Indian J. pure appl. Math., 11(3) : 384-391, March 1980

SOME SINGULAR INTEGRAL EQUATIONS INVOLVING LAGUERRE POLYNOMIALS AND FINITE PARTS OF DIVERGENT INTEGRALS

T. R. PRABHAKAR

Department of Mathematics, University of Delhi, Delhi 110007

(*Received* 8 *June* 1979)

During recent years, the integral equation

$$\int_0^t (t-u)^{\alpha} L_n^{\alpha}(t-u) f(u)\, du = g(t)$$

where $L_n^{\alpha}(t)$ is the generalized Laguerre polynomial, has been solved by several authors under the restriction $\alpha > -1$, which ensures the convergence of the integral. In this paper the author uses Mikusiński operators to discuss and solve the above integral equation when $-(n+2) < \alpha < -(n+1)$ by considering Hadamard's finite part of the divergent integral in the integral equation.

1. INTRODUCTION

Butzer (1959) solved the integral equation

$$FP \int_0^t (t-u)^{-\alpha} f(u)\, du = g(t), \quad t > 0 \qquad \ldots(1.1)$$

for $\alpha \geqslant 1$, where the left hand member denotes Hadamard's finite part (FP) of the (possibly) divergent integral. Quite recently Wiener (1968, 1971) has discussed some singular integral equations involving similar kernels and finite parts of divergent integrals.

In recent years quite a number of papers have appeared in which the integral equation (see Widder 1963, Buschman 1964, Khandekar 1965, Prabhakar 1969, 1971)

$$\int_0^t (t-u)^{\alpha} L_n^{\alpha}(t-u) f(u)\, du = g(t) \qquad \ldots(1.2)$$

or its particular cases have been discussed. Here $L_n^{\alpha}(t)$ is the generalized Laguerre polynomial (Rainville 1960, ch. 12) of degree n and α is a parameter. In order to ensure the existence (convergence) of the integral in (1.2), the parameter α was invariably restricted by all these authors by the condition $\alpha > -1$.

In this paper we discuss

$$FP \int_0^t (t-u)^\alpha L_n^\alpha (t-u) f(u)\, du = g(t), \quad t > 0 \qquad ...(1.3)$$

for $-(n+2) < \alpha < -(n+1)$. Quite recently Morton and Krall (1978) have shown by distributional techniques that for $-(n+2) < \alpha < -(n+1)$, the polynomials $\{L_m^\alpha(x)\}$ still form an orthogonal system with respect to a sesquilinear form. We show that if g is locally integrable on $t \geqslant 0$, then (1.3) has a unique solution f which is expressed explicitly by a simple integral formula as a smooth function.

2. Preliminaries

Mikusinski operators — We denote by $\mathcal{F}$ the field of convolution quotients or Mikusiński operators (Erdelyi 1962, Mikusiński 1959). Notations of Mikusiński (1959) will be used. Thus f or $\{f(t)\}$ will denote the function *per se* and $f(t)$ the value at the point t. l will denote the constant function $\{1\}$, the restriction of the unit function to $t \geqslant 0$. For Re $\mu > 0$, the function $l^\mu = \left\{\frac{t^{\mu-1}}{\Gamma(\mu)}\right\}$ in $\mathcal{F}$, can also be regarded as the fractional integration operator l^μ (Prabhakar 1969). The inverse of l, denoted by $s = 1/l$ and called the 'differential operator' plays a fundamental role in the development as indeed it does in Mikusiński calculus itself.

Space L_μ — Denoting by L the space of all (equivalence classes of) complex-valued functions f which are locally integrable on $[0, \infty)$, we set for Re $\mu > 0$,

$$L_\mu = \{f : f = l^\mu g \text{ with } g \text{ in } L\}.$$

If μ is a positive integer, then $f \in L_\mu$ iff $f^{(\mu-1)}$ is absolutely continuous and

$$f(0) = 0 = f^{(n)}(0),\ n = 1, 2, ..., \mu - 1.$$

For properties of L_μ and l^μ (see Prabhakar 1969, 1971).

