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CONTIGUOUS RELATIONS FOR THE *H*-FUNCTION OF *n* VARIABLES

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Recently, using the technique of Buschman (1972, p. 41), and Buschman and Gupta (1975, p. 1419), Tandon (1979) developed a basic set of recurrence relations, with constant coefficients, for the *H*-function of *n* variables which has been defined and studied in a series of recent papers by Srivastava and Panda (1975, 1976a, b).

In the present paper, the author intends establishing the same basic set by employing a different technique used by Srivastava (1966).

Also an $(n+2)$-term recurrence relation for Lauricella's hypergeometric function $F_A^{(n)}$ of n variables has been deduced as a particular case. This relation is believed to be new.

1. INTRODUCTION

Srivastava and Panda [1976a, p. 271, eqn. (4.1)] introduced the *H*-function of *n* variables by means of the multiple Mellin-Barnes type contour integral in the following manner (with slight change of parameters):

$$H^{0,\lambda:(\mu',\nu');\cdots;(\mu^{(n)},\nu^{(n)})}_{A,B:[C',D'];\cdots;[C^{(n)},D^{(n)}]}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]:[(c'):\phi'];\ldots;[(c^{(n)}):\phi^{(n)}];\\ [(b):\epsilon',\ldots,\epsilon^{(n)}]:[(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\; z_1,\ldots,z_n\right)$$

$$= \frac{1}{(2\pi w)^n}\int_{\mathcal{L}_1}\cdots\int_{\mathcal{L}_n} U_1(s_1)\ldots U_n(s_n)\, V(s_1,\ldots,s_n)\, z_1^{s_1}\ldots z_n^{s_n}\, ds_1\ldots ds_n,$$

$$w=\sqrt{(-1)} \qquad \ldots(1.1)$$

where

$$V(s_1,\ldots,s_n)=\frac{\prod_{j=1}^{\lambda}\Gamma(1-a_j+\sum_{i=1}^{n}\theta_j^{(i)}s_i)}{\prod_{j=\lambda+1}^{A}\Gamma(a_j-\sum_{i=1}^{n}\theta_j^{(i)}s_i)\prod_{j=1}^{B}\Gamma(1-b_j+\sum_{i=1}^{n}\epsilon_j^{(i)}s_i)} \qquad \ldots(1.2)$$

and
$$U_i(s_i) = \frac{\prod_{j=1}^{\mu^{(i)}} \Gamma(d_j^{(i)} - \delta_j^{(i)} s_i) \prod_{j=1}^{\nu^{(i)}} \Gamma(1 - c_j^{(i)} + \phi_j^{(i)} s_i)}{\prod_{j=\mu^{(i)}+1}^{D^{(i)}} \Gamma(1 - d_j^{(i)} + \delta_j^{(i)} s_i) \prod_{j=\nu^{(i)}+1}^{C^{(i)}} \Gamma(c_j^{(i)} - \phi_j^{(i)} s_i)}$$
$$(i = 1, \ldots, n). \qquad \ldots(1.3)$$

For a detailed account of the conditions under which the definition (1.1) makes sense, see the papers by Srivastava and Panda (1975, 1976a, b).

Here we assume, following Srivastava and Panda (1976a), that

(a) stands for the sequence of parameters $a_1, \ldots, a_A$;

(b) for $b_1, \ldots, b_B$; $(c^{(i)})$ for $c_1^{(i)}, \ldots, c_{C^{(i)}}^{(i)}$;

$(d^{(i)})$ for $d_1^{(i)}, \ldots, d_{D^{(i)}}^{(i)}$, etc., $i = 1, \ldots, n$, it being understood, for example, that $c^{(1)} = c'$, $c^{(2)} = c''$, and so on.

2. Notations

We shall simply write H if the parameters are as in (1.1), $H[c_1^{(1)} - 1]$ if $c_1^{(1)}$ is replaced by $c_1^{(1)} - 1$ and other parameters remain the same as in (1.1), $H[a_1 - 1]$ if a_1 is replaced by $a_1 - 1$ and other parameters remain unaltered as in (1.1), and so on.

