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ON THE DIFFERENTIAL AND INTEGRAL EQUATIONS ASSOCIATED WITH THE MODIFIED EXPONENTIAL-COSINE OPERATOR FUNCTION

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Let X be a Banach space and $B(X)$ be the space of bounded linear operators on X. A family $\{S(t) : t \in R^+\}$, $S : R^+ \to B(X)$, is called a modified exponential-cosine operator function if $S(t + s) + T(2s) S(t - s) = 2 S(t) S(s)$, $t, s \in R^+, s \leqslant t$, where $\{T(t) : t \in R^+\}$, $T : R^+ \to B(X)$, is a semigroup of operators. In this paper, this function is studied in the strong operator topology. The first and second infinitesimal generators A and B of $\{S(t)\}$ are defined by $Af = \lim_{h\to 0} (S(h)f - f)/h$, $f \in D(A)$; and $Bf = \lim_{h \to 0} 2(S(h)f - T(h)f)/h^2$, $f \in D(B)$, respectively. It is proved that, under a very general condition, A coincides with the first generator of $\{T(t)\}$. Also it is established that B is a closed linear operator with $D(B)$ dense in X. The differential and integral equations associated with $\{S(t)\}$ are obtained as

$$d^2 S(t)f/dt^2 = 2A(d\, S(t) f/dt) + (B - A^2) S(t) f,$$

and

$$B \int_0^t (t - u) S(u)f\, du = S(t)f - f + tAf + A^2 \int_0^t (t - u) S(u) f\, du$$

$$- 2A \int_0^t S(u) f\, du,\ f \in D(B) \cap D(A^2).$$

Finally some examples of modified exponential-cosine operator function are studied.

1. INTRODUCTION

Let X be a Banach space and let $B(X)$ denote the family of bounded linear operators on X. Let $R^+ = [0, \infty)$ and let I denote the identity operator. A one parameter family of operators $\{T(t);\ t \in R^+\}$, $T : R^+ \to B(X)$, is said to be a 'semigroup of operators' on X if $T(0) = I$, and $T(s + t) = T(s)\, T(t)$, $s, t \in R^+$. If $T(t)f$ is continuous at $t = 0$, for any $f \in X$, it is called a semigroup of class (C_0), (cf. Hille and Phillips 1957). The semigroups of operators have been extensively studied and the reader is referred to the treatise of Hille and Phillips 1957.

Sova introduced the 'cosine operator function', which is defined to be the family $\{C(t),\ t \in R^+\}$, $C : R^+ \to B(X)$, with $C(0) = I$ and which satisfies the functional equation

$$C(t+s) + C(t-s) = 2\, C(t)\, C(s),\ t, s \in R^+,\ t \geqslant s \text{ [ref. Sova (1968)].}$$

As a common generalization of operator semigroups and cosine operators, Buche (1975) introduced the 'exponential-cosine operator function' $\{S(t), t \in R^+\}$, $S : R^+ \to B(X)$, $S(0) = I$, defined by

$$S(t+s) - 2\, S(t)\, S(s) = (S(2s) - 2\, S^2(s))\, S(t-s),\ s, t \in R^+,\ s \leqslant t.$$

Buche studied some boundedness and continuity properties of the exponential-cosine operator function, obtained the differential equation associated with it in the uniform operator topology, and proved, that under some conditions, $S(t) = T(t)\, C(t)$, where $\{T(t), t \in R^+\}$, $T: R^+ \to B(X)$, is a semigroup of operators and $\{C(t), t \in R^+\}$, $C : R^+ \to B(X)$, is a cosine operator function.

In this paper we investigate some properties of a family of operators, which is a modified version of the above equation. The 'modified exponential-cosine operator function' was introduced by Singh and Buche (1979) as a one parameter family of operators $\{S(t), t \in R^+\}$, $S: R^+ \to B(X)$, such that

$$S(t+s) + S(t-s)\, T(2s) = 2\, S(t)\, S(s),\ s, t \in R^+,\ s \leqslant t \qquad ...(1)$$

