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ON A PROBLEM OF KATO ON THE L^2-CONTINUITY OF SOME PSEUDODIFFERENTIAL OPERATORS OF MULTIPLE SYMBOL

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Recently Kato proved the L^2-continuity of some pseudodifferential operator A associated with the symbol $a(x, \xi) \in S^0_{\rho,\rho}$, $0 \leqslant \rho \leqslant 1$, using Banach algebra techniques. He then raised the question of extending it to the symbol $a(x_1, x_2, \xi)$ of $3n$-variables. In this paper we answer the Kato's problem, using operational calculus introduced by Cordes and Kato's lemmas.

1. INTRODUCTION

Let $A : \mathcal{S} \to \mathcal{S}'$ be the pseudodifferential operator associated with the symbol $a(x_1, x_2, \xi) \in S^m_{\rho,\delta}$, $m \leqslant 0$, $0 \leqslant \rho \leqslant \delta < 1$, defined by

$$(Au, v) = (a, \omega) \qquad \ldots(1.1)$$

where

$$u, v \in \mathcal{S}, \omega = (2\pi)^{-n} e^{-i(x_1-x_2)\xi} \bar{u}(x_2)\, v(x_1)$$

or formally

$$Au(x_1) = (2\pi)^{-n} \int e^{i(x_1-x_2)\xi}\, a(x_1, x_2, \xi)\, u(x_2)\, dx_2\, d\xi. \qquad \ldots(1.2)$$

A number of authors, notably Hörmander (1967, 1971a, b), Calderón and Vaillancourt (1972), Cordes (1975), and Kumano-Go (1975), have investigated the L^2-continuity of pseudodifferential operator A defined by (1.1) or (1.2). Hörmander (1971a) showed that (1.2) is L^2-continuous if the symbol $a(x_1, x_2, \xi) \in S^m_{\rho,\delta}$ has a compact support in (x_1, x_2)-variables, satisfies the inequalities:

$$|\, \partial^\alpha_{x_1} \partial^\beta_{x_2} \partial^\gamma_\xi\, a(x_1, x_2, \xi) \,| \leqslant C_{\alpha\beta\gamma}(1 + |\,\xi\,|)^{m+\delta_1|\alpha|+\delta_2|\beta|-\rho|\gamma|} \qquad \ldots(1.3)$$

for all multi-indices α, β and γ, and $m < n\{\rho - \frac{1}{2}(\delta_1 + \delta_2)\}$ for $0 \leqslant \rho \leqslant \delta_j < 1$, $j = 1, 2$. He also disproved the L^2-continuity of (1.2) if $m > n\{\rho - \frac{1}{2}(\delta_1 + \delta_2)\}$. Later on Calderón and Vaillancourt (1972) removed the assumption that the symbol $a(x_1, x_2, \xi)$ be compactly supported in (x_1, x_2)-variables, and relaxed the order of differentiabilities to a finite number, namely $|\,\alpha\,|, |\,\beta\,| \leqslant 2m_j$, $|\,\gamma\,| \leqslant 2\,[n/2] + 2$, m_j is the smallest integer $\geqslant \frac{5n}{4(1-\delta_j)}$, $j = 1, 2$. They then settled down the critical case when $m = n\{\rho - \frac{1}{2}(\delta_1 + \delta_2)\}$. For the symbol $a(x, \xi)$ of $2n$-variables, Cordes (1975) verified (1.1) is L^2-continuous if (1.3) is bounded for $|\alpha|, |\beta| \leqslant [2/n] + 1$. He heavily uses the fact that the fundamental solution of

$$(1-\Delta)^{-\lambda},\ \lambda > 0,\ \Delta = \sum_1^n \frac{\partial^2}{\partial x_j^2},$$

the Laplacian, is given by

$$\psi_\lambda(x) = (2\pi)^{-n/2}\, 2^{1-\lambda}/\Gamma(\lambda)\ |x|^{\lambda-(n/2)}\, K_{\lambda-(n/2)}(|x|),$$

