_1}^{\infty} | a_{nk} | < 1 \quad (n = 1, 2, \ldots, n_1).$$

Further, suppose n_{m-1} and k_{m-1} $(m > 1)$ have been defined; choose n_m $(n_m > n_{m-1})$ so that

$$\left| 1 - \sum_{k=1}^{\infty} a_{nk} \right| < \frac{1}{m+1}, \sum_{k=1}^{k_{m-1}} | a_{nk} | < \frac{1}{m} \text{ hold for } n > n_m$$

and k_m $(k_m > k_{m-1})$ such that

$$\sum_{k=k_m}^{\infty} | a_{nk} | < \frac{1}{2^{m-1}\, m} \quad (n = 1, 2, \ldots, n_m).$$

With the help of the sequence of indices, $\{k_m\}$, the sequence $\{s_n\}$ is divided into zones, the mth zone consisting of the elements s_k, where

$$k = k_{m-1} + 1, k_{m-1} + 2, \ldots, k_m \ (\text{take } k_0 + 1 = 1).$$

Also, by construction,

$$\sum_{k=k_{m-1}+1}^{k_{m+1}} a_{nk} \to 1 \text{ as } n \to \infty$$

since $m \to \infty$ as $n \to \infty$.

To make the meaning of "$s_n \to \infty$ slowly enough" precise, it will be assumed that $|s_k| \leqq \sqrt{m}$ in the mth zone. Write t_n for the element of the transform, and suppose $n_m \leqq n < n_{m+1}$ ($m > 1$). Then

$$|t_n| = \left|\left(\sum_{1}^{k_{m-1}} + \sum_{k_{m-1}+1}^{k_m} + \sum_{k_m+1}^{k_{m+1}} + \dots\right) a_{nk}s_k\right|$$

$$\geqq \left|\left(\sum_{1}^{k_{m-1}} + \sum_{k_{m-1}+1}^{k_m} + \sum_{k_m+1}^{k_{m+1}}\right) a_{nk}a_k\right| - \left|\sum_{k_{m+1}+1}^{k_{m+2}} a_{nk}s_k\right| - \dots$$

$$\geqq \left|\left(\sum_{1}^{k_{m-1}} + \sum_{k_{m-1}+1}^{k_m} + \sum_{k_m+1}^{k_{m+1}}\right) a_{nk}s_k\right| - \sqrt{m+2} \sum_{k_{m+1}+1}^{k_{m+2}} |a_{nk}| - \dots$$

$$\geqq \left|\left(\sum + \sum + \sum\right) a_{nk}s_k\right| - \frac{\sqrt{m+2}}{2^m(m+1)} - \frac{\sqrt{m+3}}{2^{m+1}(m+2)} - \dots$$

$$\geqq \left|\left(\sum_{k_{m-1}+1}^{k_m} + \sum_{k_{m+1}}^{k_{m+1}}\right) a_{nk}s_k\right| - \sqrt{m} \sum_{1}^{k_{m-1}} |a_{nk}| - \frac{\sqrt{m+2}}{2^m(m+1)} - \dots$$

$$\geqq \left|\left(\sum_{k_{m-1}+1}^{k_m} + \sum_{k_m+1}^{k_{m+1}}\right) a_{nk}s_k\right| - \frac{1}{\sqrt{m}} - \frac{\sqrt{m+2}}{2^m(m+1)} - \dots$$

$$\geqq \left|\sum_{k_{m-1}+1}^{k_{m+1}} a_{nk}s_k\right| - \left(\frac{1}{\sqrt{m}} + \frac{1}{2^{m-1}}\sqrt{\frac{2}{m+1}}\right)$$

$$\geqq \left|\sum_{k_{m-1}+1}^{k_{m+1}} a_{nk}s_k\right| - \epsilon_1(m).$$

Now if $\{s_n\}$ has a block of equal elements sufficiently long to "overlap" the elements $a_{nk_{m-1}} + 1, a_{nk_{m-1}} + 2, \dots, a_{nk_m}, \dots, a_{nk_{m+1}}$, then, denoting the common value of the elements by b, write the last inequality above as

$$|t_n| \geqq |b| \cdot \left|\sum_{k_{m-1}+1}^{k_{m+1}} a_{nk}\right| - \epsilon_1(m).$$

But $\epsilon_1(m) \to 0$ since $m \to \infty$ since $n \to \infty$, and $\sum_{k_{m-1}+1}^{k_{m+1}} \to 1$, and so is greater in absolute value than, say, $\frac{3}{4}$ for n sufficiently large.

