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SOME FIXED POINT THEOREMS IN NORMED LINEAR SPACES*

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In this paper fixed point theorems are established first for mappings T of a convex subset K of a normed linear space X satisfying

$$\|Tx - Ty\| \, \|x - y\| \leqslant \|x - y\| \{a_1(x, y) \|x - y\| + a_2(x, y)(\|x - Tx\| + \|y - Ty\|) + a_3(x, y)(\|x - Ty\| + \|y - Tx\|)\} + a_4(x, y) \|x - Ty\| \, \|y - Tx\|$$

for all $x, y \in K$, $x \neq y$, and then an analogous result is obtained for a_i as constant, $i = 1, 2, 3, 4$.

1. INTRODUCTION

Let T be a selfmapping of a normed linear space X. According to Browder and Petryshyn (1966) we say T is asymptotically regular at a point $x \in X$ if $\|T^n x - T^{n+1} x\| \to 0$ as $n \to \infty$, whenever $T^n x$ is defined for all $n \in N$, where N denotes the set of positive integers. By the orbit of T at a point $x \in X$ we mean that the set $o(x, T) = \{x, Tx, T^2x, \ldots\}$, its closure is denoted by $\overline{o(x, T)}$. Let $T \mid_{\overline{o(x,T)}}$ denote the restriction of T to $\overline{o(x, T)}$. According to Jaggi (1977), we say T is called x-orbitally continuous if $T \mid_{\overline{o(x,T)}} : \overline{o(x, T)} \to X$ is continuous. From Ciric (1971, 1973), we see that T is orbitally continuous if T is x-orbitally continuous for each $x \in X$.

Let T be a selfmapping of a normed linear space X satisfying

$$\|Tx - Ty\| \leqslant \tfrac{1}{2}(\|x - Tx\| + \|y - Ty\|)$$

for all $x, y \in X$. Let $T = \lambda I + (1 - \lambda) T$ for $\lambda \in (0, 1)$. Recently, Jaggi (1977) discussed the convergence of $\{T^n x\}$ in a normed linear space, which need not necessarily be strictly convex or uniform convex, e.g. Goebel *et al.* (1973), Kannan (1971, 1973) and Petryshyn and Williamson (1973).

In this paper we shall extend and improve the results of Jaggi (1977). For other related results, we refer to Diaz and Metcalf (1969), Edelstein (1966) and Murakami and Yeh (1979).

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2. Main Results

Theorem 1 — Let K be a nonempty convex subset of a normed linear space X. Let T be a selfmapping of K satisfying

$$(C_1) \qquad \begin{aligned} \| Tx - Ty \| \, \| x - y \| \leqslant \| x - y \| \{ & a_1(x, y) \| x - y \| \\ & + a_2(x, y) (\| x - Tx \| + \| y - Ty \|) \\ & + a_3(x, y) (\| x - Ty \| + \| y - Tx \|)\} \\ & + a_4(x, y) \| x - Ty \| \, \| y - Tx \| \end{aligned}$$

for all $x, y \in K$, $x \neq y$, where $a_i : K \times K \to [0, 1)$ with

$$(C_2) \qquad \begin{cases} \sup\limits_{x, y \in K} \{a_1(x, y) + 2a_2(x, y) + 2a_3(x, y) + a_4(x, y)\} \leqslant 1, \\ \inf\limits_{x, y \in K} a_2(x, y) > 0. \end{cases}$$

Let $T_b = \sum\limits_{i=0}^{m} b_i T^i$, for some $m \in N$, where $b_i \in (0, 1)$, $\sum\limits_{i=0}^{m} b_i = 1$ and $T^0 = I$ (the identity mapping). Suppose that there is a point x_0 in K such that

(i) $\{T_b^n x_0\}_{n \in N}$ clusters to a point $u \in K$,

(ii) T_b is x_0-orbitally continuous and asymptotically regular at x_0.

Then $\{T_b^n x_0\}$ converges to u and u is the unique fixed point of T in K.

Proof: From (i), there is a subsequence $\{T_b^{n_i} x_0\}$ which converges to u. Since T_b is x_0-orbitally continuous, we have

$$(I - T_b) \, T_b^{n_i} x_0 \to (I - T_b) \, u.$$

It follows from the asymptotic regularity of T_b at x_0 that

$$\| (I - T_b) \, T_b^{n_i} x_0 \| = \| T_b^{n_i} x_0 - T_b^{n_i+1} x_0 \| \to 0$$

as $i \to \infty$. This means that $\| (I - T_b) u \| = 0$ and hence $T_b u = u$ and $Tu = u$.

Let $v \in K$ be another point of K such that $Tv = v$ and $u \neq v$. Then

$$\begin{aligned} \| u - v \| \, \| Tu - Tv \| \leqslant \| u - v \| \{ & a_1(u, v) \| u - v \| \\ & + a_2(u, v) (\| u - Tu \| + \| v - Tv \|) \\ & + a_3(u, v) (\| u - Tv \| + \| v - Tu \|)\} \\ & + a_4(u, v) \| u - Tv \| \, \| v - Tu \| \end{aligned}$$

which implies, by (C_2),

$$\|u-v\| \leqslant a_1(u,v)\|u-v\| + 2a_3(u,v)\|u-v\| + a_4(u,v)\|u-v\|$$
$$= (a_1(u,v) + 2a_3(u,v) + a_4(u,v))\|u-v\| < \|u-v\|$$

a contradiction. This contradiction proves that $u = v$.