$\Gamma(.)$ and $K_{\lambda-(n/2)}(.)$ are the Gamma and modified Bessel functions respectively, and that the integral operator with a kernel appearing in (1.1) is of trace class. Kumano-Go (1975) dealt with the space $S^{\tilde{m}_\nu; \tilde{m}_{\nu'}}_{\lambda,\rho,\sigma}$, an extension of $S^m_{\rho,\sigma}$. He then defined the pseudodifferential operator $P = p(X, D_{x_\nu}, \tilde{X}^\nu)$ associated with the symbol

$$p(x^0, \tilde{\xi}^\nu, \tilde{x}^\nu) \in S^{\tilde{m}_\nu; \tilde{m}_{\nu'}}_{\lambda,\rho,\sigma}$$

via oscillatory integral. Subsequently, Kato (1976) proved that (1.1) is L^2-continuous if the symbol $a(x, \xi) \in S^0_{\rho,\sigma}$, $0 \leqslant \rho < 1$ satisfies (1.3) for $|\alpha| \leqslant [n/2] + 2$, $|\beta| \leqslant [n/2] + 1$. In his proof, Kato used some convolution identity, and partition of unity. He then raised the question of extending his results to a symbol $a(x_1, x_2, \xi)$ of $3n$-variables.

In this paper we answer Kato's question. We used a similar technique initiated by Cordes (1975) and especially Kato (1976).

2. Operational Calculus

Following Cordes (1975), we introduce some special symbol. Let $g(x) = \psi_{r/2}(x)$ be the fundamental solution of $(1-\Delta)^{r/2}$ for $r > 0$, where

$$\Delta = \sum_1^n \partial^2/\partial x_j^2\,.$$

Then (Cordes 1975, p. 118)

$$g(x) = (2\pi)^{r/2}\, 2^{(2-r)/2}/\Gamma(r)\ |x|^{(r-n)/2}\, K_{(r-n)/2}(|x|) \qquad \text{...(2.1)}$$

where $\Gamma(.)$ and $K_{(r-n)/2}(.)$ are the Gamma and modified Bessel functions respectively. From the behaviours of $\Gamma(.)$ and $K_{(r-n)/2}(.)$ near the origin and at infinity, g has the following asymptotic properties (Jahnke *et al.* 1960, p. 201, Watson 1966, p. 202):

$$g(x) = O(\mid x \mid^{((r-n)/2) - \mid (r-n)/2 \mid}) \text{ as } \mid x \mid \to 0 \quad \text{...(2.2)}$$

$$g(x) = O(\mid x \mid^{(r-n-1)/2}\, e^{-\mid x \mid}) \text{ as } \mid x \mid \to \infty. \quad \text{...(2.3)}$$

In particular if $r = n$,

$$g(x) = O(\log \mid x \mid) \text{ as } \mid x \mid \to 0. \quad \text{...(2.4)}$$

Suppose now the symbol $a \in S^{m}_{\rho,\delta}$ satisfy (1.3) with $\mid \alpha \mid \leqslant [n/2] + 2$, $\mid \rho \mid , \mid \gamma \mid \leqslant [n/2] + 1$, and write

$$b = (1 - \Delta_{x_1})^{r/2} (1 - \Delta_{x_2})^{s/2} (1 - \Delta_{\xi})^{t/2}\, a \text{ for } r, s, t > (n/2).$$

Denote the fundamental solution of $(1 - \Delta_{x_1})^{r/2} (1 - \Delta_{x_2})^{s/2} (1 - \Delta_{\xi})^{t/2}$ by

$$g(x_1, x_2, \xi) = \psi_{r/2}(x_1)\, \psi_{s/2}(x_2)\, \psi_{t/2}(\xi) \quad \text{...(2.5)}$$

where $\psi_{r/2}(.)$ is given by (2.1). Then we have

$$a = b * g. \quad \text{...(2.6)}$$

Notice that (2.6) has a meaning even in classical sense from (2.2) - (2.5).

3. L^2-CONTINUITY

Theorem 3.1 — For the symol $a \in S'(R^{3n})$, suppose

$$b = (1 - \Delta_{x_1})^{r/2} (1 - \Delta_{x_2})^{s/2} (1 - \Delta_{\xi})^{t/2}\, a \in L^{\infty}(R^{3n})$$

for $r, s, t \leqslant [n/2] + 1$. Then the corresponding pseudodifferential operator A defined by (1.1) is L^2-continuous.