where $\{T(t); t \in R^+\}$, $T: R^+ \to B(X)$, is a known (C_0)-semigroup of operators. It may be observed that the cosine operator function becomes a particular case of the modified exponential-cosine operator function when $T(t) = I$, for all $t \in R^+$. In this paper we shall study the solution of the equation (4) in the strong operator topology. It was proved by Singh and Buche (1979) that the continuity of $\{S(t)\}$ at the origin implies continuity everywhere. It was also established that for a continuous family of operators $\{S(t)\}$ satisfying (1), there exist constants $M > 0$ and ω such that

$$\| S(t) \| \leqslant M \exp(\omega\, t),\ t \in R^+. \qquad ...(2)$$

In this paper we shall throughout assume that $S(t_1)\, T(t_2) = T(t_2)\, S(t_1)$ and $S(t_1)\, S(t_2) = S(t_2)\, S(t_1)$, for all $t_1, t_2 \in R^+$.

In section 2 we discuss the properties of the two infinitesimal generators of the modified exponential-cosine operator. The first infinitesimal generator A of $\{S(t)\}$ satisfying (1) is defined by

$$Af = \lim_{h \to 0} ((S(h) - I)/h) f,\ f \in D(A),\ h > 0 \qquad ...(3)$$

where $D(A) \subset X$ and $D(A)$ is the set of elements $f \in X$, for which the limit exists. under a general condition A coincides with the infinitesimal generator of the associated semigroup $\{T(t)\}$.

The second infinitesimal generator B of $\{S(t)\}$ satisfying (1) is defined by $Bf = \lim_{h \to 0} (2/h^2)(S(h) - T(h)) f$, $f \in D(B)$, $h > 0$, where $D(B) \subset X$, and $D(B)$ is the set of elements $f \in X$ for which the limit exists. A and B happen to be closed linear operators with domains dense in X. We also obtain the second order differential equation involving A and B.

In section 3, we discuss the representation theorem for the modified exponential-cosine operator function. In a subsequent paper, we shall deal with the generation theorems for the modified exponential-cosine operator function.

2. The Infinitesimal Generators and the Associated Differential and Integral Equation

Let $\{S(t);\ t \in R^+\}$, $S : R^+ \to B(X)$, be a modified exponential-cosine operator function, with $\{T(t)\}$ as the associated (C_0)-semigroup.

Let

$$A_h = (S(h) - I)/h, \quad h > 0 \qquad \text{...(4)}$$

$$B_h = 2\,(S(h) - T(h))/h^2, \quad h > 0. \qquad \text{...(5)}$$

The 'first infinitesimal generator' A of $\{S(t)\}$ is defined as

$$Af = \lim_{h \to 0} A_h f,\ f \in D(A) \qquad \text{...(6)}$$

where $D(A) \subset X$, and $D(A)$ is the set of elements $f \in X$, for which the limit (6) exists. Clearly $D(A)$ is a linear sub-space of X, and A is a linear operator.

The 'second infinitesimal generator' B of $\{S(t)\}$ is defined by

$$Bf = \lim_{h \to 0} B_h f,\ f \in D(B) \qquad \text{...(7)}$$

where $D(B) \subset X$, and $D(B)$ is the set of elements $f \in X$, for which the above limit exists. $D(B)$ is a linear sub-space of X and B is a linear operator.

Let A' be the infinitesimal generator of the known associated semigroup $\{T(t)\}$ of operators and is defined as

$$A'f = \lim_{h \to 0} (1/h)\,(T(h) - I)f,\ f \in D(A') \qquad \text{...(8)}$$

$D(A') \subset X$, and $D(A')$ is the set of elements $f \in X$ for which the above limit exists, (ref. Hille and Phillips 1957).

The following proposition immediately follows by using (2) and assumption of commutativity of $S(t_1)$ and $S(t_2)$.

Proposition 1 — Let $\{S(t);\ t \in R^+\}$ be a regular modified exponential-cosine operator function. Then for each $f \in D(A)$, $S(t)f \in D(A)$ and $AS(t)f = S(t)\,Af$. Also if $f \in D(B)$, then $S(t)f \in D(B)$ and $B\,S(t)\,f = S(t)\,Bf$.

Proposition 2 — Let $\{S(t), t \in R^+\}$ be a regular modified exponential-cosine operator function. If $f \in D(A)$, then $f \in D(A')$ and $A'f = Af$. Conversely, if $f \in D(A')$, and $\lim_{h \to 0} (1/h)(S(h) - T(h))f = 0$ for $f \in D(A')$, then $f \in D(A)$ and $Af = A'f$.