Hadamard's finite part — The finite part of the divergent integral appearing in (1.3) generates a distribution (in the sense of L. Schwartz), called a pseudo-function (see Zemannian 1965, §2.5; Gelfand and Shilov 1964, §1.7, §3). In view of a close relation between Schwartz distributions and operators (Mikusiński 1959) it is natural that in our treatment (see also Butzer 1959) these pseudo-functions should be expressed as convolution operators. Consequently Mikusiński calculus has been used in our analysis. On the other hand Erdélyi (1967) has used the distribution theory and fractional integration to discuss the integral equation

$$\int_a^x (x^2 - t^2)^{\lambda/2} P_\nu^{-\lambda}\left(\frac{x}{t}\right) g(t)\, dt = f(x)$$

involving (possibly) divergent integral, $P_\nu^{-\lambda}$ being the Legendre function. Surprisingly our treatment of (1.3) which is based on Butzer (1959) turns out to be much simpler and we are able to obtain an explicit solution.

3. Operator $\mathcal{L}_n^\alpha$

For a parameter α given by

$$\alpha = -(n + 1 + \beta),\ 0 < \beta < 1 \qquad \text{...(3.1)}$$

and for sufficiently smooth functions f, define

$$\mathcal{L}_n^\alpha f = \left\{FP \int_0^t (t-u)^\alpha L_n^\alpha (t-u) f(u)\, du\right\} \qquad \text{...(3.2)}$$

whenever the right member exists as Hadamard's finite part.

Theorem 1 — If $f^{(n)}(t)$ exists and is absolutely continuous on $[0, T]$, $T < \infty$ and $f(0) = 0 = f^{(\nu)}(0)$ for $\nu \leqslant n - 1$, then $\mathcal{L}_n^\alpha f$ as defined in (3.2) is given by

$$\mathcal{L}_n^\alpha f = \frac{\Gamma(\alpha + n + 2)}{(\alpha + n + 1)\, n!} (s-1)^n s^{-n-\alpha-1} f. \qquad \text{...(3.3)}$$

Proof : Using Theorem 2.1 of Butzer (1959) and result (3) of §112 of Rainville (1960) we have

$$\mathcal{L}_n^\alpha f = \left\{FP \int_0^t (t-u)^\alpha L_n^\alpha (t-u) f(u)\, du\right\}$$

$$= (1+\alpha)_n \sum_{k=0}^{n} \frac{(-1)^k}{k!\,(n-k)!\,(1+\alpha)_k} \{FP\, t^{\alpha+k}\} \{f(t)\}.$$

Thus in view of (3.1), we have

$$\mathcal{L}_n^\alpha = (1+\alpha)_n \sum_{k=0}^{n} \frac{(-1)^k}{k!\,(n-k)!\,(1+\alpha)_k} \{FP\, t^{-(n+1-k+\beta)}\}.$$

By an application of Lemma 2.2 of Butzer (1959), we get

$$\{FP\, t^{-(n+1-k+\beta)}\} = (-1)^{n+1-k} \frac{\Gamma(\beta)\,\Gamma(1-\beta)}{\Gamma(n+1-k+\beta)} s^{-k-\alpha-1}.$$

Using the standard properties of the gamma function (Rainville 1960, §17–18) and after a computation we obtain that operating on the functions f of the theorem

$$\mathcal{L}_n^\alpha = \frac{\Gamma(\alpha+n+2)}{\alpha+n+1} \sum_{k=0}^{n} (-1)^k \frac{s^{-k-\alpha-1}}{(n-k)!\,k!}.$$

That is,

$$\mathcal{L}_n^\alpha = \frac{\Gamma(\alpha+n+2)}{(\alpha+n+1)\,n!}\, s^{-\alpha-1}\left(1-\frac{1}{s}\right)^n$$

which in turn implies (3.3).

Since the right-hand member of (3.4) defines a convolution operator, we make the following definition.

Definition — $\mathcal{L}_n^\alpha$ is a Mikusiński operator defined by

$$\mathcal{L}_n^\alpha = A s^{-\alpha-1}\left(1-\frac{1}{s}\right)^n \qquad \text{...(3.5)}$$

$$A = \frac{\Gamma(\alpha+n+2)}{(\alpha+n+1)\,n!}. \qquad \text{...(3.6)}$$

Evidently for f satisfying the assumptions of Theorem 1, $\mathcal{L}_n^\alpha f$ will denote the finite part of the divergent integral in (3.2). However $\mathcal{L}_n^\alpha$ as a convolution operator, is not an ordinary function.

Theorem 2 — Let α be defined as in (3.1). Then $\mathcal{L}_n^\alpha f$ is a locally integrable function if and only if f is in $L_{-\alpha-1}$.

