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A REPRESENTATION FOR GENERAL SOLUTIONS IN THE THEORY OF ELASTICITY

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The analytical continuous solutions in the theory of elasticity can be represented in a differential form for the displacement field in harmonics. This representation is attained through the theory of the determination of the continuous analytical solutions along a boundary. In the same line, the harmonics are represented in their differential form.

We have obtained general formulae for the elastic solutions. As far as we are aware, these formulae are general, fundamental and different from those in the literature.

1. The Plane Elastic Problem

In this case we define the function q_1 and q_2 by the formulae:

$$q_1 = \frac{1}{c}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right),\ q_2 = \frac{1}{1+c}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) \qquad \ldots(1.1)$$

where u and v are the displacements, which satisfy the Cauchy-Riemann equations, and $c = 1 - 2\nu$, ν is Poisson's ratio. Let us introduce the functions φ_1 and φ_2, which satisfy the system equations:

$$\frac{\partial \varphi_1}{\partial x} + \frac{\partial \varphi_2}{\partial y} = L_1,\ \frac{\partial \varphi_1}{\partial y} - \frac{\partial \varphi_2}{\partial x} = L_2$$

where L_i satisfy Cauchy-Riemann equations. Taking $L_i = (1 + 2c)\, q_i, (i = 1, 2)$, we have

$$\frac{\partial \varphi_1}{\partial x} + \frac{\partial \varphi_2}{\partial y} = (1 + 2c)\, q_1,\ \frac{\partial \varphi_1}{\partial y} - \frac{\partial \varphi_2}{\partial x} = (1 + 2c)\, q_2. \qquad \ldots(1.2)$$

Through this system of equations we have connected the solution for the theory of elasticity to the two harmonic functions φ_1 and φ_2. We seek the solution of (1.2) in the form,

$$\left.\begin{aligned} \varphi_1 &= 2u + \frac{1}{c}\frac{\partial F}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \frac{1}{1+c}\frac{\partial F}{\partial y}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) \\ \varphi_2 &= 2v + \frac{1}{c}\frac{\partial F}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{1}{1+c}\frac{\partial F}{\partial x}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right). \end{aligned}\right\} \qquad \ldots(1.3)$$

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These functions are harmonics under the condition, that F satisfies equation

$$\triangle F = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 1$$

and u and v satisfy the homogeneous elastic equations.

Taking into consideration that, φ_1 and φ_2 are subject to eqns. (1.2) and from eqns. (1.3), it is easy to find expressions for the displacements in terms of the two harmonics φ_1 and φ_2 and F,

$$\left.\begin{aligned} 2u &= \varphi_1 - \frac{1}{1+c}\left[\frac{\partial F}{\partial x}\left(\frac{\partial \varphi_1}{\partial x} + \frac{\partial \varphi_2}{\partial y}\right) - \frac{\partial F}{\partial y}\left(\frac{\partial \varphi_1}{\partial y} - \frac{\partial \varphi_2}{\partial x}\right)\right] \\ 2v &= \varphi_2 - \frac{1}{1+2c}\left[\frac{\partial F}{\partial y}\left(\frac{\partial \varphi_1}{\partial x} + \frac{\partial \varphi_2}{\partial y}\right) + \frac{\partial F}{\partial x}\left(\frac{\partial \varphi_1}{\partial y} - \frac{\partial \varphi_2}{\partial x}\right)\right]. \end{aligned}\right\} \quad \ldots(1.4)$$

As a first application to these equations we consider the relation between the first and the second basic problems in the theory of elasticity for the incompressible material ($c = 0$).