An inequality in the opposite direction can also be obtained, giving

$$| t_n | \leqq | b | \cdot \left| \sum_{k_{m-1}+1}^{k_{m+1}} a_{nk} \right| - \epsilon_2(m)$$

where $\epsilon_2(m) \to 0$. These two inequalities prove the assertion of Lemma 2.

The function to be used in the following section was constructed by Herzog and Piranian (1955–56), who showed that if E is a set of type F_σ (a denumerable union of closed sets), then there exists a function $f(z) = \Sigma\, a_m z^n$ with the property that at each point of E the sequence of partial sums, $\{s_m(z)\}$, converges, and at each point of C-E $\{s_m(z)\}$ is unbounded. The form of the function is

$$f(z) = \sum_j j \sum_k \sum_p p^{-3} z^{h_{jkp}} g(z/t_{jkp}, [p^{3/2}]) \qquad \ldots(2)$$

where the t's are points on C, $g(z, n) = [(z^n - 1)/(z - 1)]^2$, and the h's are positive integers selected in such a way that no two terms contain like powers of z. It is shown, further, that if z is in C-E, then for each index j, except possibly one, there exist positive integers k and p, and a block of consecutive terms of the Taylor series of f (viz., the polynomial $jp^{-3}z^h g$) whose sum has a modulus greater than $A.j$ (A a universal positive constant).

3. Sets of Summability

The first theorem is suggested by Herzog and Piranian (1955–56, p. 73) and makes use of the function (2) just described.

Theorem 1 — Let E be a set of type F_σ on C, and let T be a regular Toeplitz transformation. Then there exists a function $f(z) = \Sigma\, a_m z^m$ with partial sums $s_m(z)$, and with the following properties:

(a) at each point of E, the T-transform of $\{s_m(z)\}$ converges;

(b) at each point of C-E, the T-transform of $\{s_m(z)\}$ is unbounded.

Proof : Let f be the function (2) described in the last section. Then property (a) follows from the regularity of T, for a regular method does not destroy ordinary convergence.

It must now be shown that if $z \in C$-E, then the T-transform of $\{s_m(z)\}$ is unbounded ($T(s)$ will be used to denote the sequence $\{ \Sigma\, a_{nk} s_k\}$). This will be done by showing that if the exponents h_{jkp} in (2) are chosen so as to create sufficiently long gaps between the polynomials under the summation sign in (2), then f has property (b).

Suppose $z \in C\text{-}E$. By the results quoted, there exist infinitely many triples (j, k, p) for which the corresponding polynomial in (2) has modulus greater than Aj. Corresponding to each occurrence of this phenomenon, there exists a long block of partial sums of the power series, each of modulus greater than $Aj/2$, namely, either the block of partial sums which "follow" the polynomial in question and precede the next polynomial in the series, or else the block of partial sums that follow the preceding polynomial and precede the polynomial associated with (j, k, p). That is, there exist long blocks of partial sums (equal within a block) which are large at the point z.

Now, the exponents h_{jkp} in (2) may be chosen so as to create gaps as great as desired between the polynomials, and hence to create blocks of equal partial sums as long as desired; by the same mechanism, the condition of Lemma 2 that the sequence diverges slowly enough may be satisfied. Within a block, these partial sums are all equal and have modulus greater than, say, $Aj/2$ at $z \in C\text{-}E$, where j is a positive integer belonging to one of the triples (j, k, p). By Lemma 2, the T-transform then has an element approximately equal to $Aj/2$. Since this is true for infinitely many j, the transform of $\{s_m(z)\}$ is unbounded. Since z was an arbitrary point of $C\text{-}E$, this proves Theorem 1.