PROOF : See Lee (1979).

Remark : The class of symbols a satisfying the assumptions of Theorem 3.1 contains $a(x_1, x_2, \xi)$ with compact support in ξ-variable (trivial case) or $a \in S^{0}_{0,0}$.

Theorem 3.2 (*main theorem*) — Let the symbol $a \in S^{m}_{\rho,\delta}$ for $m \leqslant 0$, $0 \leqslant \rho \leqslant \sigma < 1$, satisfy (1.3) with $\mid \alpha \mid \leqslant [n/2] + 2$, $\mid \beta \mid , \mid \gamma \mid \leqslant [n/2] + 1$. Then the corresponding pseudodifferential A defined by (1.1) is L^2-continuous.

PROOF : Let $e \in C_0^{\infty}(R')$ be a nonnegative radial function such that $e(r) > 0$ for $r < 1/2$, $e(r) = 0$ for $r > 3/4$. Let $f(\xi) = e(\mid \xi \mid)$ for $\xi \in R^n$, and define $h_j(\xi) = f(\xi - j) [\sum_{0}^{\infty} f(\xi - j)]^{-1}$. Then $\{h_j\}$ is a partition of unity for which

$$\mid \partial_{\xi}^{\gamma} h_j(\xi) \mid \leqslant C_{\gamma} \quad \text{...(3.1)}$$

is satisfied where C_γ is independent of j. Set $\phi_j(\xi) = h_j(\xi \mid \xi \mid^{-\rho})$. Then $\{\phi_j\}$ is also a partition of unity, and (3.1) implies

$$| \partial_\xi^\gamma \phi_j(\xi) | \leqslant C_\gamma(1 + | \xi |)^{-\rho |\gamma|}. \qquad \text{...(3.2)}$$

Define now $a_j(x_1, x_2, \xi) = a(x_1, x_2, \xi)\, \phi_j(\xi)$. Then $a(x_1, x_2, \xi) = \sum_0^\infty a_j(x_1, x_2, \xi)$, and each $a_j(x_1, x_2, \xi)$ has a compact support in ξ-variable. A simple computation shows for each $\xi \in$ supp $a_j(x_1, x_2, \xi)$ there exists a constant C_1 independent of j such that

$$j^{1/(1-\rho)} - C_1 \leqslant | \xi | \leqslant j^{1/(1-\rho)} + C_1 \qquad \text{...(3.3)}$$

(1.3) and (3.3) imply the following inequalities for

$$| \alpha | \leqslant \left[\frac{n}{2}\right] + 2, | \beta | , | \gamma | \leqslant \left[\frac{n}{2}\right] + 1 :$$

$$| \partial_{x_1}^\alpha \partial_{x_2}^\beta \partial_\xi^\gamma a_j(x_1, x_2, \xi) | \leqslant C_2 j^{(\sigma_1/(1-\rho)) |\alpha| + (\sigma_2/(1-\rho)) |\beta| - (\rho/(1-\rho)) |\gamma|} \chi_j(\xi) \qquad \text{...(3.4)}$$

where $\chi_j(.)$ is the characteristic function of supp $\phi_j(.)$. Set

$$\mu_j = \frac{\sigma_j}{1-\rho} \ (j = 1, 2), \ \ \sigma = \frac{\rho}{1-\rho},$$

and write

$$a_j'(x_1, x_2, \xi) = a_j(j^{-\mu_1} x_1, j^{-\mu_2} x_2, j^\sigma \xi).$$

Then (3.4) leads us to:

$$| \partial_{x_1}^\alpha \partial_{x_2}^\beta \partial_\xi^\gamma a_j'(x_1, x_2, \xi) | \leqslant C_2 \chi_j'(\xi) \qquad \text{...(3.5)}$$

where $\chi_j'(\xi)$ is the characteristic function of supp $\phi_j(j^\sigma \xi)$, namely

$$\chi_j'(\xi) = \begin{cases} 1 & j - C_3 \leqslant \xi < j + C_3 \\ 0 & \text{otherwise.} \end{cases}$$

For $r, s, t > \frac{1}{2} n$, write

$$b_j'(x_1, x_2, \xi) = (1 - \Delta_{x_1})^{r/2} (1 - \Delta_{x_2})^{s/2} (1 - \Delta_\xi)^{t/2} a_j'(x_1, x_2, \xi). \qquad \text{...(3.6)}$$

Then we have the following lemma due to Kato (1976, pp. 6–7):

Lemma 3.1 — There is a positive Radon measure μ on R^n and *a* constant C_3 independent of j such that

(i) $b_j'(x_1, x_2, \xi) \leqslant C_3 \mu * X_j'(\xi) \equiv C_3 M_j'(\xi)$

(ii) $\int (1 + | \xi |)\, d\mu(\xi) < \infty$.

