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PROXIMATE ORDER OF AN ENTIRE FUNCTION WITH INDEX PAIR (p, q)

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Proximate order and lower proximate order of an entire function with index pair (p, q) are defined and their existence established. Some properties of proximate order are also derived.

§1. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n} \qquad \text{...(1.1)}$$

be an entire function with $0 \leqslant \lambda_1 < \lambda_2 < \ldots < \lambda_n \to \infty$ as $n \to \infty$. Set

$$M(r) = M(r, f) \equiv \max_{|z|=r} |f(z)|,$$

$M(r)$ is called the maximum modulus of $f(z)$ for $|z| = r$.

The concept of (p, q)-order and lower (p, q)-order of $f(z)$ having an index pair (p, q), $(p \geqslant 1, q \geqslant 1$ and $p \geqslant q)$, was introduced by Juneja, *et al.* (1976). Thus $f(z)$ is said to be of (p, q)-order ρ and lower (p, q)-order λ, if

$$\lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log^{[p]} M(r)}{\log^{[q]} r} = \begin{matrix}\rho(p, q) \equiv \rho \\ \lambda(p, q) \equiv \lambda\end{matrix} \qquad \text{...(1.2)}$$

where $\log^{[p]} x$ stands for the pth iterate of $\log x$.

In the present paper, we define a proximate order of an entire function with index pair (p, q) and prove its existence. Further, we derive some of its properties. Our results generalize those of Levin (1968) on proximate order of an entire function for which the index pair is (2, 1).

§2. *Definition* — A real function $\rho(r)$ defined on $(0, \infty)$ is said to be a proximate order of an entire function with index pair (p, q) if it satisfies the following properties:

$$\lim_{r\to\infty} \rho(r) = \rho \qquad \text{...(2.1)}$$

$$\lim_{r\to\infty} \Lambda_{[q]}(r)\, \rho'(r) = 0 \qquad \text{...(2.2)}$$

where $\quad \Lambda_{[q]}(r) = \log^{[q]} r \log^{[q-1]} r \ldots \log r \,.\, r.$

Theorem 1 — For every function $g(r)$ $(= \log^{[p-1]} M(r))$ positive for $r > r_0$ and such that

$$\rho(p, q) = \limsup_{r\to\infty} \frac{\log g(r)}{\log^{[q]} r} < \infty \qquad \ldots(2.3)$$

there exists a proximate order $\rho(r)$ such that for all positive values of $r > r_0$

$$g(r) \leqslant (\log^{[q-1]} r)^{\rho(r)} \qquad \ldots(2.4)$$

and

$$g(r_n) = (\log^{[q-1]} r_n)^{\rho(r_n)} \qquad \ldots(2.5)$$

for some sequence $\{r_n\}_1^{\infty}$, $r_n \to \infty$.

PROOF : Consider the function

$$\alpha(r) = (\log^{[q-1]} r)^{-\rho} g(r).$$

It can be easily proved that

$$\limsup_{r\to\infty} \frac{\log \alpha(r)}{\log^{[q]} r} = 0.$$

Putting $x = \log^{[q]} r$ and $y = \log \alpha(r)$ gives us $y = \alpha_1(x)$ where

$$\alpha_1(x) = y = \log \alpha(r) = \log \alpha\, (\exp^{[q]} x)$$

or,

$$\alpha_1(x) = \log \alpha(\exp^{[q]} x).$$

So

$$\limsup_{x\to\infty} \frac{\alpha_1(x)}{x} = \limsup_{x\to\infty} \frac{\log \alpha(\exp^{[q]} x)}{x}$$

$$= \limsup_{r\to\infty} \frac{\log \alpha(r)}{\log^{[q]} r} = 0$$

which shows that for arbitrary $\epsilon > 0$ and for all values of x $(x \geqslant x_0(\epsilon))$ the entire curve $y = \alpha_1(x)$ lies below the line $y = \epsilon x$ and that on the other side there are points on the curve with arbitrarily large abscissas lying above the line $y = -\epsilon x$.

To prove the theorem we take first the case

$$\limsup_{x\to\infty} \alpha_1(x) = +\infty$$

and then we consider the general case.

Supposing that

$$\limsup_{x\to\infty} \alpha_1(x) = +\infty$$

we construct the smallest convex domain in such a fashion that it contains the positive ray of the x-axis and all the points of the curve $y = \alpha_1(x)$. The boundary of the domain (thus formed) lying above the x-axis is a continuous curve $y = \beta(x)$.

This curve possesses the following properties:

(I) The curve is convex from the above,

(II) $\lim_{x \to \infty} \frac{\beta(x)}{x} = 0,$

(III) $\alpha_1(x) \leqslant \beta(x),$

(IV) At the extreme points of the curve $y = \beta(x)$, $\alpha_1(x) = \beta(x)$,

(V) The curve $y = \beta(x)$ contains a sequence of extreme points tending to infinity.

