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STABILITY OF DIFFERENTIAL EQUATION OF NON-INTEGER ORDER THROUGH FIXED POINT IN THE LARGE

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Existence and stability for the solution of ordinary differential equation of non-integer order of the form

$$y^{(\alpha)}(x) = f(x, y),\ \alpha \in R,\ \alpha > 1$$

through the method of fixed point in the large has been discussed.

1. INTRODUCTION

Al-Abedeen and Arora (1978) proved a global existence and uniqueness theorem for the ordinary differential equation

$$y^{(\alpha)}(x) = f(x, y),\ \alpha \in R,\ 0 < \alpha \leqslant 1$$

satisfying the initial condition

$$y^{(\alpha-1)}(c) = y_0,\ y_0 \in E,\ a < c < b$$

where $f(x, y)$ is a continuous function from $(a, b) \times E$ into E, $a, b \in R$ and E is Euclidean space, by using Banach contraction principle as applied by Derrick and Janos (1976). Thus Picard's theorem was extended to ordinary differential equation of generalized order $\alpha \in R$, $0 < \alpha \leqslant 1$.

The purpose of this paper is to extend the Caratheodory existence theorem in the extended sense to ordinary differential equation

$$y^{(\alpha)}(x) = f(x, y),\ j - 1 < \alpha \leqslant j,\ j = 2, 3, \ldots, n \qquad \ldots(1.1)$$

satisfying the initial condition

$$y^{(\alpha-i)}(c) = c_i,\ c_i \in R,\ c_n = 0,\ i = 1, 2, \ldots, n \qquad \ldots(1.2)$$

and to discuss the stability of the solution of (1.1) through the method of fixed point in the large.

2. PRELIMINARIES

In this section, we set forth definitions and lemmas to be used in the main theorems. For reference see Barrett (1954).

Definition 2.1 — Let f be a function which is defined a.e. (almost everywhere) on $[a, b]$. For $\alpha > 0$, we define

$$\overset{b}{\underset{a}{I}}{}^{\alpha} f = \frac{1}{\Gamma(\alpha)} \int_a^b f(t)\,(b - t)^{\alpha-1}\,dt$$

provided that this integral (Lebesgue) exists, where Γ is the Gamma function.

Definition 2.2 — If $\alpha \leqslant 0$, and n is the smallest positive integer such that $\alpha + n > 0$, we define

$$\overset{b}{\underset{a}{I}}{}^{\alpha} f = D_x^n \overset{x}{\underset{a}{I}}{}^{\alpha+n} f, \text{ at } x = b$$

provided that $\overset{x}{\underset{a}{I}}{}^{\alpha+n} f$ and its first $(n - 1)$ derivatives exist in a segment $| b - x | < h$, and the nth derivative exists at $x = b$.

Lemma 2.1 — Let $\alpha, \beta \in R$, $\beta > -1$. If $x > a$, then

$$\overset{x}{\underset{a}{I}}{}^{\alpha} \frac{(t - a)^{\beta}}{\Gamma(\beta + 1)} = \begin{cases} \dfrac{(x - a)^{\alpha+\beta}}{\Gamma(\alpha + \beta + 1)}, & \alpha + \beta \neq \text{negative integer} \\ 0 & , \alpha + \beta = \text{negative integer.} \end{cases}$$

PROOF : The proof is straightforward.

Lemma 2.2 — (a) If $\alpha > 0$ and $f(x)$ is Lebesgue integrable on $[a, b]$, then

$$\overset{x}{\underset{a}{I}}{}^{-\alpha} \overset{t}{\underset{a}{I}}{}^{\alpha} f = f(x), \text{ a.e. on } a \leqslant x \leqslant b.$$

(b) If $\alpha \geqslant 1$ and $f(x)$ is Lebesgue integrable on $[a, b]$, then $\overset{x}{\underset{a}{I}}{}^{\alpha} f$ is absolutely continuous in $x \in [a, b]$.

PROOF : For the proof of (a) and (b) see Theorem 1.5 and Corollary 1.3.1 of Barrett (1954) respectively.

Lemma 2.3 — If $\alpha > 0$ and $f(x)$ is continuous on $[a, b]$, then $\overset{x}{\underset{a}{I}}{}^{\alpha} f$ exists.

