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ON THE EXISTENCE AND UNIQUENESS OF SOLUTIONS OF FUNCTIONAL DIFFERENTIAL EQUATIONS*

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In this note, we prove some theorems about the existence, uniqueness and continuous dependence on the given functions of solutions of the following equations:

$$y'(x) = f_n(x, y(g_1(x)), ..., y(g_m(x))), n = 0, 1, 2, ...$$

1. INTRODUCTION

Nadler (1968) discussed the existence, uniqueness and continuous dependence on the given functions of solutions of the following equations

$$y'(x) = f_n(x, y(x)), n \in N$$

where N denotes the set of nonnegative integers.

In this note, we shall extend his results to the following functional differential equations

$$y'(x) = f_n(x, y(g_1(x)), ..., y(g_m(x))), n \in N.$$

2. MAIN RESULTS

Let R be the set of real numbers. For a given point $(a_1, ..., a_{m+1}) \in R^{m+1}$ and two given positive constants a and b, let

$$I = [a_1 - a, a_1 + a]$$

$$S = \{(x_1, ..., x_{m+1}) \in R^{m+1} \mid x_1 \in I, \mid x_i - a_i \mid \leqslant b, i = 2, ..., m + 1\}.$$

Before going into discussion, we state some conditions on f_n and g_i as follows:

(C_1) $f_n \in C(S, R)$ for each $n \in N$. There exist positive number M and bounded sequences of positive numbers $\{A_n^i\}_{n \in N}$ for each $i = 1, ..., m$, such that for each $n \in N$

$$\mid f_n(x_1, ..., x_{m+1}) \mid \leqslant M \text{ for all } (x_1, ..., x_{m+1}) \in S$$

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and $$| f_n(x_1, y_1, \ldots, y_m) - f_n(x_1, z_1, \ldots, z_m) | \leqslant \sum_{i=1}^{m} A_n^i \, | y_i - z_i |$$

for all $(x_1, y_1, \ldots, y_m), (x_1, z_1, \ldots, z_m) \in S$;

(C_2) $g_i \in C(I, R)$ and $g_i(x) \leqslant x$ for all $x \in I$ and $i = 1, \ldots, m$;

(C_3) $\{f_n\}_{n=1}^{\infty}$ converges pointwise to f_0 on S;

(C_4) $g_i \in C(I, R)$ and $g_i(x) \geqslant x$ for all $x \in I$ and $i = 1, \ldots, m$;

(C_5) $g_i : I \to R$, $| g_i(x) - a_1 | \leqslant | x - a_1 |$ for $x \in I$ and $i = 1, 2, \ldots, m$.

In order to discuss our main results, we need the following lemma which is due to Nadler (1968).

Lemma — Let (Y, d) be a locally compact metric space and let $T_n : Y \to Y$ be a contraction mapping with fixed point p_n for each $n \in N$. If $\{T_n\}_{n=1}^{\infty}$ converges pointwise to T_0, then $\{p_n\}_{n=1}^{\infty}$ converges to p_0.

Now we can state and prove the following results.

Theorem 1 — Let (C_1), (C_2) and (C_3) hold. Then there exists an $h \in (0, a]$ such that for each $n \in N$ the Cauchy problem

$$\text{(CP)} \begin{cases} y'(x) = f_n(x, y(g_1(x)), \ldots, y(g_m(x))), \ x \in J \equiv [a_1 - h, a_1 + h] \\ y(x) = k_0(x), x \in K \equiv [c, a_1 - h] \ (x = a_1 - h \text{ if } c \geqslant a_1 - h) \end{cases}$$

has a unique solution y_n given as the limit of successive approximation, where $c = \min_{\substack{t \in J \\ 1 \leqslant i \leqslant m}} g_i(t)$ and $k_0 \in C(K, L \equiv [a_2 - b, a_2 + b])$ is a given function. Moreover, $\{y_n\}_{n=1}^{\infty}$ converges uniformly to y_0 on J.

PROOF : It follows from (C_1) that there is an $h \in (0, a]$ such that

$$0 < q = 2h \sum_{i=1}^{m} A_n^i < 1 \text{ for each } n \in N$$

and $Mh < b$. Let X be a space of these functions $k : J \to L$ which satisfy a Lipschitz condition with Lipschitz constant less than or equal to M and $k(a_1 - h) = k_0(a_1 - h)$. Let

$$Y = \{p(x) \in X \mid p(x) = k(x) \in X \text{ if } x \in J \text{ and } p(x) = k_0(x) \text{ if } x \in K\}.$$

Then X and Y with the supremum metric d are compact metric spaces.

The Cauchy problem (CP) is equivalent to the equation

$$\begin{cases} y(x) = k_0(a_1 - h) + \int\limits_{a_1-h}^{x} f_n(s, y(g_1(s)), \ldots, y(g_m(s)))\, ds, \; x \in J, \\ y(x) = k_0(x), \; x \in K. \end{cases}$$

For each $n \in N$ and $p \in Y$, let $T_n(p)$ be given by

$$T_n(p)(x) = \begin{cases} k_0(a_1 - h) + \int\limits_{a_1-h}^{x} f_n(s, p(g_1(s)), \ldots, p(g_m(s)))\, ds, \; x \in J, \\ k_0(x), \; x \in K. \end{cases}$$