In the neighbourhood of each angular point (if necessary) the curve $y = \beta(x)$ is made differentiable by making some inessential changes. It is thus assumed that the curve $y = \beta(x)$ is everywhere differentiable. Then (I) and (II) give that

$$\lim_{x \to \infty} \beta'(x) = 0 \qquad \text{...(2.6)}$$

and the inequality (III) is changed into

$$g(r) \leqslant (\log^{[q-1]} r)^{\rho(r)} \qquad \text{...(2.7)}$$

where

$$\rho(r) = \rho + \frac{\beta(\log^{[q]} r)}{\log^{[q]} r}.$$

Now, from (II) it follows that

$$\lim_{r \to \infty} \rho(r) = \lim_{r \to \infty} \left\{\rho + \frac{\beta(\log^{[q]} r)}{\log^{[q]} r}\right\} = \rho.$$

Properties (IV) and (V) help us to deduce that there exists a sequence of values r_n tending to infinity for which equality holds in (2.7) and

$$\lim_{r \to \infty} \Lambda_{[q]}(r)\, \rho'(r) = 0.$$

We have thus constructed the function $\rho(r)$ subject to

$$\limsup_{x \to \infty} \alpha_1(x) = \infty.$$

In order to generalize the case we construct a concave function $y = \beta_1(x)$ satisfying

$$\text{(I)} \quad \lim_{x\to\infty} \beta_1'(x) = 0, \quad \text{(II)} \quad \lim_{x\to\infty} \frac{\beta_1(x)}{x} = 0$$

and (III) $\limsup_{x\to\infty} [\alpha_1(x) + \beta_1(x)] = \infty.$

To construct the curve $y = \beta_1(x)$ we go through the following steps:

Produce a segment a_1 of the line

$$y = -\epsilon_1 x$$

from the origin to the point x_1 at which

$$\alpha_1(x_1) > -\epsilon_1 x_1 + 1.$$

Having chosen a positive number $\epsilon_2 < \epsilon_1$ we draw a segment a_2 of the line

$$y + \epsilon_1 x_1 = -\epsilon_2(x - x_1)$$

from the point $(x_1, -\epsilon_1 x_1)$ to a point $x_2 > x_1$ satisfying

$$\alpha_1(x_2) > -\epsilon_1 x_1 - \epsilon_2(x_2 - x_1) + 2.$$

Pass a segment a_3 with the slope $-\epsilon_3\,(0 < \epsilon_3 < \epsilon_2)$ etc. The sequence $\{\epsilon_n\}$ so constructed is strictly decreasing with $\epsilon_n \to 0$ but the sequence $\{x_n\}$ of the points is strictly increasing and x_n tends to infinity. The polygonal function $y = \hat{\beta}_1(x)$ constructed in the above manner satisfies the condition

$$\lim_{x\to+\infty} \frac{\hat{\beta}_1(x)}{x} = 0.$$

The function $\hat{\beta}_1(x)$ can be made everywhere differentiable by changing it in an inessential manner in the neighbourhood of each angular point. If the function $\beta_1(x)$ is defined as $\beta_1(x) = -\hat{\beta}_1(x)$, the function $y = \beta_1(x)$ has the required properties.

A convex majorant $\beta_2(x)$ for the function $\alpha_1(x) + \beta_1(x)$ is now constructed and then assuming

$$\beta(x) = \beta_2(x) - \beta_1(x),$$

yields

$$\alpha_1(x) \leqslant \beta(x).$$

Moreover,

$$\alpha_1(x_n') = \beta(x_n').$$

on some sequence $\{x_n'\}_1^\infty$ of extreme points, $x_n' \to \infty$. Also, it can be easily seen that

$$\lim_{x\to\infty} \beta'(x) = 0 \quad \text{and} \quad \lim_{x\to\infty} \frac{\beta(x)}{x} = 0. \qquad \ldots(2.8)$$

If the function $\rho(r)$ is defined as

$$\rho(r) = \rho + \frac{\beta(\log^{[q]} r)}{\log^{[q]} r},$$

we find that the properties (2.1) and (2.2) are easily satisfied in view of (2.8). $\rho(r)$ is therefore a proximate order. Furthermore

$$g(r) \leqslant (\log^{[q-1]} r)^{\rho(r)}$$

and

$$g(r_n) = (\log^{[q-1]} r_n)^{\rho(r_n)}$$

for some sequence $\{r_n\}$, $r_n \to \infty$.

The proof of the theorem is thus complete.

§3. A positive function $\phi(r)$ will be called slowly increasing if

$$\lim_{r\to\infty} \frac{\phi(jr)}{\phi(r)} = 1$$

uniformly on each interval $0 < b \leqslant j < m < \infty$.

Theorem 2 — If $\rho(r)$ is a proximate order, then the function $(\log^{[q-1]} r)^{\rho(r)-\rho}$ is slowly increasing.