From (3.6) we have

$$a'_j = b'_j * g. \qquad \ldots(3.7)$$

A scale transformation of (3.7) yields

$$a_j(x_1, x_2, \xi) = b_j * g_j(x_1, x_2, \xi) \qquad \ldots(3.8)$$

where

$$\left.\begin{aligned} b_j(x_1, x_2, \xi) &= b'_j(j^{\mu_1} x_1, j^{\mu_2} x_2, j^{-\sigma}\xi) \\ g_j(x_1, x_2, \xi) &= g(j^{\mu_1} x_1, j^{\mu_2} x_2, j^{-\sigma}\xi). \end{aligned}\right\} \qquad \ldots(3.9)$$

Insertion of (3.8) and (3.9) into (1.1) shows

$$| A_j u(x_1) | = (2\pi)^{-n} \int e^{i(x_1 - x_2)\xi} b'_j(j^{\mu_1} w, j^{\mu_2} y, j^{-\sigma} z)\, g(j^{\mu_1}(x_1 - w),$$
$$j^{\mu_2}(x_2 - y), j^{-\sigma}(\xi - z))\, u(x_2)\, dx_2\, d\xi\, dw\, dy\, dz. \qquad \ldots(3.10)$$

An application of Lemma 3.1 and (3.10) gives us

$$| \langle A_j u, u \rangle | \leqslant (2\pi)^{-n} \int e^{i(x_1 - x_2)\xi} M'_j(j^{-\sigma} z)\, \psi_{r/2}(j^{\mu_1}(x_1 - w))$$
$$\times\, \psi_{s/2}(j^{\mu_2}(x_2 - y))\, \psi_{t/2}(j^{-\sigma}(\xi - z))$$
$$\times\, u(x_2)\, \bar{u}(x_1)\, dx_2\, d\xi\, dw\, dy\, dz\, dx_1. \qquad \ldots(3.11)$$

Since

$$\int e^{i(x_1 - x_2)\xi} \psi_{t/2}(\xi)\, d\xi = (1 + | x_1 - x_2 |^2)^{-t/2} \qquad \ldots(3.12)$$

substitution of (3.12) into (3.11) yields

$$| \langle A_j u, u \rangle | \leqslant (2\pi)^{-n} j^{n\sigma} \int e^{i(x_1 - x_2)z} M_j(z)\, \psi_{r/2}(j^{\mu_1}(x_1 - w)$$
$$\times\, \psi_{s/2}(j^{\mu_2}(x_2 - y))\, (1 + j^{2\sigma} | x_1 - x_2 |^2)^{-t/2}$$
$$\times\, u(x_2)\, \bar{u}(x_1)\, dx_2\, dw\, dy\, dz\, dx_1. \qquad \ldots(3.13)$$

Notice that $M'_j(j^{-\sigma} z) \leqslant M_j(z)$ in (3.13). Let $\nu > n/2$ and write

$$u_z(x) = e^{-ixz} u(x). \qquad \ldots(3.14)$$

A straightforward computation and (3.14) reveal that (3.13) is rewritten as follows:

$$| \langle A_j, u, u \rangle | \leqslant (2\pi)^{-n} j^{n\sigma} \int (1 + j^{2\mu_1} | w |^2)^{-\nu} (1 + | y |^2)^{-\nu} M_j(z)$$
$$\times \langle K^{(j)}_{w,y} u_z, u_z \rangle\, dw\, dy\, dz \qquad \ldots(3.15)$$

where

$$K^{(j)}_{w,y} u_z(x_1) = \int k^{(j)}_{w,y}(x_1, x_2)\, u_z(x_2)\, dx_2 \qquad \ldots(3.16)$$