PROOF: From eqn. (1), we have for $h > 0$, $T(2h) = 2S^2(h) - S(2h)$.

So for, $f \in X$, $h > 0$,

$$\begin{aligned}((T(2h) - I)/2h)f &= (1/2h)(2\,S^2(h) - S(2h) - I)f \\ &= -((S(2h) - I)/2h)f + (2/2h)(S^2(h) - I)f \\ &= -((S(2h) - I)/2h)f + (1/h)(S^2(h) - I)f \\ &= -((S(2h) - I)/2h)f + (S(h) + I)((S(h) - I)/h)f\end{aligned}$$

For $f \in D(A)$, the right-hand limit exists as $h \to 0$, because of (2). So the left-hand limit as $h \to 0$ also exists, which implies that $f \in D(A')$. Taking the actual limit as $h \to 0$ in the above expression, we see that $A'f = -Af + 2Af$, or $A'f = Af$, $f \in D(A)$.

Conversely let $(1/h)(S(h) - T(h))f \to 0$ as $h \to 0$, $f \in D(A')$. Write for $h > 0$, $f \in D(A')$,

$$\begin{aligned}((S(2h) - I)/2h)f &= (2/2h)(S^2(h) - T^2(h))f + ((T(2h) - I)/2h)f \\ &= (S(h) + T(h)(1/h)(S(h) - T(h))f \\ &\quad + ((T(2h) - I)/2h)f.\end{aligned}$$

The right-hand side converges to $A'f$ as $h \to 0$. Therefore the left-hand limit as $h \to 0$ also exists. Hence $f \in D(A)$ and $Af = A'f$.

Remark: Henceforth we shall assume that $A = A'$.

Lemma 1 — Let $\{S(t), t \in R^+\}$, $S : R^+ \to B(X)$, be a regular modified exponential-cosine operator function. Then for $0 < h < t$,

$$(1/h^2)\left[\int_t^{t+h} (t - u)\,S(u)f\,du - \int_{t-h}^{t} (t - u)\,S(u)f\,du\right] \to -S(t)f,$$

as $h \to 0$, for every $f \in X$.

PROOF: Since evidently

$$(1/h^2)\left[\int_t^{t+h} (u - t)\,du + \int_{t-h}^{t} (t - u)\,du\right] = 1,$$

we may for $f \in X$, write

$$\| (1/h^2) [\int_t^{t+h} (t - u) S(u) f \, du - \int_{t-h}^{t} (t - u) S(u) f \, du] + S(t) f \|$$

$$= \| (1/h^2) [\int_t^{t+h} (u - t) (S(t) f - S(u) f) \, du$$

$$- \int_{t-h}^{t} (t - u) (S(u) f - S(t) f] \, du \|$$

$$\leqslant \sup_{t-h \leqslant u \leqslant t+h} \| S(u) f - S(t) f \|.$$

Now as $h \to 0$, the right-hand side tends to zero because of the regularity of $\{S(t)\}$. This proves the Lemma.

The following Lemma follows immediately from the regularity of $\{S(t)\}$.

Lemma 2 — Let $\{S(t); t \in R^+\}$, $S : R^+ \to B(X)$, be a regular modified exponential-cosine operator function. Then for any $f \in X$, and $t \in R^+$,

(i) $(1/h) \int_0^h S(u) f \, du \to f$, as $h \to 0$

(ii) $(1/h) \int_{t-h}^{t+h} S(u) f \, du \to 2 S(t) f$, as $h \to 0$

(iii) $(2/h^2) \int_0^h u \, S(u) f \, du \to f$, as $h \to 0$

(iv) $(2/h^2) \int_0^h (h - u) S(u) f \, du \to f$, as $h \to 0$.

PROOF : On the same lines as in the Lemma 1.

Lemma 3 — Let $\{S(t); t \in R^+\}$, $S : R^+ \to B(X)$, be a regular modified exponential-cosine operator function, satisfying the equation (1). Let $f \in D(A)$. Then, for $0 < h < t$,

$$B_h \int_0^h (t - u) S(u) f \, du \to 0, \text{ as } h \to 0.$$

PROOF : By using (2),

$$\| B_h \int_0^h (t - u) S(u) f \, du \|$$

$$\leqslant \int_0^h (t - u) \| S(u) \| \| B_h f \| \, du \leqslant M \exp (\omega h) \| B_h f \| \int_0^h (t - u) \, du$$

$$= 2\,M \exp(\omega h)\,(t - h/2)\,(\|\,(1/h)\,(S(h) - T(h))\,f\,\|).$$

Since $f \in D(A)$, so $(1/h)\,(S(h) - T(h))\,f \to 0$, as $h \to 0$, which proves the Lemma.