The components of the force, which affect the boundary are:

$$X_n = \sigma_x \cos(n, x) + \tau_{xy} \cos(n, y)$$

$$Y_n = \tau_{xy} \cos(n, x) + \sigma_y \cos(n, y).$$

Using Hooke's law and (1.4), we find

$$\begin{aligned} X_n = &- \frac{c\mu}{1+2c}\left[\left(\frac{\partial \varphi_1}{\partial x} + \frac{\partial \varphi_2}{\partial y}\right)\frac{dy}{ds} + \left(\frac{\partial \varphi_2}{\partial x} - \frac{\partial \varphi_1}{\partial y}\right)\frac{dx}{ds}\right] \\ &+ \mu \frac{\partial \varphi_1}{\partial x} \cdot \frac{dy}{ds} - \mu \frac{\partial \varphi_1}{\partial y}\frac{dx}{ds} \\ &+ \frac{\mu}{1+2c}\frac{d}{ds}\left[\frac{\partial F}{\partial y}\left(\frac{\partial \varphi_1}{\partial x} + \frac{\partial \varphi_2}{\partial y}\right) - \frac{\partial F}{\partial x}\left(\frac{\partial \varphi_2}{\partial x} - \frac{\partial \varphi_1}{\partial y}\right)\right] \quad \ldots(1.5) \end{aligned}$$

$$\begin{aligned} Y_n = &\frac{c\mu}{1+2c}\left[\left(\frac{\partial \varphi_1}{\partial x} + \frac{\partial \varphi_2}{\partial y}\right)\frac{dx}{ds} + \left(\frac{\partial \varphi_1}{\partial y} - \frac{\partial \varphi_2}{\partial x}\right)\frac{dy}{ds}\right] \\ &- \mu \frac{\partial \varphi_2}{\partial y}\frac{dx}{ds} + \mu \frac{\partial \varphi_2}{\partial x}\frac{dy}{ds} \\ &- \frac{\mu}{1+2c}\frac{d}{ds}\left[\frac{\partial F}{\partial x}\left(\frac{\partial \varphi_1}{\partial x} + \frac{\partial \varphi_2}{\partial y}\right) + \frac{\partial F}{\partial y}\left(\frac{\partial \varphi_2}{\partial x} - \frac{\partial \varphi_1}{\partial y}\right)\right]. \quad \ldots(1.6) \end{aligned}$$

where n is the out-normal to the boundary, s the arc length of the contour,

$$\cos(n, x) = \frac{dy}{ds}, \cos(n, y) = -\frac{dx}{ds}$$

and μ the shear modulus.

We introduce the harmonic functions φ_1^* and φ_2^* associated with the functions φ_1 and φ_2 respectively such that

$$\left.\begin{aligned} \frac{\partial\varphi_1}{\partial x} = \frac{\partial\varphi_1^*}{\partial y}, \frac{\partial\varphi_2}{\partial x} = \frac{\partial\varphi_2^*}{\partial y} \\ \frac{\partial\varphi_1}{\partial y} = -\frac{\partial\varphi_1^*}{\partial x}, \frac{\partial\varphi_2}{\partial y} = -\frac{\partial\varphi_2^*}{\partial x}. \end{aligned}\right\} \qquad \ldots(1.7)$$

Therefore the components of the force, take the form:

$$\begin{aligned} X_n = &-\frac{c\mu}{1+2c}\left[\left(\frac{\partial\varphi_1^*}{\partial x} + \frac{\partial\varphi_2^*}{\partial y}\right)\frac{dx}{ds} + \left(\frac{\partial\varphi_1^*}{\partial y} - \frac{\partial\varphi_2^*}{\partial x}\right)\frac{dy}{ds}\right] \\ &+ \mu\frac{d}{ds}\left[\varphi_1^* - \frac{1}{1+2c}\frac{\partial F}{\partial x}\left(\frac{\partial\varphi_1^*}{\partial x} + \frac{\partial\varphi_2^*}{\partial y}\right)\right. \\ &\left.+ \frac{1}{1+2c}\frac{\partial F}{\partial y}\left(\frac{\partial\varphi_1^*}{\partial y} - \frac{\partial\varphi_2^*}{\partial x}\right)\right] \end{aligned} \qquad \ldots(1.8)$$