$$k^{(j)}_{w,y}(x_1, x_2) = (1 + j^{2\mu_1}\,|\,w\,|^2)^{\nu}\,(1 + |\,y\,|^2)^{\nu}\,\psi_{r/2}(j^{\mu_1}(x_1 - w))$$
$$\psi_{s/2}(j^{\mu_2}(x_2 - y))\,(1 + j^{2\sigma}\,|\,x_1 - x_2\,|^2)^{-t/2}$$
$$\leqslant (1 + j^{2\mu_1}\,|\,w\,|^2)^{\nu}(1 + |\,y\,|^2)^{\nu}\psi_{r/2}(j^{\mu_1}(x_1 - w)\,\psi_{s/2}(j^{\mu_2}(x_2 - y)). \quad \ldots(3.17)$$

Lemma 3.2 — The integral operator $K^{(j)}_{w,y}$ defined by (3.16) with the kernel $k^{(j)}_{w,y}$ given by (3.17) belongs to a trace class.

Proof: It suffices, according to Gelfand-Vilenkin (1964, pp. 39–40), Kato (1966, p. 522), or Yosida (1971, pp. 280–81) to show that the kernel $k^{(j)}_{w,y}$ is expressed as a product of two Hilbert-Schmidt operators. For any positive real number $p > \frac{n}{2}$, $k^{(j)}_{w,y}$ is rewritten in view of (2.1) as follows:

$$k^{(j)}_{w,y}(x_1, x_2) = (1 + j^{2\mu_1}\,|\,w\,|^2)^{\nu}\,(1 + |\,y\,|^2)^{\nu}\,(1 - \Delta_{x_1})^{-p}\,(1 - \Delta_{x_1})^{p}$$
$$\times \{\psi_{r/2}(j^{\mu_1}(x_1 - w))\,(1 + j^{2\sigma}\,|\,x_1 - x_2\,|^2)^{-t/2}\}\,\psi_{s/2}(j^{\mu_2}(x_2 - y))$$
$$= \int (1 + j^{2\mu_1}\,|\,w\,|^2)^{\nu}\,(1 + |\,y\,|^2)^{\nu}\psi_p(x_1 - \xi)\,(1 - \Delta_\xi)^{p}$$
$$\times \{\psi_{r/2}(j^{\mu_1}(\xi - w)(1 + j^{2\sigma}\,|\,\xi - x_2\,|^2)^{-t/2}\}\,\psi_{s/2}(j^{\mu_2}(x_2 - y))\,d\xi$$
$$= \int k^{(j)}_{w,y}(x_1, \xi)\,l^{(j)}_{w,y}(\xi, x_2)\,d\xi \quad \ldots(3.18)$$

where

$$k^{(j_1)}_{w,y}(x_1, x_2) = (1 + j^{2\mu_1}\,|\,w\,|^2)^{\nu}\,(1 + |\,y\,|^2)^{\nu}$$
$$\times\,\psi_p(x_1 - x_2)\,(1 + |\,x_2\,|^2)^{-\tau/2}$$
$$l^{(j)}_{w,y}(x_1, x_2) = (1 + |\,x_1\,|^2)^{\tau/2}\,(1 - \Delta_{x_1})^{p}\,\{\psi_{r/2}(j^{\mu_1}(x_1 - w))$$
$$\times\,(1 + j^{2\sigma}\,|\,x_1 - x_2\,|^2)^{-t/2}\}\,\psi_{s/2}(j^{\mu_2}(x_2 - y)).$$