Lemma 4 — Let $\{S(t);\, t \in R^+\}$, $S : R^+ \to B(X)$, be a regular modified exponential-cosine operator function. Then, for every $f \in D(A)$,

(i) $(2/h^2) \int_0^h (S(2u)f - S(u)f)\, du \to Af$, as $h \to 0$

(ii) $(2/h^2) \int_0^h (S(u)f - f)\, du \to Af$, as $h \to 0$.

PROOF : (i) Because $f \in D(A)$, so $(1/v)\,(S(v)f - f) \to Af$, as $v \to 0$. Therefore, for any $\epsilon > 0$, there exists a $\delta > 0$, such that

$$\|\,(1/v)\,(S(v)f - f) - Af\,\| < \epsilon/3,\ 0 < v < \delta$$

or

$$\|\,S(v)f - f - vAf\,\| < (\epsilon/3)\,v,\ 0 < v < \delta. \qquad \ldots(9)$$

Now for $0 < h < \delta/2$, $f \in D(A)$,

$$(2/h^2) \int_0^h (S(2u)f - S(u)f)\, du - Af$$

$$= (2/h^2) \int_0^h (S(2u)f - S(u)f - u\,Af)\, du$$

$$= (2/h^2) \int_0^h (S(2u)f - f - 2u\,Af)\, du$$

$$- (2/h^2) \int_0^h (S(u)f - f - u\,Af)\, du.$$

By using (9), we get for $0 < h < \delta/2$,

$$\|\,(2/h^2) \int_0^h (S(2u)f - S(u)f)\, du - Af\,\| < 2(\epsilon/3) + (\epsilon/3) = \epsilon.$$

Hence

$$(2/h^2) \int_0^h (S(2u)f - S(u)f)\, du \to Af, \text{ as } h \to 0,\ f \in D(A).$$

(ii) For $f \in D(A)$, $h > 0$,

$$(2/h^2) \int_0^h (S(u)f - f)\, du - Af = (2/h^2) \int_0^h (S(u)f - f - u\,Af)\, du.$$

By using (9), we get for $0 < h < \delta$,

$$\| 2/h^2 \int_0^h (S(u)f - f)\, du - Af \| \leqslant (2/h^2)(\epsilon)(h^2/2) = \epsilon$$

which proves (ii).

Lemma 5 — Let $\{S(t);\, t \in R^+\}$, $S : R^+ \to B(X)$, be a regular modified exponential-cosine operator function satisfying eqn. (1). Then for each $f \in D(A^2)$,

$$B_h \int_0^t (t-u)\, S(u) f\, du \to S(t)f - f + t\,Af + A^2 \int_0^t (t-u)\, S(u) f\, du$$
$$- 2A \int_0^t S(u) f\, du, \text{ as } h \to 0,\ 0 < h < t.$$

Proof : By the Lemma 3, for $f \in D(A)$,

$$\lim_{h\to 0} B_h \int_0^t (t-u)\, S(u) f\, du = \lim_{h\to 0} B_h \int_h^t (t-u)\, S(u) f\, du. \quad ...(10)$$

Now
$$B_h \int_h^t (t-u)\, S(u) f\, du = (2/h^2)(S(h) - T(h)) \int_h^t (t-u)\, S(u) f\, du$$
$$= (2/h^2) \int_h^t (t-u)(S(u)\, S(h) f - T(h)\, S(u) f)\, du.$$