$$\begin{aligned} Y_n = &\frac{c\mu}{1+2c}\left[\left(\frac{\partial\varphi_1^*}{\partial y} - \frac{\partial\varphi_2^*}{\partial x}\right)\frac{dx}{ds} - \left(\frac{\partial\varphi_1^*}{\partial x} + \frac{\partial\varphi_2^*}{\partial y}\right)\frac{dy}{ds}\right] \\ &+ \mu\frac{d}{ds}\left[\varphi_2^* - \frac{1}{1+2c}\frac{\partial F}{\partial y}\left(\frac{\partial\varphi_1^*}{\partial x} + \frac{\partial\varphi_2^*}{\partial y}\right)\right. \\ &\left.- \frac{1}{1+2c}\frac{\partial F}{\partial x}\left(\frac{\partial\varphi_1^*}{\partial y} - \frac{\partial\varphi_2^*}{\partial x}\right)\right]. \end{aligned} \qquad \ldots(1.9)$$

From the representation (1.4) it follows that, the expressions

$$\begin{aligned} 2u^* &= \varphi_1^* - \frac{1}{1+2c}\frac{\partial F}{\partial x}\left(\frac{\partial\varphi_1^*}{\partial x} + \frac{\partial\varphi_2^*}{\partial y}\right) + \frac{1}{1+2c}\frac{\partial F}{\partial y}\left(\frac{\partial\varphi_1^*}{\partial y} - \frac{\partial\varphi_2^*}{\partial x}\right), \\ 2v^* &= \varphi_2^* - \frac{1}{1+2c}\frac{\partial F}{\partial y}\left(\frac{\partial\varphi_1^*}{\partial x} + \frac{\partial\varphi_1^*}{\partial y}\right) - \frac{1}{1+2c}\frac{\partial F}{\partial x}\left(\frac{\partial\varphi_1^*}{\partial y} - \frac{\partial\varphi_2^*}{\partial x}\right) \end{aligned} \qquad \ldots(1.10)$$

satisfy the homogeneous differential equations of the theory of elasticity.

The displacements u^* and v^* are associated with the displacements u and v respectively.

By introducing formulae (1.10) in (1.8) and (1.9), we obtain

$$\begin{aligned} X_n &= \frac{-c\mu}{1+2c}\left[\left(\frac{\partial\varphi_1^*}{\partial x} + \frac{\partial\varphi_2^*}{\partial y}\right)\frac{dx}{ds} + \left(\frac{\partial\varphi_1^*}{\partial y} - \frac{\partial\varphi_2^*}{\partial x}\right)\frac{dy}{ds}\right] + 2\mu\frac{du^*}{ds} \\ Y_n &= \frac{c\mu}{1+2c}\left[-\left(\frac{\partial\varphi_1^*}{\partial x} + \frac{\partial\varphi_2^*}{\partial y}\right)\frac{dy}{ds} + \left(\frac{\partial\varphi_1^*}{\partial y} - \frac{\partial\varphi_2^*}{\partial x}\right)\frac{dx}{ds}\right] \end{aligned}$$

$- \; dv^*$ ...(1.11)

therefore, for the incompressible material ($c = 0$, $\mu = E/3$) we have

$$3\frac{X_n}{E} = 2\frac{du^*}{ds}, 3\frac{Y_n}{E} = 2\frac{dv^*}{ds} \qquad ...(1.12)$$

where E is Young's modulus.

Thus, for the incompressible material the first basic problem in the theory of elasticity reduces to the second basic problem and vice versa. This means that: if we use formulae (1.12), and since we know the contour forces X_n and Y_n we find the value of the associated displacements u^*, v^* on the contour, and by analytical continuation we reconstruct u^*, v^* in the region. Knowing u^*, v^* we can find the harmonic functions φ_1^* and φ_2^*, and hence from (1.7) the functions φ_1 and φ_2 are obtained.

Formulae (1.4) give the unknown displacements u and v. The relation between the two solutions of the first and the second basic planer problem in complex variables is treated by Hill (1955).