PROOF : For proof see Barrett (1954, Corollary 1.2.1).

Lemma 2.4 — Let α, $M > 0$. If f is continuous and $| f | \leqslant M$ for all $x \in (a, b]$, then

$$\lim_{x \to a^+} \overset{x}{\underset{a}{I}}{}^{\alpha} f = 0.$$

PROOF : By Lemma 2.3, $\overset{x}{\underset{a}{I}}{}^{\alpha} f$ exists as a Lebesgue integral. The rest of the proof is straightforward.

Definition 2.3 — Under the assumption of Lemma 2.4, we define $\overset{a}{\underset{a}{I}}{}^{\alpha} f = 0$.

Definition 2.4 — If $\alpha \in R$, f is defined a.e. on $(a, b]$, we define $f^{(\alpha)}(x) = \overset{x}{\underset{a}{I}}{}^{-\alpha} f$, for all $x \in (a, b]$, provided $\overset{x}{\underset{a}{I}}{}^{-\alpha} f$ exists.

Definition 2.5 — A function $y = y(x)$ is said to be a solution for the differential equation

$$y^{(\alpha)}(x) = f(x, y), \alpha \in R, \alpha > 0 \quad \text{...(2.1)}$$

in the extended sense on the interval (a, b), $a, b \in R$ if

(i) y is absolutely continuous on (a, b) and

(ii) y satisfies eqn. (2.1) a.e. on (a, b).

This definition is given in Coddington and Levinson (1955) for $\alpha = 1$.

Definition 2.6 — A function $y(x)$ is said to be unique solution of eqns. (1.1) and (1.2) on the interval (a, b) provided that any other solution $g(x)$ differs from $y(x)$ only on a null sub-set of (a, b).

3. EXISTENCE THEOREM

Consider the initial value problem (P)

$$(P) \begin{cases} y^{(\alpha)}(x) = f(x, y), j - 1 < \alpha \leqslant j, j = 2, 3, ..., n & \text{...(3.1)} \\ y^{(\alpha - i)}(c) = c_i, c_i \in R, c_n = 0, i = 1, 2, ..., n, a < c < b & \text{...(3.2)} \end{cases}$$

where $f(x, y)$ is Lebesgue integrable function from $(a, b) \times E$ into E, $a, b \in R$ and E is Euclidean space. Let $f(x, y)$ satisfy the global Lipschitz condition

$$| f(x, y_2) - f(x, y_1) | \leqslant z(x) | y_2 - y_1 | \quad \text{...(3.3)}$$

for all $x \in (a, b)$, $y_1, y_2 \in E$ and some non-negative continuous function $z(x)$ defined on (a, b). Here $| ... |$ denotes the usual norm on E.

Let $\{I_n \mid n \geqslant 1\}$ be an increasing family of compact intervals which contain c, $a < c < b$ such that $\underset{n}{U} I_n = (a, b)$. Denote by $C(I_n)$ the Banach space of continuous functions $g : I_n \to E$ with norm

$$\| g \|_{(n,\lambda)} = \sup_{x \in I_n} \{\exp(-\lambda \mid \int_c^x (x-t)^{\alpha-1} z(t)\, dt \mid) \mid g(x) \mid \} \quad ...(3.4)$$

where λ is an arbitrary parameter. The Frechet space $C(a, b)$ may be topologized by the family of seminorms $\{\| g \|_{(n,\lambda)} : n \geqslant 1\}$. If $\lambda = 0$, the space $C(I_n)$ has the usual sup norm on I_n.

Theorem 3.1 — If the right-hand side $f(x, y)$ of the differential equation (3.1) satisfies the condition (3.3), then there exists a unique (in the sense of Definition 2.6) function $y(x)$ which is a solution of the differential equation (3.1) in the extended sense and satisfies eqn. (3.2).