Using eqn. (1), we have

$$B_h \int_h^t (t-u)\, S(u) f\, du = (2/h^2) \int_h^t (t-u)(((S(u+h) f$$
$$+ T(2h)\, S(u-h) f)/2) - T(h)\, S(u) f)\, du$$
$$= (1/h^2) \int_h^t (t-u)(S(u+h) - 2S(u) + S(u-h)) f\, du$$
$$+ (1/h^2) \int_h^t (t-u)((T(2h) - I)\, S(u-h) f$$
$$- 2(T(h) - I)\, S(u) f)\, du$$
$$= (1/h^2) \int_h^t (t-u)(S(u+h) - 2\, S(u) + S(u-h)) f\, du$$
$$+ (1/h^2) \int_h^t (t-u)(T(2h) - 2\, T(h) + I)\, S(u) f\, du$$
$$- (1/h^2) \int_h^t (t-u)(T(2h) - I)(S(u) - S(u-h)) f\, du. \quad ...(11)$$

It immediately follows that for $f \in D(A^2)$,

$$\lim_{h\to 0} (1/h^2) \int_h^t (t-u)\,(T(2h) - 2\,T(h) + I)\,S(u) f\,du$$

$$= A^2 \int_0^t (t-u)\,S(u) f\,du, \qquad \ldots(12)$$

because by using (2), it follows that

$$(1/h^2) \int_0^h (t-u)\,(T(2h) - 2\,T(h) + I)\,S(u) f\,du \to 0, \text{ as } h \to 0,$$

for $f \in D(A^2)$.

Now $$(1/h^2) \int_h^t (t-u)\,(T(2h) - I)\,(S(u) - S(u-h)) f\,du$$

$$= 2((T(2h) - I)/2h)\,(1/h) \int_h^t (t-u)\,S(u) f\,du$$

$$- (1/h) \int_0^{t-h} (t-u-h)\,S(u) f\,du)$$

$$= 2((T(2h) - I)/2h)\,((1/h) \int_0^t (t-u)\,S(u) f\,du$$

$$- (1/h) \int_0^t (t-u)\,S(u) f\,du - (1/h) \int_0^t (t-u-h)\,S(u) f\,du$$

$$+ (1/h) \int_{t-h}^t (t-u-h)\,S(u) f\,du)$$

$$= 2((T(2h) - I)/2h)\,[\int_0^t S(u) f\,du - (t/h) \int_0^h S(u) f\,du + (1/h)$$

$$\times \int_0^h u\,S(u) f\,du - (1/h) \int_{t-h}^t (t-u)\,S(u) f\,du - \int_{t-h}^t S(u) f\,du].$$

By the Lemma 2, $(1/h) \int_0^h S(u) f\,du \to f$, as $h \to 0$. By using (2) it is easy to see that as $h \to 0$,

$$(1/h) \int_0^h u\,S(u) f\,du \to 0, \int_{t-h}^t S(u) f\,du \to 0$$

and

$$(1/h) \int_{t-h}^t (t-u)\,S(u) f\,du \to 0.$$

Hence for $f \in D(A)$,

$$\lim_{h\to 0} (1/h^2) \int_h^t (t-u)\,(T(2h)-I)\,(S(u)-S(u-h)) f\,du$$

$$= 2\,A \int_0^t S(u) f\,du - 2\,t\,A f. \qquad \ldots(13)$$

Now for $f \in X$,

$$(1/h^2) \int_h^t (t-u)\,(S(u+h) - 2\,S(u) + S(u-h))\,f\,du$$

$$= (1/h^2)\,(\int_{2h}^{t+h} (t-u+h)\,S(u) f\,du - 2 \int_h^t (t-u)\,S(u) f\,du$$

$$+ \int_0^{t-h} (t-u-h)\,S(u) f\,du) = (1/h^2)\,[\int_{2h}^t (t-u)\,S(u) f\,du$$