Now we write the general solution (1.4) in a complex form. Let us take the function F in the form

$$F = \tfrac{1}{4}(x^2 + y^2) + \tfrac{1}{2}g \qquad ...(1.13)$$

(where g is a harmonic function), to obtain the Kolossova-Muskelishvilli formula. We have

$$2(u - iv) = \varphi_1 - i\varphi_2 - \frac{1}{2(1+2c)}\left(\bar{z} + \frac{\partial g}{\partial x} - i\frac{\partial g}{\partial y}\right)\left[\left(\frac{\partial \varphi_1}{\partial x} + \frac{\partial \varphi_2}{\partial y}\right) + i\left(\frac{\partial \varphi_2}{\partial x} - \frac{\partial \varphi_1}{\partial y}\right)\right]. \qquad ...(1.14)$$

Since $-\dfrac{\partial g}{\partial x}$ and $\dfrac{\partial g}{\partial y}$ are conjugate harmonic functions, then

$$-\frac{\partial g}{\partial x} + i\frac{\partial g}{\partial y} = \mathcal{F}(z)$$

where $\mathcal{F}(z)$ is an analytical function of the complex variable $z = x + iy$. Now consider the two analytical functions $\Phi_1(z)$ and $\Phi_2(z)$ which are related to the functions φ_1 and φ_2 through the relations:

$$\varphi_1 = \mathrm{Re}\,\Phi_1(z), \ \varphi_2 = \mathrm{Im}\,\Phi_2(z)$$

then the general form for the solution (1.4) takes the form

$$2(u - iv) = \tfrac{1}{2}[\Phi_1(z) - \Phi_2(z)] + \tfrac{1}{2}[\overline{\Phi_1(z)} + \overline{\Phi_2(z)}] - \frac{1}{2(1+2c)}[\bar{z} - \mathcal{F}(z)]\,[\Phi_1'(z) + \Phi_2'(z)]. \qquad ...(1.15)$$

Now when we take

$$\Phi_1(z) = (1 + 2c)\,\varphi(z) - \mathcal{F}(z)\,\varphi'(z) - \psi(z)$$

$$\Phi_2(z) = (1 + 2c)\,\varphi(z) + \mathcal{F}(z)\,\varphi'(z) + \psi(z)$$

then (1.15) gives the well-known Kolossova-Mushkelishvilli formula (Mushkelishvili 1966).

$$2(u - iv) = (1 + 2c)\,\varphi(\bar{z}) - \bar{z}\varphi'(z) - \psi(z). \qquad \text{...(1.16)}$$

On the other side in formula (1.16) if we suppose that

$$\varphi(z) = \frac{1}{2(1 + 2c)}\,[\Phi_1(z) + \Phi_2(z)]$$

$$\psi(z) = \tfrac{1}{2}\,[\Phi_2(z) - \Phi_1(z)] - \frac{1}{2(1 + 2c)}\,\mathcal{F}(z)\,[\Phi_1'(z) + \Phi_2'(z)].$$

We have a representation for the general solution of the plane elastic problem in the form (1.15).

2. The Axially Symmetric Problem

Suppose that the x-axis is the axis of symmetry, the y-axis is along the radial direction, u is the axial displacement and v the radial displacement. From the conditions of symmetry, the functions

$$q_1 = \frac{1}{c}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{v}{y}\right),\quad q_2 = \frac{y}{1 + c}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) \qquad \text{...(2.1)}$$

satisfy the system of differential equations

$$y\frac{\partial q_1}{\partial x} + \frac{\partial q_2}{\partial y} = 0,\quad y\frac{\partial q_1}{\partial y} - \frac{\partial q_2}{\partial x} = 0. \qquad \text{...(2.2)}$$

Hence we conclude, that the functions q_1 and q_2 are solutions of the second order differential equations

$$L_1\,[q_1] \equiv \frac{\partial^2 q_1}{\partial x^2} + \frac{\partial^2 q_1}{\partial y^2} + \frac{1}{y}\,\frac{\partial q_1}{\partial y} = 0 \qquad \text{...(2.3)}$$

$$L_2\,[q_2] \equiv \frac{\partial^2 q_2}{\partial x^2} + \frac{\partial^2 q_2}{\partial y^2} - \frac{1}{y}\,\frac{\partial q_2}{\partial y} = 0. \qquad \text{...(2.4)}$$