Also, for the sake of notational convenience,

$$F_A^{(n)}(\alpha, \beta_1, \ldots, \beta_n; \gamma_1, \ldots, \gamma_n; -z_1, \ldots, -z_n),$$

$$F_A^{(n)}(\alpha + 1, \beta_1, \ldots, \beta_n; \gamma_1, \ldots, \gamma_n; -z_1, \ldots, -z_n),$$

$$F_A^{(n)}(\alpha, \beta_1 + 1, \ldots, \beta_n; \gamma_1, \ldots, \gamma_n; -z_1, \ldots, -z_n)$$

and determinant

$$\begin{vmatrix} c_1^{(i)} - k & d_{D^{(i)}}^{(i)} \\ \phi_1^{(i)} & \delta_{D^{(i)}}^{(i)} \end{vmatrix}$$

will be represented by

$$F_A^{(n)}, F_A^{(n)}(\alpha+), F_A^{(n)}(\beta_1+) \text{ and } \Delta(c_1^{(i)} - k, d_{D^{(i)}}^{(i)})$$

respectively, and so on.

3. The Basic Recurrences

$$\phi^{(i)}_{C^{(i)}}\, H\,[d^{(i)}_1 + 1] - \delta^{(i)}_1\, H\,[c^{(i)}_{C^{(i)}} - 1] = \Delta(d^{(i)}_1, c^{(i)}_{C^{(i)}} - 1)\, H, \quad i = 1, \ldots, n. \qquad \ldots(3.1)$$

$$\phi^{(i)}_{C^{(i)}}\, H\,[c^{(i)}_1 - 1] + \phi^{(i)}_1\, H\,[c^{(i)}_{C^{(i)}} - 1] = -\,\Delta(c^{(i)}_1 - 1, c^{(i)}_{C^{(i)}} - 1)\, H, \quad i = 1, \ldots, n. \qquad \ldots(3.2)$$

$$\delta^{(i)}_{D^{(i)}}\, H\,[c^{(i)}_1 - 1] - \phi^{(i)}_1\, H\,[d^{(i)}_{D^{(i)}} + 1] = -\,\Delta(c^{(i)}_1 - 1, d^{(i)}_{D^{(i)}})\, H, \quad i = 1, \ldots, n. \qquad \ldots(3.3)$$

$$M_0 H\,[a_1 - 1] + M_1 H\,[c^{(1)}_1 - 1] + \ldots + M_n H\,[c^{(n)}_1 - 1] = PH, \qquad \ldots(3.4)$$

where

$$M_0 = 1,\ M_1 = -\frac{\theta^{(1)}_1}{\phi^{(1)}_1}, \ldots, M_n = -\frac{\theta^{(n)}_1}{\phi^{(n)}_1}$$

and

$$P = (1 - a_1) + \frac{\theta^{(1)}_1}{\phi^{(1)}_1}(c^{(1)}_1 - 1) + \ldots + \frac{\theta^{(n)}_1}{\phi^{(n)}_1}(c^{(n)}_1 - 1).$$

$$M_0 H\,[a_A - 1] + M_1 H\,[c^{(1)}_1 - 1] + \ldots + M_n H\,[c^{(n)}_1 - 1] = PH \qquad \ldots(3.5)$$

where

$$M_0 = 1,\ M_1 = \frac{\theta^{(1)}_A}{\phi^{(1)}_1}, \ldots, M_n = \frac{\theta^{(n)}_A}{\phi^{(n)}_1}$$

and

$$P = (a_A - 1) + \frac{\theta^{(1)}_A}{\phi^{(1)}_1}(1 - c^{(1)}_1) + \ldots + \frac{\theta^{(n)}_A}{\phi^{(n)}_1}(1 - c^{(n)}_1).$$

$$M_0 H\,[b_1 + 1] + M_1 H\,[c^{(1)}_1 - 1] + \ldots + M_n H\,[c^{(n)}_1 - 1] = PH \qquad \ldots(3.6)$$

where

$$M_0 = 1,\ M_1 = -\frac{\epsilon^{(1)}_1}{\phi^{(1)}_1}, \ldots, M_n = -\frac{\epsilon^{(n)}_1}{\phi^{(n)}_1}$$

and

$$P = -b_1 + \frac{\epsilon_1^{(1)}}{\phi_1^{(1)}} (c_1^{(1)} - 1) + \dots + \frac{\epsilon_1^{(n)}}{\phi_1^{(n)}} (c_1^{(n)} - 1).$$