For all $x \in K$, $x \neq u$, we have

$$\|T_b x - u\| = \left\| \sum_{i=0}^{m} b_i T^i(x-u) \right\| \leqslant \sum_{i=0}^{m} b_i \|T^i x - T^i u\|. \qquad \text{...(1)}$$

From (C_1), we have

$$\|Tx - Tu\|\,\|x-u\| \leqslant \|x-u\|\{a_1(x,u)\|x-u\| + a_2(x,u)\|x-Tx\| + a_3(x,u)(\|x-u\| + \|u-Tx\|)\} + a_4(x,u)\|x-u\|\,\|u-Tx\|$$

which gives

$$\|Tx - u\| \leqslant a_1(x,u)\|x-u\| + a_2(x,u)(\|x-u\| + \|u-Tx\|) + a_3(x,u)(\|x-u\| + \|u-Tx\|) + a_4(x,u)\|u-Tx\|.$$

Thus

$$\|Tx - Tu\| \leqslant \frac{a_1(x,u) + a_2(x,u) + a_3(x,u)}{1 - a_2(x,u) - a_3(x,u) - a_4(x,u)}\|x-u\| \leqslant \|x-u\|.$$

Similarly, for $i = 1, 2, \ldots, m$,

$$\|T^i x - T^i u\| \leqslant \|x-u\|. \qquad \text{...(2)}$$

It follows from (1) and (2) that

$$\|T_b x - u\| \leqslant \sum_{i=0}^{m} b_i \|x-u\| = \|x-u\|.$$

Since $x \in K - \{u\}$ is arbitrary, we have, for $x = x_0$,

$$\|T_b^{n+1} x_0 - u\| \leqslant \|T_b^n x_0 - u\|.$$

This implies $T_b^n x_0 \to u$ as $n \to \infty$, since $T_b^{n_i} x_0 \to u$ as $i \to \infty$. If $x = u$, then we also have $T_b^n x \to u$. Thus our proof is complete.

Remark: Taking $a_1 = a_3 = a_4 = 0$, $a_2 = 2^{-1}$ and $m = 1$ in our Theorem 1, we obtain Theorem of Jaggi (1977).

Theorem 2 — Let K be a nonempty closed convex bounded subset of a reflexive Banach space X and let T be a selfmapping of K satisfying

(C$_3$) $$\| Tx - Ty \| \leqslant a_1 \| x - y \| + a_2 (\| x - Tx \| + \| y - Ty \|) + a_4(\| x - Ty \| + \| y - Tx \|)$$

for all $x, y \in K$, where $a_1 \geqslant 0$, $a_2 > 0$, $a_3 \geqslant 0$ are constants and $a_1 + 2a_2 + 2a_3 \leqslant 1$;

(C$_4$) $T_b = \sum_{i=0}^{m} b_i T^i$ is asymptotically regular at some $x_0 \in K$, for some $m \in N$, where $b_i \in (0, 1)$ with $b_0 + b_1 + ... + b_m = 1$;

(C$_5$) $\sup_{y \in H} \| y - Ty \| < \delta(H)$, for every nonempty bounded closed convex subset H of K containing more than one element and mapped into itself by T, where $\delta(H)$ is the diameter of H.

Then $\{T_b^n x_0\}$ converges to a unique fixed point of T in K.

The above theorem improves and extends a recent result of Jaggi (1977, Theorem 2), who proved it by means of a result of Kannan (1971, Theorem 1). However, Goebel *et al.* (1973, p. 68) pointed out that the argument by Kannan (1971, Theorem 1) was unsuccessful. Hung and Shih (1976) extended and proved successfully Kannan's result by using a result of Smulian (1939).

PROOF OF THEOREM 2 : It follows from Theorem of Hung and Shih (1976) that T has a unique fixed point u in K. We shall prove that $u \in \bar{A}$, where $A = o(x_0, T_b)$. As in the proof of Theorem 1, we have

$$\| T_b^{n+1} x_0 - u \| \leqslant \| T_b^n x_0 - u \|. \qquad ...(3)$$

Hence A and $\bar{A}$ are bounded. Since

$$I - T_b = I - \sum_{i=0}^{m} b_i T^i = \sum_{i=0}^{m} b_i (I - T^i)$$

it follows from (C$_5$) that $I - T_b$ maps bounded closed sets into closed sets; hence $(I - T_b) \bar{A}$ is closed. Since

$$T_b^n x_0 - T_b^{n+1} x_0 = (I - T_b) T_b^n x_0 \in (I - T_b) A \subset (I - T_b) \bar{A}$$

it follows from (C$_4$) that $0 \in (I - T_b) \bar{A}$. Thus there is a $v \in \bar{A}$ such that $(I - T_b)v = 0$, that is, $T_b v = v$ and hence $Tv = v$. From the uniqueness of u, we have $u = v$. Thus $u \in \bar{A}$. Since $u \in \bar{A}$, there is a subsequence $\{T_b^{n_i} x_0\}$ converges to u. This and (3) give $T_b^n x_0 \to u$ as $n \to \infty$.

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