Proof : If

$$\phi(r) = (\log^{[q-1]} r)^{\rho(r)-\rho}$$

then

$$\log \frac{\phi(jr)}{\phi(r)} = (j-1)\, r\rho'(j'r) \log^{[q]} r + o(1),\ j < j' < 1$$

$$\leqslant (j-1)\, r \frac{\eta}{\Lambda_{[q]}(jr)} \log^{[q]} r + o(1),\ \eta > 0.$$

Thus, for $0 < j < 1$,

$$\lim_{r\to\infty} \phi(jr)/\phi(r) = 1$$

the case $j > 1$ can be easily treated similarly.

Theorem 3 — The function $(\log^{[q-1]} r)^{\rho(r)}$ is monotone increasing for all sufficiently large values of r.

PROOF :

$$\frac{d}{dr}\{\log^{[q-1]} r\}^{\rho(r)} = \{\log^{[q-1]} r\}^{\rho(r)} \left(\rho'(r) \log^{[q]} r + \frac{\rho(r)}{\Lambda_{[q-1]}(r)}\right).$$

Using (2.2) we have asymptotically

$$> \{\rho(r) - \epsilon\} \frac{(\log^{[q-1]} r)^{\rho(r)}}{\Lambda_{[q-1]}(r)}$$

$$> 0, \; 0 < \epsilon < \rho.$$

Theorem 4 — For $0 < l \leqslant k \leqslant m < \infty$ and $r \to \infty$, the asymptotic inequality

$$(1 - \epsilon)(\log^{[q-1]} r)^{\rho(r)-\rho} \{\log^{[q-1]}(kr)\}^{\rho} < \{\log^{[q-1]}(kr)\}^{\rho(kr)} < (1 + \epsilon)\{\log^{[q-1]}(kr)\}^{\rho}(\log^{[q-1]} r)^{\rho(r)-\rho}$$

holds uniformly in k.

From the definition of slowly increasing function and Theorem 2, the inequality follows atonce. Hence we omit the proof.

Theorem 5 — For $1 + \rho > \gamma$,

$$\int_{r_0}^{r} (\log^{[q-1]} t)^{\rho(t)-\gamma} \frac{dt}{\Lambda_{[q-2]}(t)} = \frac{1}{\rho - \gamma + 1}(\log^{[q-1]} r)^{\rho(r)-\gamma+1} + o(\log^{[q-1]} r)^{\rho(r)-\gamma+1}.$$

PROOF : From the definition of the proximate order we have asymptotically

$$|\rho(r) - \rho| < \epsilon/2 \quad \text{and} \quad \rho'(r)\Lambda_{[q]}(r) < \epsilon/2. \qquad \ldots(3.1)$$

Integrating by parts we have

$$\int_{r_0}^{r} (\log^{[q-1]} t)^{\rho(t)-\gamma} \cdot \frac{dt}{\Lambda_{[q-2]}(t)} = \left[\frac{1}{\rho - \gamma + 1}(\log^{[q-1]})^{\rho(t)-\gamma+1}\Big|_{r_0}^{r} - \int_{r_0}^{r} \frac{(\log^{[q-1]} t)^{\rho(t)-\gamma+1}}{\rho - \gamma + 1}\left\{\rho'(t)\log^{[q]} t + \frac{\rho(t) - \rho}{\Lambda_{[q-1]}(t)}\right\} dt\right].$$

Using (3.1) the required inequality follows.

For $\rho + 1 < \gamma$ the formula

$$\int_{r}^{\infty} (\log^{[q-1]} t)^{\rho(t)-\gamma} \frac{dt}{\Lambda_{[q-2]}(t)} = \frac{1}{\gamma - \rho - 1}(\log^{[q-1]} r)^{\rho(r)-\gamma+1} + o(\log^{[q-1]} r)^{\rho(r)-\gamma+1}$$

can be similarly obtained where $\Lambda_{[-1]}(r) = 1$.

§4. The lower proximate order can be defined as following:

Definition — A function $\lambda(r)$ defined on $(0, \infty)$ is said to be a lower proximate order of an entire function with index pair (p, q) if it possesses the following properties:

$$\lim_{r\to\infty} \lambda(r) = \lambda \quad \text{and} \quad \lim_{r\to\infty} \Lambda_{[q]}(r)\,\lambda'(r) = 0$$

Theorem 6 — For every function $g(r)$ $(= \log^{[p-1]} M(r))$ positive for $r > r_0$ and such that

$$\lambda(p, q) = \liminf_{r\to\infty} \frac{\log g(r)}{\log^{[q]}(r)} > 0$$

there exists a lower proximate order $\lambda(r)$ such that for all positive values of r

$$g(r) \geqslant (\log^{[q-1]} r)^{\lambda(r)}$$

and

$$g(r_n) = (\log^{[q-1]} r_n)^{\lambda(r_n)}$$

for some sequence $\{r_n\}_1^\infty$, $r_n \to \infty$.

The Theorem can be proved on similar lines as those of the proof of Theorem 1. Hence we omit the proof.

References

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