PROOF : Let I be a compact sub-interval containing c of (a, b). Let the norm of $g \in C(I)$, a complete subspace of $C(a, b)$, be denoted by $\| g \|_\lambda$ and defined as in (3.4). Let the restriction of the operator T on $C(a, b)$ to $C(I)$ be denoted again by T and defined as

$$Tg(x) = \sum_{i=1}^{n-1} \frac{c_i(x-c)^{\alpha-i}}{\Gamma(\alpha-i+1)} + \frac{1}{\Gamma(\alpha)} \int_c^x (x-t)^{\alpha-1} f(t, g(t))\, dt. \quad ...(3.5)$$

As proved in Al-Abedeen and Arora (1978), T is contraction mapping, that is

$$\| Tg_2 - Tg_1 \|_\lambda \leqslant \frac{1}{\lambda\Gamma(\alpha)} \| g_2 - g_1 \|_\lambda \quad ...(3.6)$$

where $\lambda > 1/\Gamma(\alpha) > 0$. Hence T has one and only one fixed point. Thus we have

$$Ty(x) = y(x), \text{ for all } x \in (a, b).$$

From (3.5) and Lemma 2.2(b), it follows that $y(x)$ is absolutely continuous in $x \in (a, b)$.

Next we prove that $y(x)$ satisfies the differential equation (3.1) a.e. From (3.5) and Definition 2.1, we have

$$y(x) = \sum_{i=1}^{n-1} \frac{c_i(x-c)^{\alpha-i}}{\Gamma(\alpha-i+1)} + \overset{x}{\underset{c}{I^{\alpha}}} f. \quad ...(3.7)$$

From (3.7) we obtain

$$\overset{x}{\underset{c}{I^{-\alpha}}} y = \sum_{i=1}^{n-1} \overset{x}{\underset{c}{I^{-\alpha}}} \frac{c_i(t-c)^{\alpha-i}}{\Gamma(\alpha-i+1)} + \overset{x}{\underset{c}{I^{-\alpha}}} \overset{t}{\underset{c}{I^{\alpha}}} f. \quad ...(3.8)$$

Applying Lemmas 2.1 and 2.2(a) and Definition 2.4 to (3.8), we get

$$y^{(\alpha)}(x) = f(x, y), \text{ a.e. on } (a, b).$$

From (3.7) we obtain

$$\overset{x}{\underset{c}{I}}{}^{p-\alpha}\, y = \sum_{p=1}^{n-1} \overset{x}{\underset{c}{I}}{}^{i-\alpha} \frac{c_p(t-c)^{\alpha-p}}{\Gamma(\alpha-p+1)} + \overset{x}{\underset{c}{I}}{}^{i-\alpha} \overset{t}{\underset{c}{I}}{}^{\alpha} f. \qquad \ldots(3.9)$$

Using Lemmas 2.2(b) and 2.1, and Definitions 2.3 and 2.4 in (3.9), we obtain

$$y^{(\alpha-i)}(c) = c_i.$$

This completes the proof.

4. Stability Theorems

Theorem 4.1 — Let the right-hand side $f(x, y)$ of the differential equation (3.1) be such that

(i) it satisfies the condition (3.3),

(ii) it is square-integrable as a function of x, and

(iii) $| f(x, y) | \leqslant z(x) \mid y \mid$, for $y \in E$, where $z(x)$ is the same as in (3.3).

Then there exists a unique solution $y(x)$ of the differential equation (3.1) such that $\| y(x) \| \leqslant K$, K being a positive constant.

Proof : We define

$S = \{g : g \in C(I);\ \| g \|_\lambda \leqslant K\}$, where $C(I)$ is the same as in Theorem 3.1.

Let the mapping T on S be defined as

$$Tg(x) = \frac{1}{\Gamma(\alpha)} \int_c^x (x-t)^{\alpha-1} f(t, g(t))\, dt. \qquad \ldots(4.1)$$

Since $(x - t)^{\alpha-1} f(t, g)$ is Lebesgue integrable, Tg is absolutely continuous and hence continuous.

Next we claim T maps S into itself. It is easy to verify that the identity

$$\left| \int_c^x \exp\left(\lambda \left| \int_c^t (x-s)^{\alpha-1} z(s)\, ds \right| \right) (x-t)^{\alpha-1} z(t)\, dt \right|$$

$$= \frac{1}{\lambda} \left(\exp\left(\lambda \left| \int_c^x (x-t)^{\alpha-1} z(t)\, dt \right| \right) - 1 \right) \qquad \ldots(4.2)$$

is valid for every $x \in (a, b)$.

Now using the definition of $\| g \|_\lambda$ as in (3.4), the hypothesis (iii) and (4.2) we can easily show that

$$\| Tg \|_\lambda \leqslant \frac{\| g \|_\lambda}{\lambda\Gamma(\alpha)} \leqslant K.$$

Thus T maps S into itself.

