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ON MIXED INTEGER SOLUTIONS TO GOAL PROGRAMMING PROBLEMS

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This paper provides a method for solving mixed integer goal programming problems with the help of cutting plane technique.

INTRODUCTION

One of the most promising techniques for multiple objective decision analysis is goal programming. Goal programming is a powerful tool which draws upon the highly developed and tested techniques of linear programming, but provides a simultaneous solution to a complex system of competing objectives. Goal programming can handle decision problems having a single goal with multiple subgoals (Lee 1972). The technique was originally introduced by Charnes and Cooper (1962), and further developed by Ijiri (1965) and Lee (1972).

In goal programming, instead of attempting to maximize or minimize the objective criterion directly, as in linear programming, the deviations between goals and what can be achieved within the given set of constraints are minimized. In the simplex algorithm of linear programming such deviations are called slack variables. These variables take on a new significance in goal programming. The deviational variable is represented in two dimensions, both positive and nagative deviations from each goal or sub goal. Then the objective function becomes the minimization of these deviations based on the relative importance or priority assigned to them.

This paper is a continuation of our paper (Sharma and Sharma 1979) where procedures are given for solving a goal programming problem with the condition that the variables are integers. In this paper, we shall study the case in which only a subset of the variables is restricted to consist of integers. The paper starts on the same lines as a paper by Gomory (1960), for the mixed integer problem.

The general goal programming (GP) model can be mathematically expressed as:

$$\text{Minimize } Z = \sum_{i=1}^{m} (d_i^+ + d_i^-)$$

$$\text{subject to} \quad Ax - Id^+ + Id^- = b$$

$$x, d^+, d^- \geqslant 0.$$

Some components of x are restricted to be integers. Here m goals are expressed by an m-component column vector b $(b_1, b_2, ..., b_m)$, A is an $m \times n$ matrix which expresses the relationship between goals and subgoals, x represents variables involved in the subgoals $(x_1, x_2, ..., x_n)$, d^+ and d^- are m-component vectors for the variable representing deviations from goals and I is an identity matrix in m-dimensions.

Theoretical Development

The method first treats the above mentioned mixed integer goal programming problem as an ordinary goal programming problem, that is, the additional condition that some of the variables are integers is not taken into consideration at the first instant. The ordinary goal programming problem is solved by the ordinary modified simplex method (Lee 1972). If all the basic variables restricted to be integers have integer values, the mixed-integer optimal solution is derived. If one or more basic variables restricted to be integers are not found to be an integer, we add a constraint (called a cut) for one of the components of x (say x_i) which does not give an integral value in the same way as Gomary has done for linear programme (Gomory 1960). If there are still some components of x which are not integers, more constraints are added one by one and the process is repeated on every addition until all x are integers.

The primary difference between the Gomory approach and the cutting plane goal programming method lies in the treatment of the multidimensional priority weights.

The Algorithm

Suppose, we obtained an optimal solution of the ordinary GP problem ignoring the integer restrictions. Let x_i be the basic variable which is restricted to take on only integer values. If the non-basic variables are denoted as y_j, $j = 1, 2, ..., n$, the basic variable x_i can be expressed as

$$x_i = b_i - \sum_{j=1}^{n} a_{ij}\, y_j. \qquad ...(1)$$

We can write

$$b_i = \hat{b}_i + \beta_i \qquad ...(2)$$

where $\hat{b}_i$ is the integer obtained by truncating the fractional part of b_i and β_i is the fractional part of b_i. By defining

$$a_{ij} = a_{ij}^{+} + a_{ij}^{-} \qquad ...(3)$$

where

$$a_{ij}^{+} = \begin{cases} a_{ij} \text{ if } a_{ij} \geqslant 0 \\ 0 \text{ if } a_{ij} < 0 \end{cases} \qquad \ldots(4)$$

and

$$a_{ij}^{-} = \begin{cases} 0 \text{ if } a_{ij} \geqslant 0 \\ a_{ij} \text{ if } a_{ij} < 0. \end{cases} \qquad \ldots(5)$$

Equation (1) can be written as

$$\sum_{j=1}^{n} (a_{ij}^{+} - a_{ij}^{-})\, y_j = \beta_i + (\hat{b}_i - x_i). \qquad \ldots(6)$$

