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ON THE GENERATION THEOREMS FOR THE MODIFIED EXPONENTIAL-COSINE OPERATOR

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Let X be a Banach space, and let $B(X)$ be the space of bounded linear operators on X. Let $R^+ = [0, \infty)$. The family $\{S(t); t \in R^+\}$, $S : R^+ \to B(X)$, is called a modified exponential-cosine operator, if it satisfies the equation

$$S(t + s) + S(t - s)\, T(2s) = 2S(t)\, S(s),\ s, t \in R^+, s \leqslant t, S(0) = I,$$

where $\{T(t); t \in R^+\}$, $T : R^+ \to B(X)$, is a known (C_0)-semigroup of operator. In this paper the generation theorems for the modified exponential-cosine operator function have been obtained. These results are analogues of the Hille-Yosida generation theorems for the semigroup of operators and Sova generation theorems for cosine operators.

1. INTRODUCTION

Let X be a Banach space, and let $B(X)$ be the space of bounded linear operators on X. Let $R^+ = [0, \infty)$. The family $\{S(t), t \in R^+\}$, $S : R^+ \to B(X)$, is called a modified exponential-cosine operator function, if it satisfies the equation

$$S(t + s) + S(t - s)\, T(2s) = 2S(t)\, S(s),\ s, t \in R^+, \quad \ldots(1)$$

$s \leqslant t$, $S(0) = I$, where $\{T(t), t \in R^+\}$, $T : R^+ \to B(X)$, is a known (C_0)-semigroup of operators. The authors (Singh and Buche 1979) have proved that the continuity at the origin of $\{S(t)\}$ implies continuity everywhere and if $\{S(t)\}$ is continuous then there exist two non-negative constants M and ω such that

$$\| S(t) \| \leqslant M \exp(\omega t). \quad \ldots(2)$$

They have introduced the two infinitesimal generator of $\{S(t)\}$ (Singh and Buche 1980). The first infinitesimal generator of $\{S(t)\}$ is defined as

$$Af = \lim_{h \to 0} ((S(h) - I)/h) f,\ f \in D(A), \quad \ldots(3)$$

where $D(A) \subset X$, and $D(A)$ is the set of elements $f \in X$ for which the above limit exists. It was proved that, under a general condition, A coincides with the infinitesimal generator of the associated semigroup $\{T(t)\}$ of operators. The second infinitesimal generator B of $\{S(t)\}$ is defined by

$$Bf = \lim_{h \to 0} (2(S(h) - T(h))/h^2) f \quad \ldots(4)$$

$f \in D(B)$, where $D(B) \subset X$, and $D(B)$ is the set of elements $f \in X$ for which the above limit exists. It has been proved that B is a closed linear operator with domain dense in X. Further for $f \in D(B) \cap D(A^2)$, $S(t)f$ is twice differentiable and

$$d^2S(t)f/dt^2 = 2AdS(t)f/dt + (B - A^2)\, S(t)f \qquad ...(5)$$

holds in the strong operator topology.

In the present paper, the generation theorems corresponding to those of Hille-Yosida (1957) and Sova (1968) are obtained for the modified exponential-cosine operator function. The appropriate resolvent turns out to be

$$R(\lambda, A)\,(I - BR^2(\lambda, A))^{-1}.$$

The results are an extension of Sova (1968) for the cosine-operator function. In the proofs, we use the properties of Laplace transforms such as uniqueness and the Post-Widder inversion formula (cf. Doetsch 1950, Widder 1946). Throughout we have assumed that $S(t_1)\,S(t_2) = S(t_2)\,S(t_1)$, and $S(t_1)\,T(t_2) = T(t_2)\,S(t_1)$, for all $t_1, t_2 \in R^+$.

2. The Generation Theorems

The generation theorems for semigroups have been obtained by Hille, Yosida, Phillips, Miyadera and Feller (ref. Hille-Phillips 1957). This theorem gives the necessary and sufficient conditions for a closed linear operator A with dense domain to generate a strongly continuous semigroup and the conditions are in terms of the boundedness of the resolvent operator $R(\lambda, A)$.

A similar theorem has been obtained for the regular cosine operator function by Sova (1968). In this case the resolvent operator to be considered is $\lambda(\lambda^2 I - B)^{-1}$, where B is the second infinitesimal generator and λ is sufficiently large.

In this section we prove a similar theorem for the modified exponential-cosine operator function. The appropriate resolvent operator now turns out to be $R(\lambda, A)$ $(I - BR^2(\lambda, A))^{-1}$, where A and B are respectively the first and the second infinitesimal generators and λ is sufficiently large. We have also indirectly proved that $S(t) = T(t)$ $C(t)$, where $\{C(t), t \in R^+\}$ is the cosine operator function generated by B.

Lemma 1 — Let $\{S(t); t \in R^+\}$, $S : R^+ \to B(X)$, be a regular modified exponential-cosine operator function with $\{T(t)\}$ as the associated (C_0)-semigroup. Let the second infinitesimal generator B of $\{S(t)\}$ generate a regular cosine operator function. Then there exist a real number ω such that $(I - BR^2(\lambda, A))$ is one to one, for all real $\lambda > \omega$.