$$+ \int_t^{t+h} (t-u)\,S(u) f\,du - 2 \int_h^{2h} (t-u)\,S(u) f\,du$$

$$- 2 \int_{2h}^t (t-u)\,S(u) f\,du + \int_0^h (t-u)\,S(u) f\,du$$

$$+ \int_h^{2h} (t-u)\,S(u) f\,du + \int_{2h}^{t-h} (t-u)\,S(u) f\,du]$$

$$+ (1/h)\,[\int_{2h}^{t+h} S(u) f\,du - \int_0^{t-h} S(u) f\,du]$$

$$= (1/h^2)\,(\int_t^{t+h} (t-u)\,S(u) f\,du - \int_{t-h}^t (t-u)\,S(u) f\,du)$$

$$+ (1/h^2)\,(\int_0^h (t-u)\,S(u) f\,du - \int_h^{2h} (t-u)\,S(u) f\,du)$$

$$+ (1/h)\,(\int_{t-h}^{t+h} S(u) f\,du - \int_0^{2h} S(u) f\,du)$$

$$= (1/h^2)\,[\int_t^{t+h} (t-u)\,S(u) f\,du - \int_{t-h}^t (t-u)\,S(u) f\,du]$$

$$- 2t(1/h^2) \int_0^h (S(2u) - S(u)) f\,du - (2/h^2) \int_0^h u\,S(u) f\,du$$

$$+ (1/h^2) \int_0^{2h} u S(u) f\,du + (1/h) \int_{t-h}^{t+h} S(u) f\,du - (1/h) \int_0^{2h} S(u) f\,du.$$

Applying the Lemmas 1, 2 and 4(ii), we see that as $h \to 0$, $f \in D(A)$,

$$(1/h^2) \int_h^t (t-u)(S(u+h) - 2S(u) + S(u-h)) f\,du \to$$

$$- S(t)f - tAf - f + 2f + 2S(t)f - 2f = S(t)f - f - tAf. \qquad \ldots(14)$$

Combining (11), (12), (13), (14), and (10), we get for $f \in D(A^2)$,

$$\lim_{h \to 0} B_h \int_0^t (t-u)\, S(u) f\,du$$

$$= S(t)f - f + tAf + A^2 \int_0^t (t-u)\, S(u) f\,du - 2A \int_0^t S(u) f\,du,$$

which proves the Lemma.

The following two Corollaries immediately follow from the above Lemma.

Corollary 1 — Let $\{S(t);\, t \in R^+\}$, $S : R^+ \to B(X)$, be a regular modified exponential-cosine operator function. Then for each $f \in D(A^2)$, $t > 0$, $\int_0^t (t-u)\, S(u) f\,du \in D(B)$, and

$$B \int_0^t (t-u)\, S(u) f\,du = S(t)f - f + tAf + A^2 \int_0^t (t-u)\, S(u) f\,du$$

$$- 2A \int_0^t S(u) f\,du.$$

Corollary 2 — Let $\{S(t);\, t \in R^+\}$, $S : R^+ \to B(X)$, be a regular modified exponential-cosine operator function. Then for each $f \in D(B) \cap D(A^2)$,

$$S(t)f = f - tAf + 2A \int_0^t S(u) f\,du + \int_0^t (t-u)\, S(u)\,(B - A^2) f\,du.$$

Theorem 1 — Let $\{S(t);\, t \in R^+\}$ be a regular modified exponential-cosine operator function. Then $D(B)$ is dense in X.

PROOF : First of all, we shall prove that $D(B)$ is dense in $D(A^2)$. Once this is established, then by the theory of semigroups (cf. Hille and Phillips 1957), $D(A^2)$ is further dense in X, since A is the infinitesimal generator of $\{T(t)\}$. So it will imply that $D(B)$ is dense in X.

Let $f \in D(A^2)$, by the Corollary 1, $\int_0^t (t-u)\, S(u) f\,du \in D(B)$, for every $t > 0$. By the Lemma 2(iv), $(2/t^2) \int_0^t (t-u)\, S(u) f\,du \to f$, as $t \to 0$. Hence for every

$f \in D(A^2)$, there exists a sequence of elements in $D(B)$, which converges to f. Hence $D(B)$ is dense in $D(A^2)$, which, by the argument already given, proves that $D(B)$ is dense in X.

Theorem 2 — Let $\{S(t); t \in R^+\}$, $S : R^+ \to B(X)$, be a regular modified exponential-cosine operator function. Then B is a closed operator.

PROOF : If $B = A^2$, there is nothing to prove. So we assume that $B \neq A^2$. Let $f_k \in D(B)$, $f_k \to f_0$, $B f_k \to g_0$. For $f_k \in D(A^2)$ and $f_0 \in D(A^2)$,

$$S(t) f_0 - T(t) f_0 = \lim_{k \to \infty} ((S(t) f_k - f_k) - (T(t) f_k - f_k)),$$

by the regularity of $\{S(t)\}$ and $\{T(t)\}$.