In this problem we introduce the two functions φ_1 and φ_2, which satisfy the system of equations:

$$y\frac{\partial \varphi_1}{\partial x} + \frac{\partial \varphi_2}{\partial y} = yq_1,\quad y\frac{\partial \varphi_1}{\partial y} - \frac{\partial \varphi_2}{\partial x} = q_2. \qquad \text{...(2.5)}$$

Through these equations we can connect the solutions of the elastic equations with the two functions φ_1 and φ_2, which satisfy the equations

$$L_1[\varphi_1] = 0, \; L_2[\varphi_2] = 0. \qquad \text{...(2.6)}$$

Also we can find the solution of (2.5) as

$$\varphi_1 = u + cxq_1 + \frac{2c-1}{2} q_2, \quad \varphi_2 = yv - \frac{2c-1}{2} y^2 q_1 + cxq_2.$$

Thus, if the function u and v satisfy the homogeneous elastic eqns. (2.2), the functions,

$$\left.\begin{aligned} \varphi_1 &= u + x\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{v}{y}\right) + \frac{2c-1}{2(1+c)} y \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) \\ \varphi_2 &= yv - \frac{2c-1}{2c} y^2 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{v}{y}\right) + \frac{c}{1+c} xy \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) \end{aligned}\right\} \qquad \text{...(2.7)}$$

satisfy eqns. (2.6).

Consider φ_1 and φ_2 subject to eqns. (2.5). From (2.7) we find the expression for the displacements in terms of the two functions φ_1 and φ_2, which satisfy eqns. (2.6), i.e.

$$\left.\begin{aligned} u &= \varphi_1 - cx\left(\frac{\partial \varphi_1}{\partial x} + \frac{1}{y}\frac{\partial \varphi_2}{\partial y}\right) + \frac{1-2c}{2}\left(y\frac{\partial \varphi_1}{\partial y} - \frac{\partial \varphi_2}{\partial x}\right) \\ v &= \frac{\varphi_2}{y} - \frac{1-2c}{c}\left(y\frac{\partial \varphi_1}{\partial x} + \frac{\partial \varphi_2}{\partial y}\right) - cx\left(\frac{\partial \varphi_1}{\partial y} - \frac{1}{y}\frac{\partial \varphi_2}{\partial x}\right). \end{aligned}\right\} \qquad \text{...(2.8)}$$

3. The Space Elastic Problem

In this case we can write the system of elastic equations in the form

$$\left.\begin{aligned} &\frac{\partial q_2}{\partial x} + \frac{\partial q_3}{\partial y} + \frac{\partial q_4}{\partial z} = 0, \quad \frac{\partial q_1}{\partial x} - \frac{\partial q_3}{\partial z} + \frac{\partial q_4}{\partial y} = 0 \\ &\frac{\partial q_1}{\partial y} + \frac{\partial q_2}{\partial z} - \frac{\partial q_4}{\partial x} = 0, \quad \frac{\partial q_1}{\partial z} - \frac{\partial q_2}{\partial y} + \frac{\partial q_3}{\partial x} = 0. \end{aligned}\right\} \qquad \text{...(3.1)}$$

Here the functions $q_i (i = 1, 2, 3, 4)$ are expressed through the displacements u, v and w by the formulae

$$\left.\begin{aligned} q_1 &= \frac{1+c}{c}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right), \quad q_2 = \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} \\ q_3 &= \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}, \quad q_4 = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}. \end{aligned}\right\} \qquad \text{...(3.2)}$$

From the system of eqns. (3.1), it follows, that the functions $q_i (i = 1, 2, 3, 4)$ are harmonic. We connect the elastic solutions with the four harmonic functions $\varphi_i (i = 1, 2, 3, 4)$ with the help of the equations

$$\left.\begin{aligned}&\frac{\partial\varphi_2}{\partial x}+\frac{\partial\varphi_3}{\partial y}+\frac{\partial\varphi_4}{\partial z}=-q_1,\ \frac{\partial\varphi_1}{\partial x}-\frac{\partial\varphi_3}{\partial z}+\frac{\partial\varphi_4}{\partial y}=q_2\\&\frac{\partial\varphi_1}{\partial y}+\frac{\partial\varphi_2}{\partial z}-\frac{\partial\varphi_4}{\partial x}=q_3,\ \frac{\partial\varphi_1}{\partial z}-\frac{\partial\varphi_2}{\partial y}+\frac{\partial\varphi_3}{\partial x}=q_4.\end{aligned}\right\}\qquad\ldots(3.3)$$