Proof : Since B generates a regular cosine operator function, there exist a real ω_1 such that $\mu^2 I - B$ is one to one for all sufficiently large real $\mu > \omega_1$. We shall prove that $\mu^2 I - \lambda^2 R^2(\lambda, A) B$ is one to one for sufficiently large real numbers λ and μ. Suppose if possible $f \neq 0$, $f \in D(B)$, and

$$(\mu^2 I - \lambda^2 R^2(\lambda, A) B) f = 0 \text{ and}$$
$$(\mu^2 I - B) f \neq 0, \lambda, \mu > \omega_1.$$

Now

$$(\mu^2 I - B) f = (\mu^2 I - \lambda^2 R^2(\lambda, A) B) f + (\lambda^2 R^2(\lambda, A) - I) Bf.$$

So

$$0 < \| (\mu^2 I - B) f \| \leqslant \| (\mu^2 I - \lambda^2 R^2(\lambda, A) B) f \| + \| \lambda^2 R^2(\lambda, A) - I \| \, \| Bf \|.$$

By the theory of semigroups (Dunford and Schwartz 1958), $\lambda R(\lambda, A) f \to f$, as $\lambda \to \infty$, and $\| \lambda R(\lambda, A) \|$ is finite. So for any $\epsilon > 0$, there exists a real number ω_2 such that

$$\| \lambda^2 R^2(\lambda, A) - I \| < \epsilon/(1 + \| Bf \|), \lambda > \omega_2.$$

Hence $0 < \| (\mu^2 I - B) f \| < \epsilon$, for $\mu > \omega = \max(\omega_1, \omega_2)$, which is a contradiction. Hence the Lemma.

Theorem 1 — Let $\{S(t) \,;\, t \in R^+\}$, $S : R^+ \to B(X)$, be a regular modified exponential-cosine operator function satisfying (1). If B generates a regular cosine operator function, then there exist real numbers $M \geqslant 0$ and ω such that

(i) $(\lambda I - A)(I - BR^2(\lambda, A)$ is one to one, and $R(\lambda, A)(I - BR^2(\lambda, A))^{-1}$ is a bounded linear operator for $\lambda > \omega$, λ real,

(ii) for each $f \in X$, $R(\lambda, A)(I - BR^2(\lambda, A))^{-1} f \to f$, as $\lambda \to \infty$, $\lambda > \omega$,

(iii) for each real $\lambda > \omega$, and

$$n = 0, 1, 2, \ldots, \| d^n(R(\lambda, A)(I - BR^2(\lambda, A))^{-1})/d\lambda^n \| \leqslant Mn! \, ((1/(\lambda + \omega))^{n+1} + (1/(\lambda - \omega))^{n+1}).$$

Proof : Let us define

$$Z(\lambda) f = \int_0^\infty \exp(-\lambda u) \, S(u) f \, du, \; \lambda > \omega_1.$$

This integral exists because by (2), $S(.) f$ is continuous on R^+ and

$$\| S(t) \| \leqslant M \exp(\omega_1 t),$$

for some $M > 0$ and some $\omega_1 > 0$. So $Z(\lambda)$ is a bounded linear operator, because $\{S(t)\}$ is linear and $\| S(t) \| \leqslant M \exp(\omega_1 t)$.

Now we shall prove that for each $f \in D(A^2)$, $Z(\lambda) f \in D(B)$ and

$$BZ(\lambda) f = (\lambda I - A)^2 Z(\lambda) f - (\lambda I - A) f.$$

We consider

$$2((S(h) - T(h))/h^2)\, Z(\lambda) f$$
$$= 2((S(h) - T(h))/h^2) \int_0^h \exp(-\lambda u)\, S(u) f\, du$$
$$+ 2((S(h) - T(h))/h^2) \int_h^\infty \exp(-\lambda u)\, S(u) f\, du.$$

It is easy to check that the first integral tends to zero as $h \to 0$, for $f \in D(A)$. So for $f \in D(A)$,

$$\lim_{h\to 0} 2((S(h) - T(h))/h^2)\, Z(\lambda) f = \lim_{h\to 0} 2((S(h) - T(h))/h^2) \times \int_h^\infty \exp(-\lambda u)\, S(u) f\, du. \qquad \ldots(6)$$

Using eqn. (1), we have

$$2((S(h) - T(h))/h^2) \int_h^\infty \exp(-\lambda u)\, S(u) f\, du$$
$$= (1/h^2) \int_h^\infty \exp(-\lambda u)\, (S(u+h) + T(2h)\, S(u-h)) f\, du$$
$$- 2(T(h)/h^2) \int_h^\infty \exp(-\lambda u)\, S(u) f\, du$$
$$= (1/h^2) \int_{2h}^\infty \exp(-\lambda(u-h))\, S(u) f\, du + (T(2h)/h^2)$$
$$\times \int_0^\infty \exp(-\lambda(u+h))\, S(u) f du - 2(T(h)/h^2) \int_h^\infty \exp(-\lambda u)\, S(u) f du$$
$$= (1/h^2) \exp(\lambda h)\, Z(\lambda) f - (\exp(\lambda h)/h^2) \int_0^{2h} \exp(-\lambda u)\, S(u) f\, du$$
$$+ (T(2h)/h^2) \exp(-\lambda h)\, Z(\lambda) f - (2T(h)/h^2)\, Z(\lambda) f$$
$$+ 2(T(h)/h^2) \int_0^h \exp(-\lambda u)\, S(u) f\, du$$
$$= ((T(2h) - 2T(h) + I)/h^2)\, Z(\lambda) f + ((\exp(\lambda h)$$
$$+ \exp(-\lambda h) - 2)/h^2)\, Z(\lambda) f + ((\exp(-\lambda h) - 1)/h^2)\, T(2h)\, Z(\lambda) f$$
$$+ (1/h^2)\, Z(\lambda) f - (1/h^2) \exp(-\lambda h)\, Z(\lambda) f +$$