Therefore

$$S(t) f_0 - T(t) f_0 = \lim_{k \to \infty} ((\int_0^t (t - u) S(u) B f_k \, du - t A f_k$$

$$- A^2 \int_0^t (t - u) S(u) f_k \, du + 2 A \int_0^t S(u) f_k \, du$$

$$- T(t) f_k + f_k), \text{ by Lemma 5.}$$

Therefore

$$S(t) f_0 - T(t) f_0 = \int_0^t (t - u) S(u) g_0 \, du - t A f_0$$

$$- A^2 \int_0^t (t - u) S(u) f_0 \, du + 2 A \int_0^t S(u) f_0 \, du$$

$$- T(t) f_0 + f_0,$$

since A and A^2 are closed operators.

Therefore

$$\lim_{t \to 0} (2/t^2) (S(t) f_0 - T(t) f_0)$$

$$= \lim_{t \to 0} (2/t^2) (\int_0^t (t - u) S(u) g_0 \, du$$

$$- A^2 \lim_{t \to 0} ((2/t^2) \int_0^t (t - u) S(u) f_0 \, du$$

$$+ \lim_{t \to 0} (2/t^2) (2 A \int_0^t (S(u) f_0 - f_0) \, du - \lim_{t \to 0} (2/t^2) (T(t) f_0$$

$$- f_0 - t A f_0) = g_0 - A^2 f_0 + 2 A^2 f_0 - A^2 f_0,$$

by the Lemmas 2(iv) and 4(ii), and the fact that

$$\lim_{t\to 0} (2/t^2)(T(t)f_0 - f_0 - tAf_0) = A^2 f_0,\ f_0 \in D(A^2).$$

Therefore $\lim_{t\to 0} (2/t^2)(S(t)f_0 - T(t)f_0) = g_0$. So $f_0 \in D(B)$ and $Bf_0 = g_0$. This proves that B restricted to $D(B) \cap D(A^2)$ is closed. Now $\overline{D(B)} \supset D(A^2)$ and A^2 is a closed operator, A being the infinitesimal generator of $T(t)$.

Also for any sequence f_k, $f_k \in D(B)$, such that $f_k \to f_0$, $Bf_k \to g_0$, we can find for each $n = 1, 2, 3, \ldots$, a sequence $\{f_{kn}\}$, $k = 1, 2, \ldots$, such that $f_{kn} \in D(B) \cap D(A^2)$, $f_{kn} \to f_k$, as $n \to \infty$. Then for the diagonal sequence $\{f_{nn}\}$, we have

$$\lim_{n\to\infty} f_{nn} = \lim_{k\to\infty} f_k = f_0,\ \lim_{n\to\infty} Bf_{nn} = \lim_{k\to\infty} Bf_k = g_0.$$

But $f_{nn} \in D(B) \cap D(A^2)$ and B is closed operator on $D(B) \cap D(A^2)$. Hence $f_0 \in D(B)$ and $Bf_0 = g_0$. This proves that B is a closed operator.

The following Theorem which follow from the Corollary 2 without much difficulty, gives the associated differential integral equations for the modified exponential-cosine operator function.

Theorem 3 — Let $\{S(t);\, t \in R^+\}$ $S : R^+ \to B(X)$, be a regular modified exponential-cosine operator function satisfying (2). Then for every $f \in D(B) \cap D(A^2)$,

(i) $S(\cdot)f$ is twice differentiable,

(ii) $dS(t)f/dt = -Af + 2AS(t)f + \int_0^t S(u)(B - A^2)f\,du$,

which converges to Af, as $t \to 0$, $t \in R^+ - \{0\}$,

(iii) $d^2S(t)f/dt^2 = 2A(dS(t)f/dt) + (B - A^2)S(t)f$,

which converges to $(B + A^2)f$, as $t \to 0$, $t \in R^+ - \{0\}$.

3. The Representation Theorem

It is known that for a semigroup $\{T(t),\, t \in R^+\}$ of operators, the following representation holds in the strong operator topology:

$$T(t)f = \lim_{h\to 0} \exp(tA_h)f,\ f \in X,$$

where $A_h = (T(h) - I)/h$ (ref. Hille and Phillips 1957).

Such a representation for the modified exponential-cosine operator function is still an open problem. However for the mixed semigroup, which is a special case of the modified exponential-cosine operator function, the representation can be directly obtained from the representation of the semigroup.