Immediately by inspection, it is possible to observe that the solutions of this system of equations are

$$\left.\begin{aligned}\varphi_1&=\frac{1}{3+4c}(xq_2+yq_3+zq_4)\\\varphi_2&=\frac{1}{3+4c}[-4(1+c)\,u-xq_1-zq_3+yq_4]\\\varphi_3&=\frac{1}{3+4c}[-4(1+c)\,v-yq_1+zq_2-xq_4]\\\varphi_4&=\frac{1}{3+4c}[-4(1+c)\,w-zq_1-yq_2+xq_3].\end{aligned}\right\}\qquad\ldots(3.4)$$

Since the functions $\varphi_i (i = 1, 2, 3, 4)$ satisfy eqns. (3.3), they must be harmonic.

If we take into consideration (3.3), and (3.4) we find the expressions for displacements

$$\left.\begin{aligned}4(1+c)\,u&=-(3+4c)\,\varphi_2+x\left(\frac{\partial\varphi_2}{\partial x}+\frac{\partial\varphi_3}{\partial y}+\frac{\partial\varphi_4}{\partial z}\right)\\&\quad-z\left(\frac{\partial\varphi_1}{\partial y}+\frac{\partial\varphi_2}{\partial z}-\frac{\partial\varphi_4}{\partial x}\right)+y\left(\frac{\partial\phi_1}{\partial z}-\frac{\partial\varphi_2}{\partial y}+\frac{\partial\varphi_3}{\partial x}\right)\\4(1+c)\,v&=-(3+4c)\,\varphi_3+y\left(\frac{\partial\varphi_2}{\partial x}+\frac{\partial\varphi_3}{\partial y}+\frac{\partial\varphi_4}{\partial z}\right)\\&\quad+z\left(\frac{\partial x_1}{\partial x}-\frac{\partial\varphi_3}{\partial z}+\frac{\partial\varphi_4}{\partial y}\right)-x\left(\frac{\partial\varphi_1}{\partial z}-\frac{\partial\varphi_2}{\partial y}+\frac{\partial\varphi_3}{\partial x}\right)\\4(1+c)\,w&=-(3+4c)\,\varphi_4+z\left(\frac{\partial\varphi_2}{\partial x}+\frac{\partial\varphi_3}{\partial y}+\frac{\partial\varphi_4}{\partial z}\right)\\&\quad-y\left(\frac{\partial\varphi_1}{\partial x}-\frac{\partial\varphi_3}{\partial z}+\frac{\partial\varphi_4}{\partial y}\right)+x\left(\frac{\partial\varphi_1}{\partial y}+\frac{\partial\varphi_2}{\partial z}-\frac{\partial\varphi_4}{\partial x}\right).\end{aligned}\right\}$$

...(3.5)

Noting that three only of the four functions $\varphi_i (i = 1, 2, 3, 4)$ are independent.

If we consider the functions φ_2, φ_3 and φ_4 to be known then, φ_1 can be determined from the first relation of (3.4)

$$(3 + 4c)\,\varphi_1 = x\left(\frac{\partial\varphi_1}{\partial x} - \frac{\partial\varphi_3}{\partial z} + \frac{\partial\varphi_4}{\partial y}\right) + y\left(\frac{\partial\varphi_1}{\partial y} + \frac{\partial\varphi_2}{\partial z} - \frac{\partial\varphi_4}{\partial x}\right)$$

$$+ z\left(\frac{\partial\varphi_1}{\partial z} - \frac{\partial\varphi_2}{\partial y} + \frac{\partial\varphi_3}{\partial x}\right). \qquad \ldots(3.6)$$

The solutions (3.5) give the representation for the analytical solutions of the space elastic problem. The harmonic functions φ_i are given by the formulae (3.4).

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