(equation continued on p. 372)

$$+ (2/h^2)\, T(h) \int_0^h \exp(-\lambda u)\, S(u) f\, du$$

$$- (1/h^2) \exp(\lambda h) \int_0^{2h} \exp(-\lambda u)\, S(u) f\, du$$

$$= (1/h^2)\,(T(2h) - 2T(h) + I)\, Z(\lambda) f + (1/h^2)\,(\exp(\lambda h)$$

$$+ \exp(-\lambda h) - 2)\, Z(\lambda) f + (1/h)\,(\exp(-\lambda h) - 1)\,(1/h)$$

$$\times (T(2h) - I)\, Z(\lambda) f + (2/h^2 (T(h) - I) \int_0^h \exp(-\lambda u)\, S(u) f\, du$$

$$- (1/h^2)\,(\exp(\lambda h) - 1) \int_0^{2h} \exp(-\lambda u)\, S(u) f\, du$$

$$- (1/h^2) \int_0^{2h} \exp(-\lambda u)\, S(u) f\, du + (2/h^2) \int_0^h \exp(-\lambda u)\, S(u) f\, du. \qquad ...(7)$$

Now, using the commutativity assumption, we observe that for $\lambda > \omega$, as $h \to 0$

$$(1/h^2)\,(T(2h) - 2T(h) + I)\, Z(\lambda) f \to A^2 Z(\lambda) f,\ f \in D(A^2) \qquad ...(8)$$

$$(2/h)\,(T(h) - I)\,(1/h) \int_0^h \exp(-\lambda u)\, S(u) f\, du \to 2Af,\ f \in D(A) \qquad ...(9)$$

$$(1/h^2)\,(\exp(\lambda h) + \exp(-\lambda h) - 2)\, Z(\lambda) f \to \lambda^2 Z(\lambda) f \qquad ...(10)$$

$$2((T(2h) - I)/2h)\,(1/h)\,(\exp(-\lambda h) - 1)\, Z(\lambda) f \to -2\lambda A Z(\lambda) f,\ f \in D(A) \qquad ...(11)$$

$$2\,((\exp(\lambda h) - 1)/h)\,(1/2h) \int_0^{2h} \exp(-\lambda u)\, S(u) f\, du \to 2\lambda f. \qquad ...(12)$$

Consider

$$(1/h^2) \int_0^{2h} \exp(-\lambda u)\, S(u) f\, du + (2/h^2) \int_0^h \exp(-\lambda u)\, S(u) f\, du$$

$$= (2/h^2) \int_0^h \exp(-\lambda u)\,(S(u) f - f)\, du$$

$$- (2/h^2) \int_0^{2h} \exp(-2\lambda u)\,(S(2u) f - f)\, du$$

$$- (2/h^2) \int_0^h (\exp(-2\lambda u) - \exp(-\lambda u)) f\, du. \qquad ...(13)$$

But $$\| (2/h^2) \int_0^h \exp(-\lambda u)\,(S(u)f - f)\,du - Af \|$$

$$= \| (2/h^2) \int_0^h (S(u)f - f - uAf)\,du + (2/h^2) \int_0^h (\exp(-\lambda u) - 1)\,(S(u)f - f) \|$$

Since $f \in D(A)$, so for any $\epsilon > 0$, there exists a $\delta > 0$, such that

$$\| S(u)f - f - uAf \| < \epsilon u,\ 0 < u < \delta.$$

Choosing $h < \delta$, we see that

$$\| (2/h^2) \int_0^h (S(u)f - f - uAf)\,du \| < \epsilon. \quad \text{Also}$$

$$(2/h^2) \int_0^h (\exp(-\lambda u))\,(S(u)f - f)\,du \to 0,$$

as $h \to 0$ by the regularity of $\{S(t)\}$. So it follows that, for $f \in D(A)$,

$$(2/h^2) \int_0^h \exp(-\lambda u)\,(S(u)f - f)\,du \to Af, \text{ as } h \to 0 \qquad \text{...(14)}$$

$$(2/h^2) \int_0^h \exp(-2\lambda u)\,(S(2u)f - f)\,du \to 2Af, \text{ as } h \to 0. \qquad \text{...(15)}$$

Also $$(2/h^2) \int_0^h (\exp(-2\lambda u) - \exp(-\lambda u))\,f\,du \to -\lambda f, \text{ as } h \to 0. \qquad \text{...(16)}$$

Applying (14) – (16), (13), we see that for $f \in D(A)$, $\lambda > \omega_1$,

$$(1/h^2) \int_0^{2h} \exp(-\lambda u)\,S(u)f\,du + (2/h^2) \int_0^h \exp(-\lambda u)\,S(u)f\,du \to \lambda f - Af. \qquad \text{...(17)}$$

Substituting (8) – (12) and (17) in (7), and using (6), it follows that for $f \in D(A^2)$, $\lambda > \omega_1$,

$$B_h \int_0^\infty \exp(-\lambda u)\,S(u)f\,du \to (\lambda I - A)^2 Z(\lambda) f - (\lambda I - A) f, \text{ as } h \to 0. \qquad \text{...(18)}$$

This implies that for each $f \in D(A^2)$, $Z(\lambda) f \in D(B)$ and

$$BZ(\lambda) f = (\lambda I - A)^2 Z(\lambda) f - (\lambda I - A) f,\ \lambda > \omega_1,$$

or $$((\lambda I - A)^2 - B)\,Z(\lambda) f = (\lambda I - A) f,\ f \in D(A^2),\ \lambda > \omega_1.$$

Since A generates a semigroup, there exists $\omega_2 > 0$, such that for $\lambda > \omega_2$, $(\lambda I - A)$ is one to one and $R(\lambda, A)$ is a bounded linear operator. Also by the Lemma, there exists $\omega_3 > 0$ such that $I - BR^2(\lambda, A)$ is one to one for $\lambda > \omega_3$. This implies that for each $f \in D(A^2)$ and $\lambda > \omega = \max(\omega_1, \omega_2, \omega_3)$,

$$R(\lambda, A)(I - BR^2(\lambda, A))^{-1} f = Z(\lambda) f.$$

Since $D(A^2)$ is dense in X and $Z(\lambda)$ is a bounded linear operator, it implies that

$$Z(\lambda) f = R(\lambda, A)(I - BR^2(\lambda, A))^{-1} f,$$

for each $f \in X$ and $\lambda > \omega$. This proves the part (i) of the Theorem. Parts (ii) and (iii) follow on the same lines as for the cosine operator functions (cf. Sova 1968).