4. Proofs

We can easily derive that

$$H[c_1^{(i)} - 1] = (1 - c_1^{(i)}) H + \phi_1^{(i)} z_i \frac{\partial H}{\partial z_i}, \quad i = 1, \dots, n. \qquad \dots(4.1)$$

$$H[c_{C(i)}^{(i)} - 1] = (c_{C(i)}^{(i)} - 1) H - \phi_{C(i)}^{(i)} z_i \frac{\partial H}{\partial z_i}, \quad i = 1, \dots, n. \qquad \dots(4.2)$$

$$H[d_1^{(i)} + 1] = d_1^{(i)} H - \delta_1^{(i)} z_i \frac{\partial H}{\partial z_i}, \quad i = 1, \dots, n. \qquad \dots(4.3)$$

$$H[d_{D(i)}^{(i)} + 1] = - d_{D(i)}^{(i)} H + \delta_{D(i)}^{(i)} z_i \frac{\partial H}{\partial z_i}, \quad i = 1, \dots, n. \qquad \dots(4.4)$$

$$H[a_1 - 1] = (1 - a_1) H + \sum_{i=1}^{n} \theta_1^{(i)} z_i \frac{\partial H}{\partial z_i} \qquad \dots(4.5)$$

$$H[a_A - 1] = (a_A - 1) H - \sum_{i=1}^{n} \theta_A^{(i)} z_i \frac{\partial H}{\partial z_i} \qquad \dots(4.6)$$

$$H[b_1 + 1] = - b_1 H + \sum_{i=1}^{n} \epsilon_1^{(i)} z_i \frac{\partial H}{\partial z_i}. \qquad \dots(4.7)$$

Eliminating $z_i \frac{\partial H}{\partial z_i}$ between (4.2) and (4.3); (4.1) and (4.2); (4.1) and (4.4); (4.1) and (4.5); (4.1) and (4.6); (4.1) and (4.7), the desired recurrences (3.1) – (3.6) will follow respectively.

We assert here that the $3n$ recurrences (3.1) – (3.3) together with 3 recurrences (3.4) – (3.6), form a basic set of recurrences for the H-function of n variables.

5. Particular Cases

If we choose all θ's, ϵ's, δ's and ϕ's to be 1 of the H-functions involved in (3.4) – (3.6), the corresponding $(n + 2)$-term recurrences for Meijer's G-function of n variables will follow.

And, further, by suitably specializing the parameters of the *G*-functions involved in the recurrence thus obtained from (3.4) and applying the obvious relation (see, for example, Srivastava and Panda 1976a, p. 272, eqn. (4.7)):

$$G^{0,1:(1,1);\cdots;(1,1)}_{1,0:[1,2];\cdots;[1,2]}\left[\begin{matrix}1-\alpha: & 1-\beta_1 \;;\ldots;\; 1-\beta_n \;; \\ \text{———}: & 0,\, 1-\gamma_1;\ldots;\, 0,\, 1-\gamma_n;\end{matrix}\; z_1,\ldots,z_n\right]$$

$$= \frac{\Gamma(\alpha)\,\Gamma(\beta_1)\ldots\Gamma(\beta_n)}{\Gamma(\gamma_1)\ldots\Gamma(\gamma_n)} \times F_A^{(n)}(\alpha, \beta_1, \ldots, \beta_n; \gamma_1, \ldots, \gamma_n; -z_1, \ldots, -z_n) \qquad \ldots(5.1)$$

we get, after a little simplification, the corresponding $(n+2)$-term recurrence relation for $F_A^{(n)}$, the Lauricella's hypergeometric function of n variables:

$$-\alpha F_A^{(n)}(\alpha+) + \beta_1 F_A^{(n)}(\beta_1+) + \ldots + \beta_n F_A^{(n)}(\beta_n+)$$

$$= (-\alpha + \beta_1 + \ldots + \beta_n)\, F_A^{(n)}. \qquad \ldots(5.2)$$

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