The fact that $k^{(j)}_{w,y}$ and $l^{(j)}_{w,y}$ belong to the kernels of Hilbert-Schmidt operator follow from (2.2) - (2.3), and by taking $\tau > (3p/2) + t - r$. This completes the proof in view of (3.18). Set

$$k^{(j)}(x_1, x_2) = \max_{k,\,l}\ \sup_{w,\,y}\left|\,k^{(j)}_{w,y}(x_1, x_2),\ l^{(j)}_{w,y}(x_1, x_2)\right| \quad \ldots(3.19)$$

and let $K^{(j)}$ be the integral operator (3.16) with the kernel $k^{(j)}$. Then the Fourier series expansion of $K^{(j)}$ is given by

$$K^{(j)} = \sum \lambda_q\,\langle\,.\,,\varphi_{jq}\,\rangle\,\varphi_{jq} \quad \ldots(3.20)$$

where $\{\lambda_q\}$ are the eigenvalues of $K^{(j)}$ and $\{\varphi_{jq}\}$ are the corresponding normalized eigenvectors. It follows from (3.15), (3.13) and (3.20) that

$$\left| \langle K^{(j)}_{w,y}\, u_z, u_z \rangle \right| \leqslant \sum_q \lambda_q \langle u_z, \varphi_{jq} \rangle \langle \varphi_{jq}, u_z \rangle$$

$$= \sum_q \lambda_q \left| \langle \hat{u}(z+\eta),\, j^{-n\mu_1/2}\, \hat{\varphi}_q(j^{-\mu_1}\eta) \rangle \right|^2 \text{(Plancherel)}$$

$$\leqslant \sum_q \lambda_q \int |\hat{u}(z+\eta)|^2\, |\hat{\varphi}_q(j^{-\mu_1}\eta)|^2\, d\eta$$

$$= \sum_q \lambda_q \int |\hat{u}(z+j^{\mu_1}\eta)|^2\, |\hat{\varphi}_q(\eta)|^2\, d\eta. \qquad \ldots(3.21)$$

Observe that $\hat{u}_z(\eta) = \hat{u}(z+\eta)$, $\hat{\varphi}_{jq}(\eta) = (j^{-n\mu_1/2}\, \hat{\varphi}_q(j^{-\mu_1}\eta)$ in (3.21). Insertion of (3.21) into (3.15) gives us

$$| \langle A_j u, u \rangle | \leqslant C(2\pi)^{-n} j^{n\sigma} \sum_q \lambda_q \int M_j(z)\,(1+j^{2\mu_1}|w|^2)^{-\nu}$$

$$\times (1+|y|^2)^{-\nu}\, |\hat{u}(z+j^{\mu_1}\eta)|^2\, |\hat{\varphi}_q(\eta)|^2\, dw\, dy\, dz\, d\eta$$

$$\leqslant C(2\pi)^{-n} \sum_q \lambda_q \int M_j(\xi - j^{\mu_1}\eta)\, |\hat{u}(\xi)|^2\, |\hat{\varphi}_q(\eta)|^2\, d\xi\, d\eta.$$

$$\ldots(3.22)$$

In (3.22) we have used a change of variables and the fact that $\nu > n/2$ and $\mu_1 > 0$. The following lemma is found in Kato (1976, pp. 8–9).

Lemma 3.3 (Kato 1976, pp. 8–9) — There is a constant C such that

$$\sum_j M_j(\xi - j^{\mu_1}\eta) \leqslant C(1+|\eta|).$$

Finally, since $(1+|D|)K$ also belongs to the trace class (Kato 1976, p. 4), (3.22) and Lemma 3.3 imply

$$| \langle Au, u \rangle | = \sum_j \langle A_j u, u \rangle$$

$$\leqslant C(2\pi)^{-n} \sum_q \lambda_q \int |\hat{u}(\xi)|^2 (1+|\eta|)\, |\hat{\varphi}_q(\eta)|^2\, d\xi\, d\eta$$

(equation continued on p. 286)

$$\leqslant C'(2\pi)^{-n} \| u \|_2^2 \sum_q \lambda_q \langle (1 + | D |) \varphi_q, \varphi_q \rangle$$

$$\leqslant C''(2\pi)^{-n}\, tr((1 + | D |) K) \| u \|_2^2 .$$

This completes the proof of our main theorem.

The problem of proving Kato's question via Banach algebra technique is still open. The following convolution operator whose meaning should be interpreted in distributional sense might be an appropriate one:

$$b * g(x, ., D) = \int b(w, y, \xi)\, e^{i(y+\xi)x} e^{-iwD} g(x, ., D)\, e^{iwD} e^{-i(y+\xi)x}\, dw\, dy\, d\xi.$$

However, we were not successful in proving the L^2-continuity of (1.2) under the assumptions of Theorem 3.2, using the above operator.

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