Theorem 2 — Let $A : D(A) \to X$, $D(A) \subset X$ and $B : D(B) \to X$, $D(B) \subset X$, be two closed linear operators which commute on $D(AB)$ and let $D(B) \cap D(A^2)$ be dense in X. Suppose A generates a (C_0)-semigroup $\{T(t),\ t \in R^+\}$, $T : R^+ \to B(X)$, of operators. Let M and ω be two non-negative constants. If

(i) for each $\lambda > \omega$, the operator $(\lambda I - A)(I - BR^2(\lambda, A))$ is one to one and $R(\lambda, A)(I - BR^2(\lambda, A))^{-1}$ is a bounded linear operator,

(ii) $\lambda R(\lambda, A)(I - BR^2(\lambda, A))^{-1} f \to f$, as $\lambda \to \infty$, $f \in X$, $\lambda > \omega$,

(iii) For each $\lambda > \omega$, and $n = 0, 1, 2, \ldots$,

$$\| (d^n/d\lambda^n) R(\lambda, A)(I - BR^2(\lambda, A))^{-1} \|$$
$$\leqslant \tfrac{1}{2} M n! \, ((1/(\lambda + \omega))^{n+1} + (1/(\lambda - \omega))^{n+1},$$

then there exists a regular modified, exponential-cosine operator function $\{S(t) : t \in R^+\}$, $S : R^+ \to B(X)$, such that

(1) the first infinitesimal generator of $\{S(t)\}$ is A,

(2) the second infinitesimal generator of $\{S(t)\}$ is B,

(3) for every $t \in R^+$, $\| S(t) \| \leqslant M \exp(\omega t)$.

PROOF: Define for $\lambda > \omega$, $f \in X$, $l \in X^*$, the space of bounded linear functionals on X,

$$\pi(f, l)(\lambda) = l(R(\lambda, A)(I - BR^2(\lambda, A))^{-1} f). \qquad \ldots(19)$$

Further we write

$$\pi^{(n)}(f, l)(\lambda) = (d^n/d\lambda^n)\, \pi(f, l)(\lambda),\ n = 0, 1, 2, \ldots . \qquad \ldots(20)$$

By the assumption (iii) and by the same arguments as in Sova (1968), there exists a real-valued measurable function $\phi(f, l)$ on R^*, such that

$$| \phi(f, l)(t) | \leqslant M \exp(\omega t) \| f \| \| l \| \qquad \ldots(21)$$

and for $\lambda > \omega$,

$$\int_0^\infty \exp(-\lambda u)\, \phi(f, l)(u)\, du = \pi(f, l)(\lambda). \qquad \ldots(22)$$

For $f \in D(B) \cap D(A^2)$, and $l \in X^*$, we define a new function $\psi(f, l)(t)$ as follows; for $t > 0$,

$$\psi(f, l)(t) = l(f) - tl(Af) + 2\int_0^t \phi(Af, l)(u)\, du$$
$$+ \int_0^t \int_0^s \phi((B - A^2)f, l)(u)\, du\, ds. \qquad \ldots(23)$$

It is easy to verify that $\psi(f, l)$ is continuous on R^+ for $l \in X^*$ and $f \in D(B) \cap D(A^2)$. Moreover for $f \in D(B) \cap D(A^2)$, $l \in X^*$ and $\lambda > \omega$, by using (19) and (22),

$$\int_0^\infty \exp(-\lambda u)\, \psi(f, l)(u)\, du = (1/\lambda) \cdot l(f) - (1/\lambda^2)\, l(Af)$$
$$+ (2/\lambda) \int_{v=0}^\infty \exp(-\lambda v)\, \phi(Af, l)(v)\, dv + (1/\lambda^2)$$
$$\times \int_{\omega=0}^\infty \exp(-\lambda\omega)\, \phi((B - A^2)f, l)(\omega)\, d\omega$$
$$= (1/\lambda)\, l(f) - (1/\lambda^2)\, l(Af) + (2/\lambda)\, l(R(\lambda, A)(I - BR^2(\lambda, A))^{-1} Af)$$
$$+ (1/\lambda^2)\, l(R(\lambda, A)(I - BR^2(\lambda, A))^{-1}(B - A^2)f)$$
$$= l[(f/\lambda) - (1/\lambda^2)Af + (2/\lambda)R(\lambda, A)(I - BR^2(\lambda, A))^{-1}$$
$$\times (\lambda f - (\lambda I - A)f) + (1/\lambda^2)R(\lambda, A)(I - BR^2(\lambda, A))^{-1}$$
$$\times (\lambda^2 I - 2\lambda A - ((\lambda I - A)^2 - B)f]$$
$$= l[(f/\lambda) - (1/\lambda^2)Af + 2R(\lambda, A)(I - BR^2(\lambda, A))^{-1}f$$
$$- (2/\lambda)(I - BR^2(\lambda, A))^{-1} + (1/\lambda)R(\lambda, A)(I - BR^2(\lambda, A))^{-1}$$
$$\times (\lambda I - 2A)f - (1/\lambda^2)(\lambda I - A)f]$$
$$= l[3R(\lambda, A)(I - BR^2(\lambda, A))^{-1}f - (2/\lambda)$$
$$\times (I - BR^2(\lambda, A))^{-1}(f + R(\lambda, A)Af)]$$
$$= l[3R(\lambda, A)(I - BR^2(\lambda, A))^{-1}f - 2R(\lambda, A)(I - BR^2(\lambda, A))^{-1}f]$$
$$= l[R(\lambda, A)(I - BR^2(\lambda, A))^{-1}f].$$