Here, by assumption, x_i is restricted to integer values while b_i is not an integer. Since $0 < \beta_i < 1$, and $\hat{b}_i$ is an integer, we can have the value of $\beta_i + (\hat{b}_i - x_i)$ either $\geqslant 0$ or < 0. First, we consider the case where

$$\beta_i + (\hat{b}_i - x_i) \geqslant 0. \qquad \ldots(7)$$

In this case, in order for x_i to be an integer, we must have

$$\beta_i + (\hat{b}_i - x_i) = \beta_i \text{ or } \beta_i + 1 \text{ or } \beta_i + 2, \ldots . \qquad \ldots(8)$$

Thus eqn. (6) gives

$$\sum_{j=1}^{n} (a_{ij}^{+} + a_{ij}^{-})\, y_i \geqslant \beta_i. \qquad \ldots(9)$$

Since a_{ij} are non-positive and y_j are non-negative by definition, we have

$$\sum_{j=1}^{n} a_{ij}^{+} y_j \geqslant \sum_{j=1}^{n} (a_{ij}^{+} + a_{ij}^{-})\, y_j \qquad \ldots(10)$$

and hence

$$\sum_{j=1}^{n} a_{ij}^{+}\, y_j \geqslant \beta_i. \qquad \ldots(11)$$

Next we consider the case where

$$\beta_i + (\hat{b}_i - x_i) < 0 \qquad \ldots(12)$$

In order for x_i to be an integer, we must have (since $0 < \beta_i < 1$)

$$\beta_i + (\hat{b}_i - x_i) = -1 + \beta_i, \text{ or } -2 + \beta_i, \ldots . \qquad \ldots(13)$$

Thus eqn. (6) yields

$$\sum_{j=1}^{n} (a_{ij}^{+} + a_{ij}^{-})\, y_j \leqslant \beta_i - 1. \qquad \text{...(14)}$$

Since

$$\sum_{j=1}^{n} a_{ij}^{-} y_j \leqslant \sum_{j=1}^{n} (a_{ij}^{+} + a_{ij}^{-})\, y_j,$$

we obtain

$$\sum_{j=1}^{n} a_{ij}^{-} y_j \leqslant \beta_i - 1 \qquad \text{...(15)}$$

or

$$\frac{1}{\beta_i - 1} \sum_{j=1}^{n} a_{ij}^{-}\, y_j \geqslant 1 \qquad \text{...(16)}$$

[by dividing both sides by the negative quantity $(\beta_i - 1)$]. Multiplying both sides of (16) by $\beta_i > 0$, we can write the inequality (16) as

$$\frac{\beta_i}{\beta_i - 1} \sum_{j=1}^{n} a_{ij}^{-} y_j \geqslant \beta_i \qquad \text{...(17)}$$

Since one of the inequalities out of (11) and (17) must be satisfied, the following inequality must hold true:

$$\sum_{j=1}^{n} a_{ij}^{+}\, y_j + \left(\frac{\beta_i}{\beta_i - 1}\right) \sum_{j=1}^{n} a_{ij}^{-}\, y_j \geqslant \beta_i \qquad \text{...(18)}$$

or

$$-\sum_{j=1}^{n} a_{ij}^{+}\, y_j - \left(\frac{\beta_i}{\beta_i - 1}\right) \sum_{j=1}^{n} a_{ij}^{-}\, y_j \leqslant -\beta_i. \qquad \text{...(19)}$$

This is the required Gomory cut.

Dual simplex method is now applied. If there are still some other components of x which are restricted to be integers are not integers, more constraints are added one by one and the process is repeated on each addition until all basic variables restricted to be integers are integers.

Example

$$\text{Minimize } Z = P_1 d_3^+ + P_2 d_1^- + P_3 d_2^- + P_4 d_3^-$$

$$\text{subject to } 2x_1 + 3x_2 + d_1^- = 14$$

$$2x_1 + x_2 + d_2^- = 9$$

$$3x_1 + 2x_2 + d_3^- - d_3^+ = 14$$

x_2 is required to be an integer.