PROOF : From (3.1), $-\alpha-1 = n+\beta$ where $0<\beta<1$. From (3.5) we have

$$\mathcal{L}_n^\alpha f = A\left(1-\frac{1}{s}\right)^n s^{-\alpha-1} f.$$

$$= A\left[\left(1-\frac{1}{s}\right)^n - 1\right] s^{-\alpha-1} f + A s^{-\alpha-1} f.$$

For f in $L_{-\alpha-1}$,

$$s^{-\alpha-1} f = I^{\alpha+1} f$$

exists in L. Also $\left(1-\frac{1}{s}\right)^n - 1$ is a function in L (Mikusiński 1959, p. 369; Erdelyi 1962, p. 49). Hence $\mathcal{L}_n^\alpha f$ is in L if and only if f is in $L_{-\alpha-1}$.

Theorem 3 — If $f \in L_{n+1}$, then

$$\mathcal{L}_n^\alpha f = \left\{ FP \int_0^t (t-u)^\alpha L_n^\alpha (t-u) f(u)\, du \right\}$$

exists as a function in L.

PROOF: Recall that $f \in L_{n+1}$ if and only if $f^{(n)}(t)$ is absolutely continuous for $t \geqslant 0$ and $f(0) = 0 = f^{(\nu)}(0)$ for $\nu \leqslant n$. The proof then follows from Theorem 2 combined with Theorem 1 and the fact that L_{n+1} is properly contained in $L_{-\alpha-1}$ (Prabhakar 1971).

Corollary — A necessary condition for (1.3) to have a solution $f \in L_{n+1}$ is that g is locally integrable.

4. Integral Equation (1.3)

It is easy to see that for any non-zero element g in the field $\mathcal{F}$, the operator equation

$$\mathcal{L}_n^\alpha f = g$$

will have a unique solution $f \in \mathcal{F}$. Our main interest is in the solutions of eqn. (1.3) where g and f are ordinary functions.

Theorem 4 — If $\alpha = -(n + \beta + 1)$ with $0 < \beta < 1$ and g is in L, then the equation

$$\mathcal{L}_n^\alpha f(t) = g(t),\ t > 0 \qquad \text{...(4.1)}$$

has a unique solution f in L expressible by

$$f(t) = \frac{(\alpha + n + 1)\, n!}{\Gamma(\alpha + n + 2)} \int_0^t \frac{(t-u)^{-\alpha-2}}{\Gamma(-\alpha-1)}\, {}_1F_1(n; -\alpha-1; t-u)\, g(u)\, du. \qquad \text{...(4.2)}$$

Moreover the solution f is in $L_{-\alpha-1}$, that is, in $L_{n+\beta}$.

PROOF: In view of the isomorphism between locally integrable functions and the analytic functions (of variable s) constructed by the Laplace transformation (Mikusiński 1959, p. 377) and using 4.23(1) of Erdelyi (1954), we get

$$s^{n+\alpha+1}(s-1)^{-n} = \left\{ \frac{t^{-\alpha-2}}{\Gamma(-\alpha-1)}\, {}_1F_1(n; -\alpha-1; t) \right\} \qquad \text{...(4.3)}$$

where ${}_1F_1$ is Kummer's confluent hypergeometric function (Rainville 1960, p. 123). If g is locally integrable, then indeed

$$f = \frac{1}{A} s^{n+\alpha+1}(s-1)^{-n} g$$

$$= \left\{ \frac{1}{A} \int_0^t \frac{(t-u)^{-\alpha-2}}{\Gamma(-\alpha-1)} \, {}_1F_1(n; -\alpha-1; t-u) \, g(u) \, du \right\} \quad \ldots (4.4)$$

satisfies the equation

$$A(s-1)^n s^{-n-\alpha-1} f = g$$

that is $\qquad \mathcal{L}_n^\alpha f = g.$

In the notation of Prabhakar (1969), the integral in (4.4) is $K_{n,-\alpha-1}g(t)$ which (see Prabhakar 1969, 1971) is in $L_{-\alpha-1}$ if g is in L.