So for $f \in D(B) \cap D(A^2)$, $\lambda > \omega$,

$$\int_0^\infty \exp(-\lambda u)\, \psi(f, l)(u)\, du = \int_0^\infty \exp(-\lambda u)\, \phi(f, l)(u)\, du. \qquad \ldots(24)$$

This shows that $\psi(f, l) = \phi(f, l)$ almost everywhere, for $f \in D(B) \cap D(A^2)$, $l \in X^*$. By the same arguments as in Sova (1968), this implies that for each $f \in D(B) \cap D(A^2)$, $t \in R^+$ and $\epsilon > 0$, there exists a natural number $n_0(f, t, \epsilon)$ such that $n_0(f, t, \epsilon) > t\omega$ and for each $n \geqslant n_0(f, t, \epsilon)$ and each $l \in X^*$,

$$| (1/n!) (-1)^n (n/t)^{n+1} \pi^{(n)}(f, 1) (n/t) - \psi(f, l) (t) | < (\epsilon/2) \| l \|. \quad \text{...(25)}$$

For $\lambda > \omega$ and $n = 0, 1, 2, \ldots$, we define

$$V_n(\lambda) = ((-1)^n/n!) \lambda^{n+1} d^n[R(\lambda, A) (I - BR^2(\lambda, A))^{-1}]/d\lambda^n. \quad \text{...(26)}$$

So, by (20),

$$(-1)^n/n! \, \lambda^{n+1}\pi^{(n)}(f, l) (\lambda) = l(V_n(\lambda)), \quad \text{...(27)}$$

for $\lambda > \omega$, $f \in D(B) \cap D(A^2)$, $n = 0, 1, 2, \ldots$.

It immediately follows from (25) and the assumption (iii) that $\{V_n(n/t)f\}$ is Cauchy for every $f \in D(B) \cap D(A^2)$. By the completeness of X, it converges.

Let

$$S_0(t) f = \lim_{\substack{n \to \infty \\ n > t\omega}} V_n(n/t) f, \; f \in D(B) \cap D(A^2). \quad \text{...(28)}$$

By (25) and (26), we see that for each $f \in D(B) \cap D(A^2)$, $l \in X^*$ and $t \in R^+$,

$$l(S_0(t) f) = \psi(f, l) (t). \quad \text{...(29)}$$

It is easy to see that for each $f \in D(B) \cap D(A^2)$, the function $S_0(t) f$ is continuous on R^+, because $\psi(f, l) (t)$ is continuous. By the assumption (iii), we have

$$\| V_n(n/t) f \| \leqslant ((n/t)^{n+1}/((n/t) + \omega)^{n+1}) + ((n/t)^{n+1}/((n/t) - \omega)^{n+1})) \| f \|,$$

the right-hand side tending to $M \cosh(\omega t) \| f \|$, as $n \to \infty$. So $\| S_0(t) \| \leqslant M \exp(\omega t)$. Also by (29) (24) and (22), for each $f \in D(B) \cap D(A^2)$,

$$\int_0^\infty \exp(-\lambda u) S_0(u) f \, du = R(\lambda, A) (I - BR^2(\lambda, A))^{-1} f. \quad \text{...(30)}$$

Since $D(B) \cap D(A^2)$ is dense in X and $\{S_0(t)\}$ is continuous on R^+, there exists a one and only one continuous extension of $\{S_0(t)\}$ to the whole space X. We denote this extension by $\{S(t)\}$. Clearly $\| S(t) \| \leqslant M \exp(\omega t)$, and for each $f \in X$, $\lambda > \omega$,

$$\int_0^\infty \exp(-\lambda u) S(u) f \, du = R(\lambda, A) (I - BR^2(\lambda, A))^{-1} f. \quad \text{...(31)}$$

Let us write

$$P(\lambda) = R(\lambda, A) (I - BR^2(\lambda, A))^{-1} f, \; \lambda > \omega. \quad \text{...(32)}$$

To prove that $\{S(t)\}$ is modified exponential-cosine operator function, with $\{T(t)\}$ as the associated semigroup, we shall need the following results.

Result 1 — For $f \in X$, $\lambda > \omega$, $\mu > \omega$, $\lambda \neq \mu$

(a) $\int_0^\infty \exp(-\mu s)\, S(t+s) f\, ds$ is continuous in $t \in R^+$.

(b) $\| \int_0^\infty \exp(-\mu s)\, S(t+s) f\, ds \| \leqslant M/(\mu - \omega) \exp(\omega t) \| f \|$,

(c) $$\int_0^\infty \exp(-\lambda t) \left(\int_0^\infty \exp(-\mu s)\, S(t+s) f\, ds \right) dt$$
$$= (1/(\lambda - \mu))\, (P(\mu) - P(\lambda)) f.$$

(a) and (b) are easy to check by using the fact that $\| S(t) \| \leqslant M \exp(\omega t)$, and the continuity of $\{S(t)\}$.

