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A NOTE ON FIXED POINT IN COMPACT METRIC SPACES

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In this paper the results of Fisher (1978) and Khan (1978) have been extended to a more general case.

The results of this paper are inspired by two recent papers of Fisher (1978) and Khan (1978). They proved that a continuous mapping T of a compact metric space (X, d) has a unique fixed point if T satisfies

$$d(Tx, Ty) < \tfrac{1}{2}(d(x, Ty) + d(y, Tx))$$

or

$$d(Tx, Ty) < (d(x, Tx)\, d(y, Ty))^{1/2}$$

for all x, y in X with $x \neq y$.

The main purpose of this paper is to extend their results to a more general case. For related results, we refer to Ciric (1976) and Yeh (1978).

Theorem — Let T be a continuous mapping of a nonempty compact metric space (X, d) satisfying

$$\text{(C)} \qquad d(Tx, Ty) < h(d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx), (d(x, y))^{-1} d(x, Tx)\, d(y, Ty), a(x, y)\, d(x, Ty)\, d(y, Tx), b(x, y)\, (d(y, Tx)\, d(x, Ty))^{1/2})$$

for all x, y in X with $x \neq y$, where

(i) $a(x, y)$ and $b(x, y)$ are nonnegative real functions;

(ii) $h : (R_+)^8 \to R_+ \equiv [0, \infty)$ is nondecreasing in each coordinate variable; and $g(t) = h(t, t, t, c_1 t, c_2 t, t, t, t) \leqslant t$ for each $t > 0$, where $c_1 + c_2 \leqslant 2$.

Then T has a fixed point. If, in addition, $a(x, y) \leqslant (d(x, y))^{-1}$ and $b(x, y) \leqslant 1$, then T has a unique fixed point.

PROOF : Define a real value function f on X by $f(x) = d(x, Tx)$. Since d and T are continuous functions, it follows that f is a continuous function on X. Since X

is compact, it attains its minimum value and so there is a point u in X such that $f(u) = \inf\{f(x) : x \in X\}$. If $u \neq Tu$, then it follows from (C) that

$$\begin{aligned} d(Tu, T^2u) &< h(d(u, Tu)\, d(u, Tu), d(Tu, T^2u), d(u, T^2u), d(Tu, Tu), \\ &\qquad (d(u, Tu))^{-1}\, d(u, Tu)\, d(Tu, T^2u), 0, \\ &\qquad b(u, Tu)\, (d(Tu, Tu)\, d(u, T^2u))^{1/2}) \\ &\leqslant h(d(Tu, T^2u), d(Tu, T^2u), d(Tu, T^2u), 2d(Tu, T^2u), 0, \\ &\qquad d(Tu, T^2u), 0, 0) \leqslant g(d(Tu, T^2u)) \leqslant d(Tu, T^2u) \end{aligned}$$

a contradiction. This contradiction proves that $Tu = u$.

Next we prove that u is unique for $a(x, y) \leqslant (d(x, y))^{-1}$ and $b(x, y) \leqslant 1$. Suppose that $v(\neq u)$ is a fixed point of T. Then

$$\begin{aligned} d(u, v) = d(Tu, Tv) &< h(d(u, v), d(u, u), d(v, v), d(u, v), d(v, u), \\ &\qquad 0, d(u, v)\, d(u, v)) \leqslant g(d(u, v)) \leqslant d(u, v) \end{aligned}$$

a contradiction. This contradiction proves the uniqueness of u. Thus our proof is complete.

Corollary — Let T be a continuous mapping of a nonempty compact metric space (x, d) satisfying.

$$\begin{aligned} d(Tx, Ty) < \max\{& d(x, y), \tfrac{1}{2}(d(x, Tx) + d(y, Ty), \\ & \tfrac{1}{2}(d(x, Ty) + d(y, Tx)), (d(x, y))^{-1} d(x, Tx)\, d(y, Ty), \\ & a(x, y)\, d(x, Ty)\, d(y, Tx), (d(x, Tx)\, d(y, Ty))^{1/2}, \\ & b(x, y)\, (d(y, Tx)\, d(x, Ty))^{1/2}\} \end{aligned}$$

for all x, y in X with $x \neq y$, where $a(x, y)$ and $b(x, y)$ are nonnegative real functions, then T has a fixed point. If, in addition, $a(x, y) \leqslant (d(x, y))^{-1}$ and $b(x, y) \leqslant 1$, then T has a unique fixed point.

References

Ciric, Lj. (1976). A certain class of maps and fixed point theorems. *Publ. Inst. Math.*, **20**(34), 73–77.

Fisher, B. (1978). A fixed point theorem for compact metric spaces. *Publ. Inst. Math.*, **25**, 193–94.

Khan, M. S. (1978). On fixed point theorems. *Math. Japonica*, **23**, 201–204.

Yeh, C. C. (1978). A fixed point theorem in orbitally complete metric spaces. *Publ. Inst. Math.*, **24**, 197-99.