Remark : Theorem 1 continues to hold if the regular sequence-to-sequence transform is replaced by a regular sequence-to-function transform; the proof is essentially unchanged. Such a transform satisfies virtually the same regularity conditions; Lemma 2 can also be stated for sequence-to-sequence transforms.

Theorem 1 and the above remark may be restated in the following form.

Theorem 2 — A sufficient condition for a point set E on C to be a set of summability for a regular Toeplitz transform or a regular sequence-to-function transform of the sequence $\{s_m(z)\}$ is that E be of type F_σ.

Theorem 3 — A necessary and sufficient condition for a point set E on C to be a set of boundedness of the transform of the sequence $\{s_m(z)\}$ by a regular row-finite Toeplitz matrix is that E be of type F_σ.

PROOF : Theorem 2 proves the sufficiency of the condition. A proof of the necessity is given by Hausdorff (1958, pp. 286–87, 305–307). Only the continuity of the elements, $t_n(z)$, of the transform is needed in that proof, and this is insured by the row-finiteness of the transform.

Theorem 4 — If T is a regular row-finite Toeplitz matrix and E is a set on C such that some Taylor series is summable-T on E and not summable-T on $C\text{-}E$, then E is of type $F_{\sigma\delta}$ (a denumerable intersection of sets of type F_σ).

PROOF : For any positive integers m, n, k let $H(m, n, k)$ be the set of points z on C at which

$$| t_m(z) - t_n(z) | \leqq \frac{1}{k}$$

where $t_m(z) = \Sigma\, a_{mn}s_n(z)$. Let $H(m, k)$ be the intersection of the sets $H(m, m + p, k)$ $(p = 1, 2, \ldots)$. Since the sets $H(m, n, k)$ are closed, the sets $H(m, k)$ are closed. Note that the set $H(m, k)$ is the totality of points on C at which the difference of any two terms after the mth term of the transformed sequence does not exceed $1/k$.

Let $H(k)$ be the union of the sets $H(m, k)$ $(m = 1, 2, \ldots)$; then $H(k)$ is the set of all points on C at which the difference of any two terms of the transformed sequence is ultimately not greater than $1/k$. Finally, the intersection of the sets $H(k)$ $(k = 1, 2, \ldots)$ is the set of convergence of the sequence $\{t_m(z)\}$, and since each $H(k)$ is of type F_σ, the set of summability is of type $F_{\sigma\delta}$.

Theorem 5 — If T is a regular, row-finite Toeplitz matrix and E is a set on C such that some Taylor series is summable uniformly by T on E and not summable on C-E, then E is a closed set.

PROOF : Suppose $t_m = \Sigma\, a_{mn}s_n(z)$ denotes the mth element of the transformed sequence which is uniformly summable on E. Then, for every positive integer, k, there exists an integer $N(k)$ such that

$$| t_n(z) - t_{N(k)}(z) | \leqq \frac{1}{k} \text{ if } n > N(k) \qquad \ldots(3)$$

whenever z lies in E. Let E_{kn} denote the set

$$E\left[z \mid z \in C, \; | t_n(z) - t_{N(k)}(z) | \leqq \frac{1}{k} \right]$$

and note that E_{kn} is closed. Then $E_k = \bigcap_{n > N(k)} E_{kn}$ is closed and E_k denotes the totality of points on C at which (3) is satisfied. Further, E_k contains E because of the choice of $N(k)$. Therefore, E is contained in the intersection $E^* = \bigcap_k E_k$. But $\{t_m(z)\}$ converges uniformly in E and therefore $E = E^*$. Since E^* is a closed set, E is a closed set.

Theorem 6 — The hypothesis that T is row-finite may be removed from Theorems 3, 4, and 5 if the partial sums of the Taylor series are bounded on C.

PROOF : By Lemma 1, there exists a matrix T' which is absolutely equivalent to T for all bounded sequences. The sequence $\{t_m\}$ may be replaced by the sequence $\{t'_m\}$ of the transform by T' and the calculations will go as before, for absolute equivalence implies that either t_n and t'_n have the same limit or else neither has a limit but in either case their difference may be made arbitrarily small.

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