For (c), $$\int_0^\infty \exp(-\lambda t) \left(\int_0^\infty \exp(-\mu s)\, S(t+s) f\, ds \right) dt$$

$$= \int_0^\infty \exp(-\lambda t) \exp(\mu t) \left(\int_0^\infty \exp(-\mu u)\, S(u) f\, du \right) dt$$

$$= \int_0^\infty \exp(-(\lambda - \mu)\, t) \left(\int_0^\infty \exp(-\mu u)\, S(u) f\, du - \int_0^t \exp(-\mu u)\, S(u) f\, du \right) dt$$

$$= (1/(\lambda - \mu))\, R(\mu, A)\, (I - BR^2(\mu, A))^{-1} f - \int_{u=0}^\infty \exp(-\lambda u)\, S(u) f \int_{t=u}^\infty \exp(-(\lambda - \mu)(t-u))\, dt\, du$$

$$= (1/(\lambda - \mu))\, R(\mu, A)\, (I - BR^2(\mu, A))^{-1} f - (1/(\lambda - \mu))\, R(\lambda, A)\, (I - BR^2(\lambda, A))^{-1} f$$

$= (1/(\lambda - \mu))\, (P(\mu) - P(\lambda)) f$, which proves (c).

Result 2 — For $f \in X$, $\lambda > \omega$, $\mu > \omega$, $s > t > 0$,

(a) $\int_t^\infty \exp(-\mu s)\, S(s-t) f\, ds$ is continuous in t,

(b) $\| \int_t^\infty \exp(-\mu s)\, S(s-t) f\, ds \| \leqslant (M/(\mu - \omega)) \exp(-\mu t) \| f \|$,

(c) $\int_0^\infty \exp(-\lambda t) \int_t^\infty \exp(-\mu s)\, S(s-t)\, T(2t) f\, ds\, dt$

$$= R(\lambda + \mu, 2A)\, P(\mu) f.$$

A change of variables gives

$$\int_t^\infty \exp(-\mu s)\, S(s-t) f\, ds = \exp(-\mu t) \int_0^\infty \exp(-\mu u)\, S(u) f\, du.$$

(a) and (b) are easy to verify from this. Now for $\lambda > \omega$, $\mu > \omega$, $s > t > 0$, $f \in X$,

$$\int_0^\infty \exp(-\lambda t) \left(\int_t^\infty \exp(-\mu s)\, T(2t)\, S(s-t) f\, ds \right) dt$$

$$= \int_0^\infty \exp(-(\lambda + \mu) t)\, T(2t) \left(\int_0^\infty \exp(-\mu u)\, S(u) f\, du \right) dt$$

$$= R(\lambda + \mu, 2A)\, R(\mu, A)\, (I - BR^2(\mu, A))^{-1} f$$

$$= R(\lambda + \mu, 2A)\, P(\mu) f.$$

Result 3 — If $f \in X$, $\lambda > \omega$, $\mu > \omega$,

(a) $\int_0^t \exp(-\mu s)\, S(t-s)\, T(2s) f\, ds$ is continuous in $t \in R^+$,

(b) $\left\| \int_0^t \exp(-\mu s)\, S(t-s)\, T(2s) f\, ds \right\| \leqslant (M^2/(\mu - \omega)) \exp(\omega t) \| f \|$,

(c) $\int_0^\infty \exp(-\lambda t) \int_0^t \exp(-\mu s)\, S(t-s)\, T(2s) f\, ds\, dt$

$$= R(\lambda + \mu, 2A)\, P(\lambda) f.$$

(a) and (b) are easy to check, by using $\| S(t) \| \leqslant M \exp(\omega t)$, and the continuity of $\{S(t)\}$.

Now for $f \in X$, $\lambda > \omega$, $\mu > \omega$,

$$\int_0^\infty \exp(-\lambda t) \int_0^t \exp(-\mu s)\, S(t-s)\, T(2s) f\, ds\, dt$$

$$= \int_0^\infty \exp(-(\lambda + \mu) t) \int_0^t \exp(\mu u)\, S(u)\, T(2(t-u)) f\, du\, dt$$

$$= \int_0^\infty \exp(-\lambda u)\, S(u) \left(\int_{v=0}^\infty \exp(-(\lambda + \mu) v)\, T(2v) f\, dv \right) du$$

$$= R(\lambda + \mu, 2A)\, R(\lambda, A)\, (I - BR^2(\lambda, A))^{-1} f$$
$$= R(\lambda + \mu, 2A)\, P(\lambda) f.$$

Result 4 — For every $\lambda > \omega$, $\mu > \omega$, $\lambda \neq \mu$, we have

$$2P(\lambda)\, P(\mu) = R(\lambda + \mu, 2A)\, (P(\lambda) + P(\mu)) - (1/(\lambda - \mu))\, (P(\lambda) - P(\mu)).$$

We have

$$(\lambda I - A)^2 - ((\lambda I - A)^2 - B) = (\mu I - A)^2 - ((\mu I - A)^2 - B),$$

Multiplying both sides by $P(\lambda)\, P(\mu)$, we get

$$(\lambda I - A)^2 P(\lambda)\, P(\mu) - (\lambda I - A)\, P(\mu) = (\mu I - A)^2 P(\lambda)\, P(\mu) - (\mu I - A)\, P(\lambda),$$

or

$$((\lambda I - A)^2 - (\mu I - A)^2)\, P(\lambda)\, P(\mu) = (\lambda I - A)\, P(\mu) - (\mu I - A)\, P(\lambda),$$

i.e.

$$2(\lambda^2 I - \mu^2 I - 2(\lambda - \mu)\, A)\, P(\lambda)\, P(\mu) = 2(\lambda I - A)\, P(\mu) - 2(\mu I - A)\, P(\lambda),$$

or

$$(\lambda - \mu)\, (\lambda I + \mu I - 2A)\, P(\lambda)\, P(\mu) = (\lambda - \mu)\, (P(\lambda) + P(\mu)) - (\lambda + \mu - 2A)(P(\lambda) - P(\mu)).$$

Hence

$$2P(\lambda)\, P(\mu) = R(\lambda + \mu, 2A)\, (P(\lambda) + P(\mu)) - 1/(\lambda - \mu)\, (P(\lambda) - P(\mu))$$

for $\lambda, \mu > \omega$, which proves the Result 4.