Step 1 — Solve the GP problem by modified simplex method by neglecting the integer requirement. This gives the following optimal tableau:

C_j					P_2	P_3	P_4	P_1
C_B	B	*rhs*	x_1	x_2	d_1^-	d_2^-	d_3^-	d_3^+
P_3	x_2	14/5	0	1	3/5	0	− 2/5	2/5
	d_2^-	11/15	0	0	1/5	1	− 4/5	4/5
	x_1	14/5	1	0	− 2/5	0	3/5	− 3/5
$Z_j - C_j$	P_4	0					− 1	
	P_3	11/15			1/5		− 4/5	4/5
	P_2	0			− 1			
	P_1	0						− 1

The optimal non-integer solution is $x_1 = 14/5$, $x_2 = 14/5$, $d_2^- = 11/15$, $P_1 = 0$ (under attainment of priority 1 goal), $P_2 = 0$, $P_3 = 11/15$, $P_4 = 0$. Since the model requires mixed integer solution, this solution is not acceptable. Therefore a 'cut' must be developed and a further solution procedure is required.

Step 2 : *Formulate a Gomory constraint* (*cut*) — Since x_2 is the only variable that is restricted to take integer values, we construct the Gomory constraint for x_2. From above table, we obtain

$$x_2 = b_2 - a_{21}y_1 - a_{22}y_2 - a_{23}y_3$$

where $b_2 = 2 + 4/5, a_{21} = + 3/5, a_{22} = -2/5, a_{23} = 2/5.$

According to (2), we write b_2 as $\hat{b}_2 + \beta_2$, where $\hat{b}_2 = 2$ and $\beta_2 = 4/5$. Similarly, we write from eqn. (3)

$$a_{21} = a_{21}^{+} + a_{21}^{-}$$

$$a_{22} = a_{22}^{+} + a_{22}^{-}$$

$$a_{23} = a_{23}^{+} + a_{23}^{-}$$

where

$$a_{21}^{-} = 0, a_{21}^{+} = +3/5$$

$$a_{22}^{-} = 3/5, a_{22}^{+} = 0$$

$$a_{23}^{+} = 2/5, a_{23}^{-} = 0.$$

The following Gomory cut (or constraint) can be obtained as [from eqn. (19)]

$$-3/5y_1 - 8/5y_2 - 2/5y_3 \leqslant -4/5.$$

The above constraint can be transformed to a goal constraint

$$-3/5y_1 - 8/5y_2 - 2/5y_3 + d_4^{-} = -4/5.$$

When this constraint is added to the above table, we obtain the following:

C_j					P_2	P_3	P_4	P_1	
C_B	B	*rhs*	x_1	x_2	d_1^-	d_2^-	d_3^-	d_3^+	d_4^-
P_3	x_2	14/5	0	1	3/5	0	− 2/5	+ 2/5	0
	d_2^-	11/15	0	0	1/5	1	− 4/5	4/5	0
	x_1	14/5	1	0	− 2/5	0	3/5	− 3/5	0
	d_4^-	− 4/5	0	0	− 3/5	0	− 8/5	− 2/5	1
$Z_j - C_j$	P_4	0					− 1		
	P_3	11/15			1/5		− 4/5	4/5	
	P_2	0			− 1				
	P_1	0						− 1	

Step 3 : *Apply dual modified simplex method to find a new optimum solution* — Since − 4/5 is the only negative b_i term, the pivot operation has to be done in d_4^- row.

C_j					P_2	P_3	P_4	P_1	
C_B	B	rhs	x_1	x_2	d_1^-	d_2^-	d_3^-	d_3^+	d_4^-
P_3	x_2	3	0	1	3/4	0	0	1/2	− 1/4
	d_2^-	17/15	0	0	1/2	1	0	1	− 1/2
P_4	x_1	5/2	1	0	− 5/8	0	0	− 3/4	3/8
	d_3^-	1/2	0	0	3/8	0	1	1/4	− 5/8

This tableau gives the desired mixed-integer solution as $x_1 = 5/2$, $x_2 = 3$, $d_2^- = 17/15$, $d_3^- = 1/2$, $P_1 = 0$, $P_2 = 0$, $P_3 = 17/15$, $P_4 = 1/2$.

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