Thus we have shown that for $g \in L$, the operator equation $\mathcal{L}_n^\alpha f = g$ has a unique solution f in $L_{n+\beta}$. In order that (4.2) be a solution of (1.3), there has to be a tacit assumption that f is in L_{n+1}. We prefer to state it clearly in the following:

Theorem 5 — Let $f \in L_{n+1}$ and $g \in L$. Then the solution of

$$FP \int_0^t (t-u)^\alpha L_n^\alpha (t-u) f(u) \, du = g(t) \quad \ldots (4.5)$$

is given by

$$f(t) = \frac{(n+\alpha+1)\, n!}{\Gamma(n+\alpha+2)} \int_0^t \frac{(t-u)^{-\alpha-2}}{\Gamma(-\alpha-1)} \, {}_1F_1(n; -\alpha-1; t-u) \, g(u) \, du. \quad \ldots (4.6)$$

PROOF : Indeed the left-hand member of (4.5) can be written as

$$\mathcal{L}_n^\alpha f = g$$

and so we can apply Theorem 4 to get the solution (4.6).

Remark 1 : In view of Theorem 1, the assumption that $f \in L_{n+1}$ can be replaced by the assumption that $f^{(n)}$ is absolutely continuous and $f(0) = 0 = f^{(\nu)}(0)$, $\nu \leqslant n-1$.

Remark 2 : If in (3.1) and subsequent discussion α is replaced by a smaller number, all the results proved still hold.

Particular Cases

(1) For $n = 0$, the above theorem gives the following result :

If $f \in L_1$ and $g \in L$, then the solution of

$$FP \int_0^t (t-u)^{\alpha} f(u)\, du = g(t)$$

is given by

$$f(t) = \frac{1}{\Gamma(\alpha+1)} \int_0^t \frac{(t-u)^{-\alpha-2}}{\Gamma(-\alpha-1)} g(u)\, du$$

where $-2 < \alpha < -1$. This is precisely the Theorem 3.2 of Butzer (1959) for $k = 1$.

(2) For $n = 1$, we have :

If $f \in L_2$ and $g \in L$, then the solution of

$$FP \int_0^t (t-u)^{\alpha} (1 + \alpha - t + u) f(u)\, du = g(t)$$

is given by

$$f(t) = \frac{(\alpha+2)}{\Gamma(\alpha+3)} \int_0^t \frac{(t-u)^{-\alpha-2}}{\Gamma(-\alpha-1)} \, {}_1F_1(1, -\alpha-1; t-u)\, g(u)\, du,$$

where $-3 < \alpha < -2$.

REFERENCES

Buschman, R. G. (1964). Convolution equations with generalized Laguerre polynomial kernels. *SIAM Rev.*, **6**, 166–67.

Butzer, P. L. (1959). Singular integral equations of Volterra type and the finite part of divergent integrals. *Arch. ration. Mech. Analysis*, **3**, 194–205.

Erdelyi, A. (1954). Tables of Integrals Transforms, Vol. I. McGraw-Hill Book Co., Inc., New York.

——— (1962). Operational Calculus and Generalized Functions. Holt, Rinehart and Winston, New York.

——— (1964). An integral equation involving Legendre functions. *J. Soc. Indust. appl. Math.*, **12**, 15–30.

——— (1967). Some integral equations involving finite parts of divergent integrals. *Glasgow Math. J.*, **8**, 50–54.

Gelfand, I. M., and Shilov, G. E. (1964). Generalized Functions, Vol. I. Academic Press, New York.

Khandekar, P. R. (1965). On a convolution transform involving generalized Laguerre polynomial as kernel. *J. Math. Pures Appl.*, **44**, 195–97.

Mikusiński, J. (1959). Operational Calculus. Pergamon Press, New York.

Morton, R. D., and Krall, A. M. (1978). Distributional weight functions for orthogonal polynomials. *SIAM J. Math. Anal.*, **9**, 604–26.

Prabhakar, T. R. (1969). Two singular integral equations involving confluent hypergeometric functions. *Proc. Camb. phil. Soc.*, **66**, 71–89.

——— (1971). Some integral equations with Kummer's functions in the kernels. *Canad. Math. Bull.*, **14**, 391–404.

Rainville, E. D. (1960). Special Functions. Macmillan & Co., New York.

Widder, D. V. (1963). The inversion of convolution transform whose kernel is a Laguerre polynomial. *Am. Math.* Monthly, **70**, 291–93.

Wiener, K. (1968). Uber die Losungen Abelscher integralglichungen mit Hadamard-integralen. *Math. Nachr.*, **48**, 335–39.

——— (1971). Uber integralglichungen mit singularen rechten Seiten und Hadamard-integralen mit singularitaten von ganz zahliger ordung. *Math. Nachr.*, **51**, 111–22, 363–76.

Zemannian, A. H. (1965). Distribution Theory and Transform Analysis. McGraw-Hill Book Co., Inc., New York.