With the help of these four Results, we shall prove that $\{S(t); t \in R^+\}$ is a modified exponential-cosine operator function, with $\{T(t)\}$ as the associated semi-group. Let $f \in X$ be fixed, and let us define an auxiliary function $K(t, s)$, for $s, t \in R^+$, with values on X as follows :

$$K(t, s) = \begin{cases} S(t + s) + T(2s)\, S(t - s), & \text{if } t > s, \\ S(t + s) + T(2t)\, S(s - t), & \text{if } s > t, \\ S(2t) + T^2(t), & \text{if } s = t. \end{cases} \qquad \ldots(33)$$

It immediately follows that (a_1) $K(t, s) = K(s, t)$, for every $s, t \in R^+$. (a_2) For every fixed $t \in R^+$, the function $K(t, .)$ is continuous on R^+, since both S and T are continuous. (a_3) $\| K(t, s) \| \leqslant 2M^2 \exp(\omega t) \exp(\omega s)$ because $\| S(t) \| \leqslant M \exp(\omega t)$, $t \in R^+$.

Now let us define for every $\mu > \omega$, and $t \in R^+$,

$$\Lambda_1(t, \mu) = \int_0^\infty \exp(-\mu s)\, K(t, s)\, ds. \qquad \ldots(34)$$

This definition is meaningful by (a_2) and (a_3). Fot every fixed $\mu > \omega$, the function $\Lambda_1(\cdot, \mu)$ is continuous on R^+ because of (a_2), and

$$\| \Lambda_1(t, \mu) \| \leqslant 2M^2 \exp(\omega t)\, (1/(\mu - \omega)),$$

because of (a_3).

Now let

$$\Lambda_2(\lambda, \mu) = \int_0^\infty \exp(-\lambda t)\, \Lambda_1(t, \mu)\, dt,\ \lambda, \mu > \omega. \qquad \ldots(35)$$

This definition is evidently meaningful by the above mentioned properties of $\Lambda_1(t, \mu)$.

Now for $\lambda, \mu > \omega$,

$$\begin{aligned}\Lambda_2(\lambda, \mu) = \int_0^\infty \exp(-\lambda t) [\int_t^\infty \exp(-\mu s) \{S(s + t) \\ + T(2t)\, S(s - t)\} f\, ds + \int_0^t \exp(-\mu s) \{S(s + t) \\ + T(2s)\, S(t - s)\} f\, ds]\, dt \\ = \int_0^\infty \exp(-\lambda t) (\int_0^\infty \exp(-\mu s)\, S(t + s) f\, ds)\, dt \\ + \int_0^\infty \exp(-\lambda t) (\int_t^\infty \exp(-\mu s)\, T(2t)\, S(s - t) f\, ds)\, dt \\ + \int_0^\infty \exp(-\lambda t) \int_0^t \exp(-\mu s)\, T(2s)\, S(t - s) f\, ds\, dt.\end{aligned}$$

By using the Results 1, 2, 3, and 4, we get

$$\begin{aligned}\Lambda_2(\lambda, \mu) &= (1/(\lambda - \mu))\, (P(\mu) - P(\lambda)) f + R(\lambda + \mu, 2A)\, P(\mu) f \\ &\quad + R(\lambda + \mu, 2A)\, P(\lambda) f \\ &= R(\lambda + \mu, 2A)\, (P(\lambda) + P(\mu)) f - (1/(\lambda - \mu)\, (P(\lambda) - P(\mu)) f \\ &= 2P(\lambda)\, P(\mu) f.\end{aligned}$$

By (35), for every $t \in R^+$ and $\mu > \omega$, we have

$$\Lambda_1(t, \mu\,.) = 2S(t)\, P(\mu).$$

Repeating the argument, we obtain $K(t, s) = 2S(t)\, S(s)$, for every $t, s \in R^+$.

Hence for every $t > s > 0$, we obtain

$$S(t+s) + T(2s)\,S(t-s) = 2S(t)\,S(s).$$

Thus $\{S(t),\ t \in R^+\}$ is a modified exponential-cosine operator function with $\{T(t)\}$ as the associated semigroup.

Using (23), we obtain for any $l \in X^*$, $f \in D(B) \cap D(A^2)$,

$$l(S(t)f) = \psi(f, l)(t) = l(f) - tl(Af) + 2\int_0^t \phi(Af, l)(u)\,du$$
$$+ \int_0^t \int_0^s \phi((B - A^2)f, l)(u)\,du\,ds.$$

For $f \in D(B) \cap D(A^2)$, such that $Af \in D(B) \cap D(A^2)$ and $(B - A^2)f \in D(B) \cap D(A^2)$, we have

$$l(S(t)f) = l(f) - tl(Af) + 2\int_0^t \psi(Af, l)(u)\,du$$
$$+ \int_0^t \int_0^s \psi(B - A^2)f, l)(u)\,du\,ds. \qquad ...(36)$$

This implies that for such an f,

$$S(t)f = f - tAf + 2\int_0^t S(u)\,Af\,du + \int_0^t \int_0^s S(u)\,(B - A^2)f\,du\,ds. \qquad ...(37)$$

From (37) it can be easily checked that the first infinitesimal generator of $\{S(t)\}$ is A and the second infinitesimal generator is B.