It follows from (C_1) and (C_2) that T_n is a selfmapping of Y for each $n \in N$. For p_1, $p_2 \in Y$ and $n \in N$,

$$\begin{aligned} &d(T_n(p_1), T_n(p_2)) \\ &\leqslant \sup_{x \in J \cup K} \Big| \int\limits_{a_1-h}^{x} [f_n(s, p_1(g_1(s)), \ldots, p_1(g_m(s)) \\ &\qquad - f_n(s, p_2(g_1(s)), \ldots, p_2(g_m(s))]\, ds \Big| \\ &\leqslant 2h(A_n^1 + \ldots + A_n^m) \sup_{x \in J} | p_1(x) - p_2(x) | =: qd\,(p_1, p_2). \end{aligned}$$

Hence T_n, $n \in N$, is a contraction mapping of Y into itself. From the well-known theorem of Banach fixed point principle there is a unique fixed point of T_n for each $n \in N$, i.e. a unique solution $y_n \in Y$ of eqn. (CP) given as the limit of successive approximations.

Next we prove that $y_n \to y_0$, uniformly on J as $n \to \infty$. It follows from (C_1) and (C_3) that $f_n \to f_0$ pointwise and $| f_n | \leqslant M$ for $n = 1, 2, \ldots$. By the Lebesgue bounded convergence theorem

$$\int\limits_{a_1-h}^{x} f_n(s, p(g_1(s)), \ldots, p(g_m(s)))\, ds \to \int\limits_{a_1-h}^{x} f_0(s, p(g_1(s)), \ldots, p(g_m(s)))\, ds$$

as $n \to \infty$. Thus $\{T_n(p)\}_{n=1}^{\infty}$ converges pointwise to T_0 on $J \cup K$. Since

$$| T_n(p)(x) - T_n(p)(y) | \leqslant M \, | x - y | , \; x, y \in J \cup K.$$

the sequence $\{T_n(p)\}_{n=1}^{\infty}$ is equicontinuity on the compact set $J \cup K$. Thus $\{T_n(p)\}_{n=1}^{\infty}$ converges uniformly to T_0 on $J \cup K$ and hence $\{T_n\}_{n=1}^{\infty}$ converges pointwise to T_0 on Y. From Lemma, the sequence of unique fixed points of T_n for $n = 1, 2, \ldots$, tends to the unique point of T_0. Since $T_n y_n = y_n$ for $n \in N$, we have $y_n \to y_0$ uniformly on $J \cup K$. Thus our proof is complete.

Theorem 2 — Let (C_1), (C_3) and (C_4) hold. Then there is an $h \in (0, a]$ such that for each $n \in N$ the Cauchy problem

$$\begin{cases} y'(x) = f_n(x, y(g_1(x)), \ldots, y(g_m(x))), \ x \in J \\ y(x) = k_1(x), \ x \in [a_1 + h, d] \end{cases}$$

has a unique solution y_n, where $d = \max\limits_{\substack{1 \leq i \leq m \\ x \in J}} g_i(x)$ and $k_1 : [a_1 + h, d] \to [a_2 - b, a_2 + b]$ is a given continuous function. Moreover, $\{y_n\}_{n=1}^{\infty}$ converges uniformly to y_0 on J.

PROOF: Let X_1 be a space of these functions $k : J \to [a_2 - b, a_2 + b]$ which satisfy a Lipschitz condition with Lipschitz constant less than or equal to M and $k(a_1 + h) = k_1(a_1 + h)$. Let Y_1 be a space of all functions p such that

$$p(x) = \begin{cases} k(x), \ x \in J \\ k_1(x), \ x \in [a_1 + h, d] \end{cases}$$

where $k \in X_1$. We see easily that for each $n \in N, p \in Y_1$

$$T_n(p)(x) = \begin{cases} \int_{a_1+h}^{x} f_n(s, p(g_1(s)), \ldots, p(g_m(s)))\, ds, \ x \in J \\ k_1(x), \ x \in [a_1 + h, d] \end{cases}$$

is a contraction selfmapping of Y_1. As in the proof of Theorem 1, we can show that $\{T_n\}$ converges pointwise to T_0 on Y_1 and from Lemma we complete our proof.

Theorem 3 — Let (C_1), (C_3) and (C_5) hold. Then there is an $h \in (0, a]$ such that for each $n \in N$ the Cauchy problem

$$\begin{cases} y'(x) = f_n(x, y(g_1(x)), \ldots, y(g_m(x))), \ x \in J \\ y(a_1) = a_2 \end{cases}$$

has a unique solution y_n and $\{y_n\}_{n=1}^{\infty}$ converges uniformly to y_0 on J.

PROOF : Let Y_2 be the space of these functions $k : J \to [a_2 - b, a_2 + b]$ which satisfy a Lipschitz condition with Lipschitz constant less than or equal to M. For $k \in Y_2$, by (C_5)

$$k(g_i(x)) \in [a_2 - b, a_2 + b], \text{ for } x \in J, i = 1, 2, \ldots, m.$$

We can prove that for each $n \in N$ and $k \in Y_2$,

$$T_n(k)(x) = a_2 + \int_{a_1}^{x} f_n(s, k(g_1(s)), \ldots, k(g_m(s)))\, ds, \; x \in J$$

is a contraction selfmapping of Y_2 and $\{T_n\}_{n=1}^{\infty}$ converges pointwise to T_0 on Y_2. From Lemma, we complete our proof.

REFERENCE

Nadler, S. B. (Jr) (1968). Sequences of contractions and fixed points. *Pacific J. Math.*, **27**, 579–85.