As shown in the Theorem 3.1, T is a contraction mapping and its fixed point $y(x)$ is a solution of the differential equation (3.1). Hence $\| y(x) \| \leqslant K$, for $x \in (a, b)$.

This completes the proof.

Following Brydak (1970) we define stability as follows. The differential equation (3.1) is called stable if there is a $K > 0$ such that for every $g \in C(a, b)$ which satisfies

$$\| g - T^n g \| < \epsilon, \ (n = 1, 2, \ldots) \qquad \ldots(4.3)$$

where T is a mapping $T : C(a, b) \to C(a, b)$, there is a solution $y(x)$ of the differential equation (3.1) such that

$$\| g(x) - y(x) \| < K\epsilon \text{ a.e. on } (a, b). \qquad \ldots(4.4)$$

Here $\| \ldots \|$ is the $C(a, b)$ norm as defined in (3.4).

Theorem 4.2 — If the right-hand side $f(x, y)$ of the differential equation (3.1) satisfies the hypotheses of Theorem 4.1, then the differential equation is stable in the above sense.

PROOF: Let the mapping T on $C(a, b)$, where the Frechet space $C(a, b)$ being topologized by the family of semi-norms $\{\| g \|_{(n,\lambda)} \mid n \geqslant 1\}$, be defined as in (4.1). Consider an arbitrarily fixed element $g \in C(a, b)$ and set

$$Tg = g_1, \ T^2 g = g_2, \ \ldots, \ T^n g = g_n.$$

It is straightforward to prove that $\{g_n\}$ is a Cauchy sequence. Since the Frechet space is complete, there is an element $y(x)$ in $C(a, b)$, limit of this sequence, that is,

$$\operatorname*{limit}_{n} T^n g(x) = y(x). \qquad \ldots(4.5)$$

As shown in Theorem 4.1, T is a contraction mapping. From (4.5) and T being a contraction mapping and hence continuous it follows that

$$Ty(x) = y(x), \ x \in (a, b). \qquad \ldots(4.6)$$

Thus the limit point $y(x)$ of the sequence $\{T^n g\}$ turns out to be the (unique) fixed point of the mapping T. $y(x)$, as shown in Theorem 3.1, satisfies the differential equation (3.1) a.e.

We can easily obtain the classical error estimate in the nth approximation

$$\| T^n g - y \|_\lambda \leqslant \frac{1}{[(\lambda\Gamma(\alpha))^n - (\lambda\Gamma(\alpha))^{n-1}]} \| Tg - g \|_\lambda. \qquad ...(4.7)$$

From (4.7) we have

$$\| T^n g - y \| < M\epsilon, \text{ a.e.} \qquad ...(4.8)$$

[(4.8) holds a.e. because $y(x)$ is the limit of the sequence $\{T^n g\}$ as well as it satisfies the differential equation (3.1) a.e.].

Now from (4.3) and (4.8), we have

$$\| g(x) - y(x) \| < \epsilon K, \text{ a.e. on } (a, b), \text{ where } K = 1 + M > 0.$$

This completes the proof.

We make the following remark in conclusion:

We have proved Theorems 4.1 and 4.2 for $\alpha > 1$. These two theorems can be proved in the same way for α, $0 < \alpha \leqslant 1$ and in this case (4.4) and (4.8) hold for all $x \in (a, b)$.

References

Al-Abedeen, A. Z., and Arora, H. L. (1978). A global existence and uniqueness theorem for ordinary differential equation of generalized order. *Canad Math. Bull.*, **21**(3), 267–71.

Barrett, J. H. (1954). Differential equations of non-integer order. *Canad. J. Math.*, **6**, 529–41.

Brydak, D. (1970). On the stability of the linear functional equation $\phi[f(x)] = g(x)\phi(x) + F(x)$. *Proc. Am. math. Soc.*, **26**, 455–60.

Coddington, E. A., and Levinson, N. (1955). Theory of Ordinary Differential Equations. McGraw-Hill Book Company, Inc., New York, p. 42.

Derrick, W., and Janos, L. (1976). A global existence and uniqueness theorem for ordinary differential equations. *Canad. Math. Bull.*, **19**(1), 105–7.