Example 1 — Let $X = C(R)$, the class of bounded uniformly continuous real-valued functions on the real line, with sup norm. Let $\{T(t)\}$ be the translation semigroup, which is defined as

$$T(t)\, f(x) = f(x + at) \qquad \ldots(15)$$

where $a > 0$ is some constant, $f \in C(R)$ and $x \in R$ (cf. Hille and Phillips 1957).

Let us define the family of bounded linear operators $\{S(t) : t \in R^+\}$ on X as follows

$$S(t) f(x) = (1/2)\,(f(x + (a - b)\, t) + f(x + (a - b)\, t)) \qquad \ldots(16)$$

where b is some constant $0 < b < a$. It can be checked that $\{S(t) : t \in R^+\}$ is a regular modified exponential-cosine operator function with $\{T(t)\}$ defined by (15) as the associated semigroup. The first and the second infinitesimal generators A and B of (16) are easily found to be

$$A = a(d/dx),$$

$$B = b^2(d^2/dx^2).$$

Hence the partial differential equation associated with (16) given by the Theorem 3 is

$$(\partial^2 u(x, t)/\partial t^2) - 2a(\partial^2 u(x, t)/\partial x\, \partial t) - (b^2 - a^2)\,(\partial^2 u(x, t)/\partial x^2) = 0 \qquad \ldots(17)$$

where $u(x, 0) = f(x)$, $(\partial u(x, t)/\partial t)_{t=0} = af'(x)$.

Here it is assumed that f has an absolutely continuous derivative. The equation (17) is the differential equation associated with Shock waves in one dimensional gas flow (ref. Landau and Lifshitz 1959).

Example 2 — Let X be the same Banach space as in Example 1. Let $\{T(t) : t \in R^+\}$ be the C_0-semigroup defined as follows. For each $f \in X$, $x \in R$,

$$T(t)\, f(x) = \exp\,(-\alpha t)\, f(x),\ \alpha > 0. \qquad \ldots(18)$$

Let the one parameter family of operators $\{S(t) : t \in R^+\}$ be defined as

$$S(t) f(x) = \exp\,(-\alpha t) \sum_{k=0}^{\infty} (\alpha t)^{2k}/(2k)!\, f(x - 2k\mu) \qquad \ldots(19)$$

where $\mu > 0$ is some constant (cf. Buche 1975). It can be easily checked that $\{S(t) : t \in R^+\}$ satisfies eqn. (1) with (18) as the associated semigroup. The first and the second infinitesimal generators of $\{S(t)\}$ defined by (19) are given by

$$Af(x) = -\alpha\, f(x)$$

and

$$Bf(x) = \alpha^2 f(x - 2\mu).$$

By the Theorem 3, the associated differential equation is given by

$$d^2u(x, t)/dt^2 + 2\alpha(du(x, t)/dt) = \alpha^2(u(x - 2\mu, t) - u(x, t)),$$

where $u(x, 0) = f(x)$, $(du(x, t)/dt)_{t=0} = -\alpha f(x)$.

Example 3 : Mixed Semigroup (ref. Singh 1978) — Let X be a Banach space and $B(X)$ the space of bounded linear operators on X. Let $\{T_1(t) : t \in R^+\}$, $T_1 : R^+ \to B(X)$, and $\{T_2(t) : t \in R^+\}$, $T_2 : R^+ \to B(X)$ be two (C_0)-semigroups of operators. Let $\{T_1(t)\}$ and $\{T_2(t)\}$ commute and satisfy Trotter's (1959) condition so that

$$T(t) = T_1(t/2)\, T_2(t/2),\ t \in R^+ \qquad ...(20)$$

is a (C_0)-semigroup. The family $\{S(t) : t \in R^+\}$, $S : R^+ \to B(X)$, defined by

$$S(t) = (T_1((t) + T_2(t))/2,\ t \in R^+ \qquad ...(21)$$

is a modified exponential-cosine operator function with $\{T(t)\}$ defined by (20) as the associated (C_0)-semigroup. The first and the second infinitesimal generators of this $\{S(t)\}$ come out to be

$$A = \tfrac{1}{2}(A_1 + A_2) \text{ and } B = \tfrac{1}{4}(A_2 - A_1)^2.$$

If B does not generate a cosine operator function, then $\{S(t)\}$ defined by (21) can not be written in the form $\{T(t)\, C(t)\}$, where $\{C(t)\}$ is a cosine operator function.

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