Theorem 3 — Let $\{S(t) : t \in R^+\}$, $S : R^+ \to B(X)$, be a regular modified exponential-cosine operator function, with $\{T(t)\}$ as the associated (C_0)-semigroup. Let the second infinitesimal generator B of $\{S(t)\}$ generates a regular cosine operator function $\{C(t)\}$. Then

$$S(t) = T(t)\,C(t),\ t \in R^+.$$

PROOF: It is easy to check (cf. Buche 1975) that $\{T(t)\,C(t)\}$ is a regular modified exponential-cosine operator function with the first infinitesimal generator A and the second infinitesimal generator B. Thus the Laplace-transform of $\{T(t)\,C(t)\}$ will coincide with that of $S(t)$. Hence $S(t) = T(t)\,C(t)$, $t \in R^+$.

3. EXAMPLES

1. In Example 1 of Singh and Buche (1979), $\{S(t),\ t \in R^+\}$ is defined on $C(R)$ as

$$S(t)\,f(x) = (\tfrac{1}{2})\,(f(x + (a+b)\,t) + f(x + (a-b)\,t))$$

where a and b are constants, $0 < b < a$. It has been proved there that $\{S(t)\}$ is a regular modified exponential-cosine operator function, and $\| S(t) \| \leqslant 1$.

Now we shall find the Laplace transform of this modified exponential-cosine operator function and the approximating sequence given by (26).

Let us write for $\lambda > 0$,

$$L(\lambda) f(x) = \int_0^\infty \exp(-\lambda u)\, S(u) f(x)\, du$$

$$= (\tfrac{1}{2}) \int_0^\infty \exp(-\lambda u) f(x + (a+b)\, u)\, du$$

$$+ (\tfrac{1}{2}) \int_0^\infty \exp(-\lambda u) f(x + (a-b)\, u)\, du$$

$$= (\tfrac{1}{2}(a+b)) \int_x^\infty \exp(-\lambda(u-x)/(a+b)) f(u)\, du$$

$$+ (\tfrac{1}{2}(a-b)) \int_x^\infty \exp(-\lambda(u-x)/(a-b)) f(u)\, du$$

$$= (\tfrac{1}{2}) \int_x^\infty ((1/(a+b)) \exp(-\lambda(u-x))/(a+b))$$

$$+ (1/(a-b)) \exp(-\lambda(u-x)/(a-b))) f(u)\, du,$$

which is the Laplace transform of $\{S(t)\}$ given by (26).

Differentiating $L(\lambda)$, n times we get

$$(d^n/d\lambda^n)\, L(\lambda) f(x) = ((-1)^n/2)((1/(a+b))^{n+1} \int_x^\infty \exp(-\lambda(u-x)/(a+b))$$

$$+ (1/(a-b))^{n+1} \exp(-\lambda(u-x)/(a-b))\, (u-x)^n) f(u)\, du).$$

So the approximating sequence for $\{S(t)\}$ is given by

$$V_n(n/t) f(x) = ((n/t)^{n+1}/2(n!)) \left(\int_x^\infty ((1/(a+b))^{n+1} \right.$$

$$\times \exp(-(n/t)(u-x)/(a+b)) + (1/(a-b))^{n+1}$$

$$\times \exp(-(n/t)(u-x)/(a-b)))\, (u-x)^n f(u)\, du),$$

$$n = 0, 1, 2, \ldots, t \in R^+, -\infty < x < \infty.$$

Example 2 — In Example 2 of Singh and Buche (1979), $\{S(t);\ t \in R^+\}$ is defined on $C(R)$ as

$$S(t) f(x) = \exp(-\beta t)) \sum_{k=0}^{\infty} ((\alpha t)^{2k}/(2k)!) f(x - 2k\mu)$$

where $\mu > 0$ is some constant. It has been proved that $\{S(t)\}$ is a regular modified exponential-cosine operator function with $\| S(t) \| \leqslant 1$.

The Laplace transform $L(\lambda)$ of $\{S(t)\}$, for $\lambda > 0$ is given by

$$L(\lambda) f(x) = \int_0^\infty \exp(-\lambda t)\, S(t) f(x)\, dt$$

$$= \int_0^\infty \exp(-(\lambda + \alpha) t) \sum_{k=0}^\infty ((\alpha t)^{2k}/(2k)!) f(x - 2k\mu)\, dt.$$

Because of the uniform convergence of the series inside the integral, it is term by term integrable.

Hence

$$L(\lambda) fx = \sum_{k=0}^\infty (\alpha^{2k}/(2k)!) \int_0^\infty t^{2k} \exp(-(\lambda + \alpha) t) f(x - 2k\mu)\, dt$$

$$= (1/(\lambda + \alpha)) \sum_{k=0}^\infty (\alpha/(\lambda + \alpha))^{2k} f(x - 2k\mu).$$

Differentiating $L(\lambda)$ n times, we get

$$(d^n/d\lambda^n) L(\lambda) f(x) = (-1)^n (1/(\lambda + \alpha)^{n+1} \sum_{k=0}^\infty ((2k + n)!/(2k)!) (\alpha/(\lambda + \alpha))^{2k}$$

$$\times f(x - 2k\mu),\ n = 0, 1, 2, \ldots .$$

So the approximating sequence of $\{S(t)\}$ as given by (26) is

$$V_n(n/t) f(x) = (n/(n + \alpha t))^{n+1} \sum_{k=0}^\infty ((2k + n)!/n!\,(2k)!)$$

$$\times (\alpha t/(n + \alpha t))^{2k} f(x - 2k\mu).$$

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